Teacher Guide
Description of the lesson series

Title	**Algebraic expressions and their addition and subtraction using tokens**
Time	*5-8 school hours (depending on the students' pace and learning level)*
Grade	*Grades 6-8 (students 12-15 years old) or Grade 9 (for students with difficulties in learning mathematics)*
Aim of the lesson cycle and its brief description	*The aim of this series of lessons is to shape the concept of an algebraic expression and its opposition, as well as the addition and subtraction of such expressions using tokens.* *The scenario can be used both in younger grades as an introduction to algebraic expressions and for repetition lessons with students in older grades.* *As students play with the concrete model (tokens), they build up the concept of the algebraic expression and its opposition, and develop an understanding of the operation of addition as adding tokens, and subtraction as taking away tokens.* *Through this, students undertake mathematical modelling.*
Teaching materials	*Each student is given 10 tokens of each colour (white/black) and each shape: (round/oblong/square), for a total set of 60 tokens, to use as tools during the lessons.*

A linguistic note on working with tokens in the context of integers and algebraic expressions:

In our scenarios, we are careful to keep the two worlds - the world of mathematics, i.e. abstractions, and the world of real objects - in our case tokens - linguistically separate. Thus, in the context of tokens, we use terms that describe their appearance: white/black round/ oblong/square token rather than the short-form white circle/rectangle/square. Similarly, in the context of tokens, we mention placing and taking away tokens – while in the context of mathematics, we discuss addition and subtraction operations. We also make a point of verbally reading action signs as add/subtract, rather than just naming them plus/minus signs. We believe that modelling arithmetic and algebraic expressions with clarity and linguistic correctness in mind is of great value and is highly recommended.

PART 1

This material is provided by the AMMA Team, responsible institution: Pedagogical University of Krakow

Part 1

Topic: Introduction to algebraic expressions

ACTIVITY 1: Introduction to models. Rules of the game - opposite tokens

In the next lessons, we will talk about algebraic expressions (in a different way than during previous lessons)*
(*) add if the topic has been discussed before.

We will be using new models and the tokens you are already familiar with.

Regarding the tokens, we already know that:

⊕ ***The white token → represents the number 1***

⊖ ***The black token → represents the number −1***

NOTE!
We detach ourselves from the terms positive/negative and deliberately DO NOT USE them, so as not to build up erroneous associations of the type: $x > 0$, $-x < 0$.

A reminder of the token agreement:

How did we describe the white and black tokens?

- *white and black form a neutral pair*
- *white and black form a pair that does not change value*
- *white with black form a pair that represents 0*
- *white and black form a pair of opposing tokens*
- *black and white eat each other*
- ***black with white (two tokens of different colours) cancel each other out***

DISCUSSION INTRODUCING FURTHER MODELS

We remember that two tokens of different colours add up to a total of zero (representing the value 0), which is why we call them opposite tokens.

Let us remind each other of the definition and notation of opposite numbers (the flow of the dialogue may be similar to the following as in Part 3 of the series of lessons on negative numbers).

- We have so far discussed black and white tokens. What was the most important rule regarding taking away white and black tokens at the same time?

S: They cancel each other out.

- How can we write this using the numbers 1 and -1?

S: 1 + (-1) = 0

- What could such tokens be called? What are they to each other as they cancel each other out? Please provide your ideas.

(Students suggest different names - brainstorming).

- You have made various interesting suggestions. From now on, let's agree to call them ***OPPOSITE TOKENS*** *because that's what mathematicians agreed upon:*

Two numbers that add up to zero are called OPPOSITE NUMBERS*, just as opposite tokens cancel each other out.*

- What number is the opposite of 2?

S: -2

- Why?

S: Because when we add 2 and -2, it results in 0.

- What number is opposite to -2?

S: 2

- Why?

S: Because the sum of -2 and 2 gives 0.

- Provide the number opposite to -100.

S: 100

- Why?

(....)

[The number of examples is chosen by the teacher as needed].

- How does the number and its opposite differ in the notation?

S: We write a minus sign before the number

-Let's pay attention to this notation. How do we write the number opposite to 2?

S: -2

Mathematicians also agreed to write the opposite number to a given number by adding a minus sign in front of the number. *It allows a lazy mathematician to not have to calculate in his head what the reciprocal of a given number is, because he can automatically write such a number down by adding a minus sign.*

- *Then how do we write the number opposite to -2?*

S: - (-2)

If the answer was 2:

- *And how can we do it differently when we know that we are supposed to add a minus sign at the front when we create the opposite number?*

S: - (-2)

If the students do not say this, the teacher will direct them:

- *When I say the opposite of a number, what sign do I add? What does it mean when I write the opposite to -2?*

- *Yes, and we know that the number opposite to -2 is what?*

S: 2

- *What can we notice about the numbers 2 and - (-2)?*

S: - (-2) = 2

- *We had tokens representing the number* 1 *and the number opposite to* 1 *- round tokens.* *Now we want to have any numbers represented in some way* - ***any numbers***, *not just* 1 *and* -1. *Provide examples of such arbitrary numbers.*

S: $4,\ 7,\ -12,\ 0,\ -23,\ -13{,}4\,;\ -\frac{3}{5}, 2\pi, -\sqrt{3};$

- *Any we can think of in general, as for numbers whose values we don't agree on for now - let's label them with the letter* x.

So that we don't get confused by the round tokens, let's now use elongated models, in the shape of a rectangle like such:

x

The white oblong token→ represents x

[Teacher pins the model to the board].

NOTE to students:
We can sometimes use different letters to write certain arbitrary numbers - for example a, b.

- *What number can be* x*?*

S: Any

- *Can* x *can be a negative number?*

S: Yes

- We want to have similar models as for integers. The white and black tokens were opposing tokens. So what will the black oblong token stand for?

$-x$

Black oblong token→ represents -x

[Teacher pins the model on the board].

S: opposite number to x

- How do we write the number opposite to x?

S: $-x$

- Does $-x$ mean a negative number?

S: No, not always, it depends what number x is.

[In case of difficulties:

- And if we substitute x with the number -3, then how much is $-x$?

The teacher accepts both answers 3 or - (-3), in case of answer - (-3) asks:

- What is the number?]

- Let's summarise:

CONCLUSION: $-x$ ***, i.e. the expression x with a minus sign means the expression opposite to*** x***.***   *(From time to time during the lesson, the teacher should ask the students what $-x$ means, to consolidate this).*

- And how we have written the expression opposite to $-x$?

U: $-(-x)$

- On the other hand, we agreed that the opposite of the black oblong token that represents -x is white, and the white rectangular token represents what expression?

S: x

- That is, on the one hand, we know that it is x and at the same time - (-x), so what can we say about the expressions x and -(-x)?

S: $-(-x) = x$

- *We would also like to have a token that denotes any number* x^2*, which is the square of the number* x*. What shape do you propose to make it easy for us to remember and not get confused?*

S: square

- *As before, we agree that:*

x^2 *means* x^2 *i.e. the square of a certain number*

$-x^2$ *means* $-x^2$ *i.e. the number opposite to the square of a certain number*

[Teacher pins models to the board].

Let us summarise:

A white token and a black token of the same shape form a neutral pair, i.e. - they cancel each other out, in other words - they give us a total of 0; and therefore - as for other models - ***two tokens of the same shape but of different colours are called opposite tokens.***

MINUS reads as "the opposite number/expression to".

ACTIVITY 2: Writing algebraic expressions from a model

ACTIVITY 2a

The teacher places tokens [magnetic or virtual] on the board.
The students respond and justify the expressions represented by these tokens.

N	The teacher places the tokens on the board	Students respond	Notes for the teacher / Issues for discussion
1.	x x x x	$4x$	*NOTES:* *- We arrange the tokens in a disorderly, 'chaotic' manner.*
2.	x^2 x^2	$2x^2$	

ACTIVITY 2b

- What is the expression opposite to 4x?

S: -4x

- How do we represent it on the tokens?

S: $-x$ $-x$ $-x$ $-x$

- What is the expression opposite to $2x^2$?

S: $-2x^2$

- How do we represent it with tokens?

S: $-x^2$ $-x^2$

The teacher places tokens [magnetic or virtual] on the board.
The students respond and justify which expressions these tokens represent

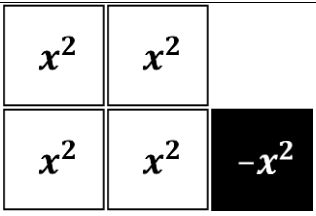
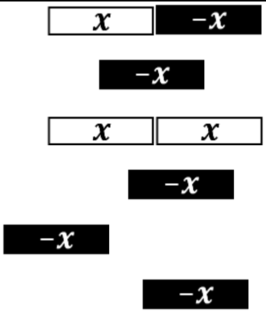
1.	$-x$ $-x$ $-x$	$-3x$
2.	$-x^2$ $-x^2$ $-x^2$	$-3x^2$
3.	x $-x$	0
4.	$-x^2$ x^2	0

ACTIVITY 2c

The teacher tells the students which tokens to put (can put themselves at the same time). The students **arrange them on their desks independently** (the tokens take up a lot of space but can be arranged so that they overlap). The students answer and justify **which algebraic expression is shown.**

A discussion can be held on the most effective way of stacking.

Co-funded by the Erasmus+ Programme of the European Union

Teacher says	Students arrange the tokens	Student response	*Notes for the teacher* / Issues for discussion
Take square tokens: 4 white and 1 black		$3x^2$	*- The students place the tokens however they want on the desk; if some place it chaotically and others neatly, one below the other, then there is an opportunity to discuss approaches - which is more useful*
Take the oblong tokens: 3 white and 5 black		$-2x$	*- If ordering (putting white over black or vice versa) does not come out naturally and spontaneously from the students, deliberately ask questions about the most efficient way of stacking.*

We note (or remind ourselves) that if we have tokens of the same shape and size then the rules we learned when working with tokens on negative numbers carry over.