A discriminant criterion of irreducibility

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Introduction

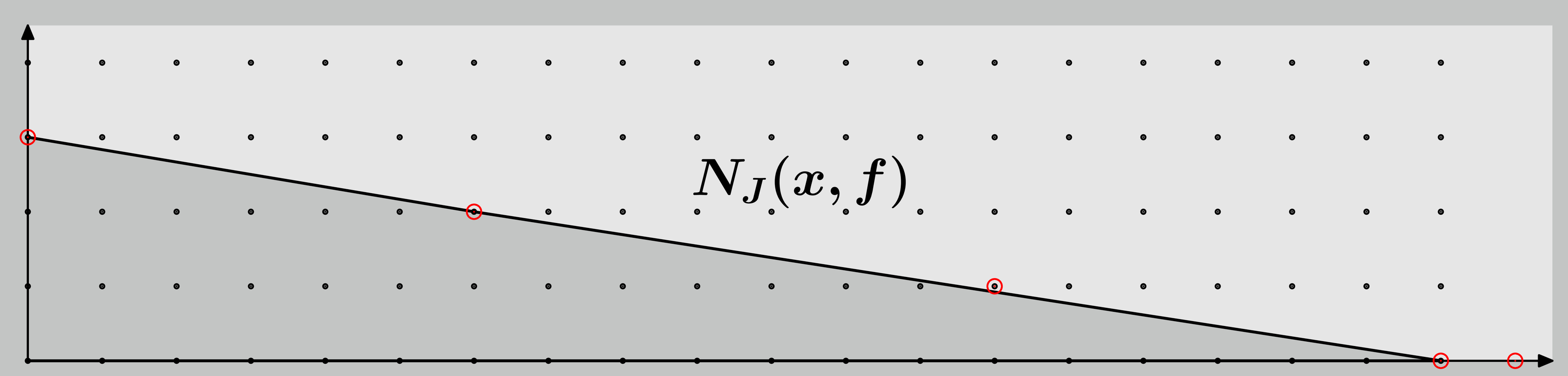
We use the discriminant of a polynomial in one variable to verify if a plane complex analytic curve $f(x, y) = 0$ is analytically irreducible at the origin. Throughout the poster we assume that $f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$ is a distinguished polynomial of degree larger than 1.

- Let $D(u, v) = \mathrm{Discriminant}_y(f(u, y) - v)$.
  - The discriminant curve $D(u, v) = 0$ is geometrically the set of critical values of the mapping $(x, f) : (C^2, 0) \to (C^2, 0)$.
  - The Newton diagram of $D(u, v)$ is called the jacobian Newton diagram of a pair (x, f) and denoted $N_J(x, f)$.

Example

Let $f(x, y) = (y^2 - x^3)^2 - x^5 y$. Then

$$D(u, v) = -256v^3 + 256u^6v^2 + 288u^{13}v - 256u^{19} - 27u^{20}.$$



Red dots are the points of the support of $D(u, v)$.

Merle type diagrams

By definition Δ is the Merle type diagram if there exists an irreducible curve $f = 0$ such that $\Delta = N_J(x, f)$.

- Properties of Merle type diagrams
  - $N_J(x, f)$ is a convenient Newton diagram which is determined by the intersection multiplicity $(f, x)_0$ and the embedded topological type of the curve $f = 0$. The interested reader can see Appendix B.
  - The jacobian Newton diagram $N_J(x, f)$ is a complete topological invariant of the singularity of the curve $xf(x, y) = 0$; in particular $N_J(x, f)$ determines the embedded topological type of the curve $f = 0$.
  - We show an easy way of checking if the given Newton diagram Δ is a Merle type diagram (see Appendix C).

Irreducibility criterion

Theorem. The curve $f(x, y) = 0$ is analytically irreducible at the origin if and only if $N_J(x, f)$ is a Merle type diagram.

The implication $\Rightarrow$ is obvious. The opposite implication follows from

Theorem. Let $f = 0$ and $g = 0$ be two curves such that $N_J(x, f) = N_J(x, g)$. Suppose that $f = 0$ is an irreducible curve. Then $g = 0$ is also irreducible.

Kuo's examples

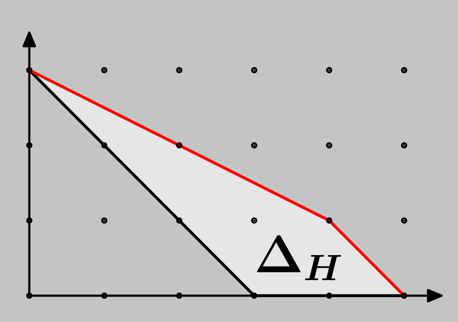
- Take $f(x, y)$ from the previous example. Using notations of Appendix A for convenient Newton diagrams we have $N_J(x, f) = \{\frac{6}{1}\} + \{\frac{13}{2}\}$ which is a Merle type diagram. Hence the curve $f(x, y) = 0$ is irreducible.
- Let $g(x, y) = (y^2 - x^3)^2 - x^7$. Then $D(u, v) = -256(v - u^6 + u^7)(v + u^7)^2$. By drawing the Newton diagram of $D(u, v)$, one can check that $N_J(x, g) = \{\frac{6}{1}\} + \{\frac{14}{2}\}$. Since $N_J(x, g)$ is not a Merle type diagram, $g(x, y) = 0$ is a reducible curve.

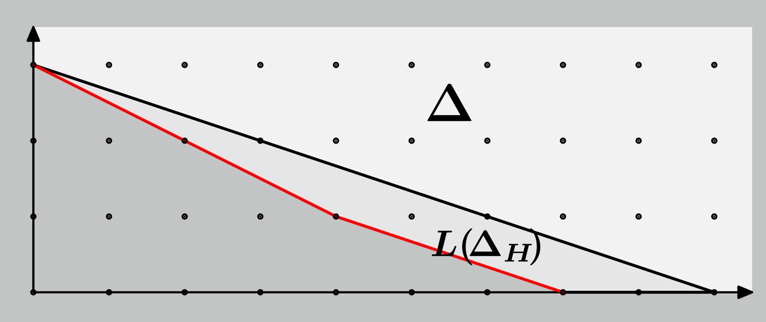
Local irreducibility criterion for algebraic curves

Let $F(x, y)$ be a square free complex polynomial such that the origin is a singular point of the curve $F(x, y) = 0$. Let c be a variable and let $H(u, v) \in C[c][u, v]$ be the polynomial given by $H(u, v) = \mathrm{Discriminant}_y(F(u + cy, y) - v)$. Then the curve $F(x, y) = 0$ is analytically irreducible at the origin if and only if the Newton diagram of $H(u, v)$ is a Merle type diagram.

Irreducibility criterion at infinity

Let $F(x, y) = y^n +$ *terms of lower degrees* be a complex polynomial. Let $H(x, t) = \mathrm{Discriminant}_y(F(x, y) - t)$ and let $L : R^2 \to R^2$ be an affine transformation given by $L(i, k) = (n(n - 1) - i - nk, k)$. We construct the Newton diagram Δ using $L(\Delta_H)$ where Δ_H is the Newton polygon of $H(x, t)$ as shown in the figure below. Then the projective closure of the curve $F(x, y) = 0$ is irreducible at infinity if and only Δ is a Merle type diagram.





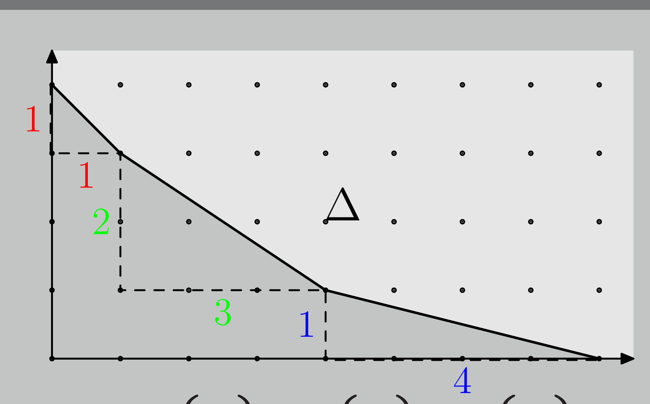
Δ is the region above the red polygon

Conclusions

- Our irreducibility criterion is a two-step procedure: first compute the equation of the discriminant curve, then check if the Newton diagram of the discriminant curve passes the arithmetical test for Merle type diagrams.
- All other known methods are multi-step procedures, for example using Abhyankar's irreducibility criterion (see [1]) we are forced to compute approximate roots and use G-adic expansions with respect to these roots.
- Our method gives an effective criterion of local irreducibility of plane algebraic curves.
- For an affine plane curve with one point at infinity, we obtain a criterion for analytical irreducibility in terms of the Newton diagram of a discriminant.

Appendix A: Teissier's fractions

- The Newton diagram is called convenient if it intersects both coordinate axes.
- Every convenient Newton diagram is a sum of Teissier's fractions.
- Compact edges of Δ are in one-to-one correspondence with Teissier's fractions.



$$\Delta = \left\{\frac{1}{1}\right\} + \left\{\frac{3}{2}\right\} + \left\{\frac{4}{1}\right\}$$

Appendix B: Smith-Merle-Ephraim formula

- Let $f = 0$ be an irreducible singular curve and $l = 0$ be a smooth curve.
- The semigroup $S(f)$ is the set of intersection multiplicities
$$S(f) = \{(f, g)_0 : f \text{ is not a factor of } g\}.$$
- The $(f, l)_0$-minimal system of generators of $S(f)$ is the sequence $\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_h$ determined by conditions:
  - $\bar{b}_0 = (f, l)_0$,
  - $\bar{b}_k = \min(S(f) \setminus (N\,\bar{b}_0 + \cdots + N\,\bar{b}_{k-1}))$,
  - $S(f) = N\,\bar{b}_0 + \cdots + N\,\bar{b}_h$.
- Let $n_1, \ldots, n_h$ be the integers defined as $n_k = \frac{\gcd(\bar{b}_0,\ldots,\bar{b}_{k-1})}{\gcd(\bar{b}_0,\ldots,\bar{b}_k)}$ for $k = 1, \ldots, h$. Then
$$N_J(l, f) = \sum_{k=1}^{h} \left\{\frac{(n_k - 1)\bar{b}_k}{(n_k - 1)n_1 \ldots n_{k-1}}\right\} .$$

Appendix C: Arithmetical test for Merle type diagrams

Let $\Delta = \sum_{i=1}^{h} \{\frac{L_i}{M_i}\}$ where $\frac{L_1}{M_1} < \cdots < \frac{L_h}{M_h}$. Let $H_0 = 1$, $H_i = 1 + M_1 + \cdots + M_i$ for $i \in \{1, \ldots, h\}$ and $C_0 = H_h$, $C_i = H_{i-1}L_i/M_i$ for $i \in \{1, \ldots, h\}$. Then Δ is a Merle type diagram if and only if the following conditions are satisfied:

- H_i/H_{i-1} are integers for $i \in \{2, \ldots, h\}$,
- C_i are integers for $i \in \{1, \ldots, h\}$,
- $\gcd(C_0, \ldots, C_i) = C_0/H_i$ for $i \in \{1, \ldots, h\}$.

Assume that $\Delta = N_J(l, f)$. Then $C_0, \ldots, C_h$ is the $(f, l)_0$-minimal system of generators of the semigroup $S(f)$.

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