The d-Weddle locus for a finite set of points in projective space

Seminarium z geometrii algebraicznej i algebry przemiennej

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Motivation

Definition

A finite set $Z \subset \mathbb{P}^3$ of points whose projection to $\mathbb{P}^2$ from a general point is a complete intersection of a curve of degree a with a curve of degree $b \geqslant a$ is called (a, b)-*geproci*.

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For such Z there are cones (with vertex in a general point) of degree a and b containing Z.

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Problem

Study locus of points occuring as a vertex of degree d cone containing Z.

Classical example

[W] T. Weddle, *On the theorems in space analogous to those of Pascal and Brianchon in a plane*. Part II, Cambridge and Dublin Mathematical Journal, 5 (1850), pp. 58 - 69.

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Let Z be a set of 6 points in $\mathbb{P}^3$ in **L**inear **G**eneral **P**osition and let $d = 2$. Then the vertex locus is a surface, now known as classical Weddle surface.

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[E] A. Emch,*On the Weddle surface and analogous loci*, Transactions of the American Mathematical Society, 27 (1925), pp. 270 - 278.

For larger d and $Z \subset \mathbb{P}^3$ of other cardinality.

The d-Weddle locus

Let $Z = \{P_1, \dots, P_r\} \subset \mathbb{P}^n$ let $P \notin Z$ be a point and let d be a positive number. Let

$$I = I(Z) \cap I(P)^d \subset R = \mathbb{C}[\mathbb{P}^n] = \mathbb{C}[x_0, \dots, x_n],$$

$$\delta(Z, P, d, t) = \dim_{\mathbb{C}}[I]_t.$$

For Z, t and d we define

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Definition

The *d-Weddle locus* of Z is the closure of the set of points $P \in \mathbb{P}^n \setminus Z$ (if any) for which $\delta(Z, P, d, d) > \delta(Z, d, d)$.

The classical Weddle surface

Example

Let $Z \subset \mathbb{P}^3$ be a set of 6 points in LGP. Then the 2-Weddle locus is the classical Weddle surface, i.e., the closure of the locus of points $P \notin Z$ in $\mathbb{P}^3$ that are the vertices of quadric cones in $\mathbb{P}^3$ containing Z.

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The general projection does not have this property and indeed for a point P of the classical Weddle surface we have

$$\dim[I(Z_P)]_2 = 1 > 0 = \delta(Z, Q, 2, 2),$$

where Q is general.

Two approaches to finding the d-Weddle scheme

Interpolation matrix

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The ideal I is saturated and defines a subscheme $X = s_1 P_1 + \cdots + s_r P_r$.
$[I]_t$ is given for each degree t by the kernel of a matrix

$$\Lambda(X, t)$$

with entries in $\mathbb{C}$, known as the *interpolation matrix* for X in degree t.

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- $X = (k+1)P_1$
  $\Lambda((k+1)P_1, t)$ is the $\binom{n+k}{n} \times \binom{n+t}{n}$-matrix whose entries are

$$\Lambda((k+1)P_1, t)_{ij} = \frac{\partial M_j}{\partial m_i}(P_1) = \partial_{m_i} M_j(P_1),$$

  where m_i's are monomials of degree k and

$$\text{for } m = x_0^{i_0} \cdots x_n^{i_n} \text{ denote } \partial_m = \frac{\partial^{i_0}}{\partial x_0^{i_0}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}}.$$

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$Z = P_1 + \cdots + P_r$ and we have an additional fat point dP
The $(r + \binom{n+d-1}{n}) \times \binom{n+d}{n}$-matrix relevant to the d-Weddle locus is

$$\Lambda(Z + dP, d) = \begin{pmatrix} \Lambda(P_1, d) \\ \vdots \\ \Lambda(P_r, d) \\ \Lambda(dP, d) \end{pmatrix} = \begin{pmatrix} \Lambda(Z, d) \\ \Lambda(dP, d) \end{pmatrix}.$$

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The d-Weddle locus is the closure of the locus of points $P \notin Z$ such that $\mathrm{rank}(\Lambda(Z + dP, d)) < \rho(Z, d, d)$, where $\rho(Z, d, d)$ is the maximal rank of $\Lambda(Z + dP, d)$ and it is achieved when P is general.

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Definition

This locus is defined by the ideal $I_{\rho(Z,d,d)}(\Lambda(Z + dP, d))$ of $\rho(Z, d, d) \times \rho(Z, d, d)$ minors of $\Lambda(Z + dP, d)$. We call this ideal the *d-Weddle ideal* for Z.

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Since

$$I_{\rho(Z,d,d)}(\Lambda(Z+dQ,d)) = I_{\rho(Z,d,d)-\alpha}(\Lambda'_{Z+dP,d}),$$

both define the d-Weddle scheme.

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- $L_P = p_0x_0 + \cdots + p_nx_n \in [R]_1$
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More generally, for $0 \leqslant k_i \leqslant t$ and $0 \leqslant d \leqslant t$, we have

$$[I(P_1)^{k_1} \cap \cdots \cap I(P_r)^{k_r}]_t \cong [R^*/(\partial_{L_{P_1}}^{t-k_1+1}, \dots, \partial_{L_{P_r}}^{t-k_r+1})]_t$$

and

$$[I(P_1)^{k_1} \cap \cdots \cap I(P_r)^{k_r} \cap I(P)^d]_t \cong [R^*/(\partial_{L_{P_1}}^{t-k_1+1}, \dots, \partial_{L_{P_r}}^{t-k_r+1}, \partial_{L_P}^{t-d+1})]_t.$$

Macaulay duality

Now have the exact sequence

$$\left[\frac{R^*}{(\partial^t_{L_{P_1}},\dots,\partial^t_{L_{P_r}})}\right]_{d-1} \xrightarrow{\times\partial^{t-d+1}_{L_P}} \left[\frac{R^*}{(\partial^t_{L_{P_1}},\dots,\partial^t_{L_{P_r}})}\right]_t \to \left[\frac{R^*}{(\partial^t_{L_{P_1}},\dots,\partial^t_{L_{P_r}},\partial^{t-d+1}_{L_P})}\right]_t \to 0$$

where we have

$$[R]_{d-1} \cong [R^*]_{d-1} = [R^*/(\partial^t_{L_{P_1}},\dots,\partial^t_{L_{P_r}})]_{d-1},$$

$$[R^*/(\partial^t_{L_{P_1}},\dots,\partial^t_{L_{P_r}})]_t \cong [I(P_1)\cap\cdots\cap I(P_r)]_t$$

and

$$[R^*/(\partial^t_{L_{P_1}},\dots,\partial^t_{L_{P_r}},\partial^{t-d+1}_{L_P})]_t \cong [I(P_1)\cap\cdots\cap I(P_r)\cap I(P)^d]_t.$$

In particular, as a vector space, $[I(P_1)\cap\cdots\cap I(P_r)\cap I(P)^d]_t$ is isomorphic to the cokernel of the map $\times\partial^{t-d+1}_{L_P}$.

Macaulay duality

For the d-Weddle locus we want $t = d$

$$\left[\frac{R^*}{(\partial_{L_{P_1}}^d, \ldots, \partial_{L_{P_r}}^d)}\right]_{d-1} \xrightarrow{\times \partial_{L_P}} \left[\frac{R^*}{(\partial_{L_{P_1}}^d, \ldots, \partial_{L_{P_r}}^d)}\right]_{d} \to \left[\frac{R^*}{(\partial_{L_{P_1}}^d, \ldots, \partial_{L_{P_r}}^d, \partial_{L_P})}\right]_{d} \to 0.$$

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It can be rewritten as

$$([R^*]_0)^r \oplus [R^*]_{d-1} \xrightarrow{D \oplus (\times \partial_{L_P})} [R^*]_d \to [R^*/(\partial^d_{L_{P_1}}, \dots, \partial^d_{L_{P_r}}, \partial_{L_P})]_d \to 0$$

where

$$([R^*]_0)^r \xrightarrow{D} [R^*]_d \quad \text{and} \quad [R^*]_{d-1} \xrightarrow{\times \partial_{L_P}} [R^*]_d$$

$v = (a_1, \dots, a_r) \in ([R^*]_0)^r \mapsto D(v) = a_1 \partial^d_{L_{P_1}} + \dots + a_r \partial^d_{L_{P_r}}$,
$w \in [R^*]_{d-1} \mapsto (\times \partial_{L_P})(w) = w \partial_{L_P}$,
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So now $[I(P_1) \cap \cdots \cap I(P_r) \cap I(P)^d]_d$ is isomorphic to the vector space cokernel of the map $D \oplus (\times\partial_{L_P})$.

Macaulay duality

If we regard $[R^*]_{d-1}$ as being the sum $\oplus_m[R^*]_0$ over all monomials m of degree $d-1$ and $[R^*]_d$ as being the sum $\oplus_M[R^*]_0$ over all monomials M of degree d, then

$$([R^*]_0)^r \oplus [R^*]_{d-1} \xrightarrow{D\oplus(\times\partial_{L_P})} [R^*]_d$$

can (in terms of the bases of monomials m and M) be written as a matrix map $T = T(Z, dP)$

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$$(T_1)_{M,i} = c_M M(P_i),$$

where c_M comes from $\partial^d_{L_{P_i}} = (p_{0i}\partial_{x_0} + \cdots + p_{ni}\partial_{x_n})^d = \sum_M c_M M(P_i)\partial_M$.

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$(T_2)_{M,m} = 0$ unless $mx_i = M$, and then $(T_2)_{M,m} = p_i$.

Interpolation matrix and Macaulay duality

Note that we have

$$\binom{d+n}{n} - \operatorname{rank} \Lambda(Z+dP,d) = \dim\ker\Lambda(Z+dP,d)$$
$$= \dim\operatorname{coker} T(Z+dP) = \binom{d+n}{n} - \operatorname{rank} T(Z+dP)$$

since both the kernel and cokernel are isomorphic to $[I(P_1)\cap\cdots\cap I(P_r)\cap I(P)^d]_d$, and hence $\Lambda(Z+dP,d)$ has the same rank as $T(Z+dP)$.

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- $(N_1)_{ij} = M_i(P_j)$ and $(T_1)_{ij} = c_{M_i} M_i(P_j) = d! M_i(P_j)/e_{M_i}$, so

$$(T_1)_{ij} = c_{M_i}(N_1)_{ij} = d!(N_1)_{ij}/e_{M_i}$$

[for $M = x_0^{i_0} \cdots x_n^{i_n}$: $\quad e_M = i_0! \cdots i_n!$ and $c_M = \dfrac{(i_0 + \cdots + i_n)!}{i_0! \cdots i_n!} = \dfrac{d!}{e_M}$.]

Interpolation matrix and Macaulay duality

$$I_q(T(Z, dQ)) = I_q(\Lambda(Z + dQ, d)), \ \textit{for all } q$$

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- $(N_2)_{ij} = \partial_{m_j} M_i(Q)$ and this is 0 if $m_j \not| M_i$ and it is $e_{M_i} x_{k_{ij}}$ if $m_j | M_i$ where $x_{k_{ij}} = M_i/m_j$ and $(T_2)_{ij}$ is 0 if $m_j \not| M_i$ and it is $x_{k_{ij}}$ if $m_j | M_i$ where $x_{k_{ij}} = M_i/m_j$, so

$$(N_2)_{ij} = e_{M_i}(T_2)_{ij}.$$

Interpolation matrix and Macaulay duality

Theorem

Given a finite set of points $Z \subset \mathbb{P}^n$ and a degree d, let A be a minor of $T = T(Z + dP)$, coming from a given choice of s rows and s columns of T. Assume that the rows correspond to $\partial_{M_{i_j}}$ for monomials $M_{i_1}, \dots, M_{i_s}$, and that j of the chosen columns come from T_1. Let B be the corresponding minor of $N = (\Lambda(Z + dP, d))^t$. Then

$$B = \frac{e_{M_{i_1}} \cdots e_{M_{i_s}}}{(d!)^j} A$$

and thus $I_s(T(Z + dP)) = I_s(\Lambda(Z + dP, d))$.

Example

- $n = 3,\ d = 3$

Example

- $n = 3$, $d = 3$
- Z consists of the points:
  $P_1 = [1 : 0 : 0 : 0]$, $P_2 = [0 : 1 : 0 : 0]$, $P_3 = [0 : 0 : 1 : 0]$,
  $P_4 = [0 : 0 : 0 : 1]$, $P_5 = [1 : 1 : 1 : 1]$, $P_6 = [2 : 3 : 5 : 7]$,

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  $P_4 = [0:0:0:1]$, $P_5 = [1:1:1:1]$, $P_6 = [2:3:5:7]$,
- basis for $[R]_3$:
  $M_1 = x_0^3$, $M_2 = x_0^2x_1$, $M_3 = x_0^2x_2$, $M_4 = x_0^2x_3$, $M_5 = x_0x_1^2$,
  $M_6 = x_0x_1x_2$, $M_7 = x_0x_1x_3$, $M_8 = x_0x_2^2$, $M_9 = x_0x_2x_3$, $M_{10} = x_0x_3^2$,
  $M_{11} = x_1^3$, $M_{12} = x_1^2x_2$, $M_{13} = x_1^2x_3$, $M_{14} = x_1x_2^2$, $M_{15} = x_1x_2x_3$,
  $M_{16} = x_1x_3^2$, $M_{17} = x_2^3$, $M_{18} = x_2^2x_3$, $M_{19} = x_2x_3^2$, $M_{20} = x_3^3$,

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  $M_6 = x_0x_1x_2$, $M_7 = x_0x_1x_3$, $M_8 = x_0x_2^2$, $M_9 = x_0x_2x_3$, $M_{10} = x_0x_3^2$,
  $M_{11} = x_1^3$, $M_{12} = x_1^2x_2$, $M_{13} = x_1^2x_3$, $M_{14} = x_1x_2^2$, $M_{15} = x_1x_2x_3$,
  $M_{16} = x_1x_3^2$, $M_{17} = x_2^3$, $M_{18} = x_2^2x_3$, $M_{19} = x_2x_3^2$, $M_{20} = x_3^3$,
- basis for $[R]_2$:
  $m_1 = x_0^2$, $m_2 = x_0x_1$, $m_3 = x_0x_2$, $m_4 = x_0x_3$, $m_5 = x_1^2$, $m_6 = x_1x_2$,
  $m_7 = x_1x_3$, $m_8 = x_2^2$, $m_9 = x_2x_3$, $m_{10} = x_3^2$.

Example - transpose of interpolation matrix

$$N = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 8 & 6x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \boxed{12} & 2x_1 & 2x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 20 & 2x_2 & 0 & 2x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 28 & 2x_3 & 0 & 0 & 2x_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 18 & 0 & 2x_1 & 0 & 0 & 2x_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 30 & 0 & x_2 & x_1 & 0 & 0 & x_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 42 & 0 & x_3 & 0 & x_1 & 0 & 0 & x_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 50 & 0 & 0 & 2x_2 & 0 & 0 & 0 & 0 & 2x_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 70 & 0 & 0 & x_3 & x_2 & 0 & 0 & 0 & 0 & x_0 & 0 \\
0 & 0 & 0 & 0 & 1 & 98 & 0 & 0 & 0 & 2x_3 & 0 & 0 & 0 & 0 & 0 & 2x_0 \\
0 & 1 & 0 & 0 & 1 & 27 & 0 & 0 & 0 & 0 & \boxed{6x_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 45 & 0 & 0 & 0 & 0 & 2x_2 & 2x_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 63 & 0 & 0 & 0 & 0 & 2x_3 & 0 & 2x_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 75 & 0 & 0 & 0 & 0 & 0 & 2x_2 & 0 & 2x_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 105 & 0 & 0 & 0 & 0 & 0 & x_3 & x_2 & 0 & x_1 & 0 \\
0 & 0 & 0 & 0 & 1 & 147 & 0 & 0 & 0 & 0 & 0 & 0 & 2x_3 & 0 & 0 & 2x_1 \\
0 & 0 & 1 & 0 & 1 & 125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6x_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 175 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2x_3 & 2x_2 & 0 \\
0 & 0 & 0 & 0 & 1 & 245 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2x_3 & 2x_2 \\
0 & 0 & 0 & 1 & 1 & 343 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6x_3
\end{pmatrix}$$

$$N_{2,6} = M_2(P_6) = x_0^2 x_1([2:3:5:7]) = 2^2 3 = 12$$

$$N_{11,11} = \partial_{m_5} M_{11} = \partial_{x_1^2}(x_1^3) = 6x_1$$

Example - Macaulay duality matrix

$$T = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 8 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & \boxed{36} & x_1 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 60 & x_2 & 0 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 84 & x_3 & 0 & 0 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 54 & 0 & x_1 & 0 & 0 & x_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 180 & 0 & x_2 & x_1 & 0 & 0 & x_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 252 & 0 & x_3 & 0 & x_1 & 0 & 0 & x_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 150 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 & x_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 420 & 0 & 0 & x_3 & x_2 & 0 & 0 & 0 & 0 & x_0 & 0 \\
0 & 0 & 0 & 0 & 3 & 294 & 0 & 0 & 0 & x_3 & 0 & 0 & 0 & 0 & 0 & x_0 \\
0 & 1 & 0 & 0 & 1 & 27 & 0 & 0 & 0 & 0 & \boxed{x_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 135 & 0 & 0 & 0 & 0 & x_2 & x_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 189 & 0 & 0 & 0 & 0 & x_3 & 0 & x_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 225 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 630 & 0 & 0 & 0 & 0 & 0 & x_3 & x_2 & 0 & x_1 & 0 \\
0 & 0 & 0 & 0 & 3 & 441 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 & x_1 \\
0 & 0 & 1 & 0 & 1 & 125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 525 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & x_2 & 0 \\
0 & 0 & 0 & 0 & 3 & 735 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & x_2 \\
0 & 0 & 0 & 1 & 1 & 343 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3
\end{pmatrix}$$

$$T_{2,6} = c_{M_2} M_2(P_6) = \frac{3!}{2!1!}(x_0^2 x_1)([2:3:5:7]) = 36,$$

$$T_{11,11} = M_{11}/m_5 = x_1^3/(x_1^2) = x_1$$

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So the 3-Weddle scheme is not reduced and thus is not equal to the 3-Weddle locus.
In fact, the scheme defined by the 3-Weddle ideal consists of the union of the 15 lines together with embedded components at each of the six points of Z.
More precisely, a primary decomposition for the ideal of the 3-Weddle scheme is given by the intersection of the ideals of the 15 lines with the cubes of the ideals of the six points.

Reduced Weddle surface

Theorem

Let L_i be three noncoplanar lines concurrent at a point O. Let $Z = \{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$ be a set of six points in $\mathbb{P}^3$ away from O, distributed in pairs P_i, Q_i on the lines L_i. Then the Weddle surface $\mathcal{W}(Z)$ consists of four planes: the three planes generated by pairs of the lines L_i and the plane spanned by H_1, H_2, H_3, where H_i is the point on L_i such that (P_i, Q_i, O, H_i) are harmonic, for $i = 1, 2, 3$.

Reduced Weddle surface

We may assume that

$$O = [0:0:0:1],\ P_1 = [1:0:0:0],\ P_2 = [0:1:0:0],\ P_3 = [0:0:1:0]$$

$$Q_1 = [a:0:0:1],\ Q_2 = [0:b:0:1],\ Q_3 = [0:0:c:1]$$

for some nonzero a, b, c. Then the interpolation matrix defining the Weddle surface has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a^2 & 0 & 0 & 1 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & b^2 & 0 & 1 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & c^2 & 1 & 0 & 0 & 0 & 0 & 0 & c \\ 2x & 0 & 0 & 0 & y & z & w & 0 & 0 & 0 \\ 0 & 2y & 0 & 0 & x & 0 & 0 & z & w & 0 \\ 0 & 0 & 2z & 0 & 0 & x & 0 & y & 0 & w \\ 0 & 0 & 0 & 2w & 0 & 0 & x & 0 & y & z \end{pmatrix}.$$

Its determinant is $2xyz(bcx + acy + abz - 2abcw)$,

$$H_1 = [2a:0:0:1],\ H_2 = [0:2b:0:1],\ H_3 = [0:0:2c:1]$$

such that (P_i, Q_i, O, H_i) are harmonic.

$(2,3)$-grid

$$Z = \{[1:0:0:0],[0:1:0:0],[1:1:0:0],\\ [0:0:1:0],[0:0:0:1],[0:0:1:1]\}.$$

The Macaulay duality matrix $T'(Z,2P)$ is

$$\begin{bmatrix} z & 0 & x & 0 \\ w & 0 & 0 & x \\ 0 & z & y & 0 \\ 0 & w & 0 & y \end{bmatrix}, \qquad \det(T'(Z,2P)) = 0.$$

The ideal of 3×3 minors of $T'(Z,2P)$ is:

$$(xzw, xw^2, -yzw, -yw^2, -xz^2, -xzw, yz^2, yzw, -xyz, -xyw,\\ y^2z, y^2w, x^2z, x^2w, -xyz, -xyw),\\ \mathrm{sat}(I) = (yw, xw, yz, xz) = (x,y) \cap (z,w),$$

so the 2-Weddle scheme consists of the two lines, $x = y = 0$ and $w = z = 0$, which are grid lines for the (2,3)-grid Z. Hence the 2-Weddle scheme is the same as the 2-Weddle locus.