Notes on Jordan type of an Artinian algebra

Pedro Macias Marques

Departamento de Matemática, ECT, CIMA, IIFA, Universidade de Évora, Rua Romão Ramalho, 59, P–7000–671 Évora, Portugal

Email address: pmm@uevora.pt

Partially supported by WELCOME TO POLAND program of NAWA grant no. PPI/WTP/2022/1/00063

Contents

CHAPTER 1

Introduction

These notes were made for the Preparatory School on Lefschetz Properties, to be held from 6th to 10th May 2024, in Kraków, ahead of the conference "Lefschetz properties in algebra, geometry, topology, and combinatorics". They are an introduction to Jordan type of an Artinian algebra, and collect basic results known so far, with examples to illustrate them along the way.

The study of Jordan types as invariants has a long history, but their relations with the weak and the strong Lefschetz properties form a new subject of study that is drawing attention in the commutative algebra community. The Jordan type of an Artinan algebra tells us whether the algebra has the weak or strong Lefschetz properties, so it is a finer invariant.

The two main sources used to write theses notes were the introductory paper "Artinian algebras and Jordan type" written by the author, with Anthony Iarrobino and Chris McDaniel [**IMMM22**], and "Jordan type of an Artinian algebra, a survey" written with Nasrin Altafi and Anthony Iarrobino [**AIMM24**]. Both papers are attached at the end.

Acknowledgements. The author wishes to thank Nancy Abdallah for her comments and proofreading of this text, and Tony Iarrobino for very useful discussions on how to approach the subject.

CHAPTER 2

Preliminaries

Let us start by defining Jodan type. The definition below is the one we are going to adopt in these notes, but clearly it could be extended to any nilpotent endomorphism of a finite-dimensional vector space (see Definition 1.1 in [**AIMM24**]). Let us fix notation and conventions.

SETTING 2.1. We consider a field k, of any characteristic, and either the local regular ring $\mathcal{R} = \mathsf{k}\{x_1, \dots, x_r\}$ or the polynomial ring $R = \mathsf{k}[x_1, \dots, x_r]$. Let A be an Artinian algebra, either quotient of $\mathcal{R}$ by an Artinian ideal I, or quotient of R by a homogeneous Artinian ideal I. In case A is local, $A = \mathcal{R}/I$, we denote its maximal ideal by m; if A is graded, $A = R/I$, we take $\mathsf{m} = \bigoplus_{i\geq 1} A_i$. If A is graded, but we consider a non-standard grading, we assume $A_0 = \mathsf{k}$.

The *socle* of A is the ideal $(0 : \mathsf{m})$. The *socle degree* of A is the unique integer j such that $\mathsf{m}^j \neq 0 = \mathsf{m}^{j+1}$. If A is graded, the socle degree is the highest degree of a non-zero element in the socle.

DEFINITION 2.2 (Jordan type). Let M be a finitely generated module over an Artinian algebra A as above, and let $\ell \in \mathsf{m}$. The Jordan type of ℓ in M is the partition of $\dim_{\mathsf{k}} M$, denoted

$$P_\ell = P_{\ell,M} = (p_1, \dots, p_s),$$

where $p_1 \geq \cdots \geq p_s$, whose parts p_i are the block sizes in the Jordan canonical form matrix of the multiplication map

$$m_\ell : M \to M, \quad x \mapsto \ell x.$$

EXAMPLE 2.3. Let $A = \mathsf{k}[x, y]/(x^2, xy^2, y^5)$. Then A is a graded Artinian algebra admitting a monomial basis

$$\begin{array}{ccccc} 1 & x & xy & y^3 & y^4 \\ & y & y^2 & & \end{array}$$

as a vector space over k. Being monomial, this is in particular a homogeneous basis, in the sense that all its elements are homogeneous, and therefore it can be partitioned into bases for each homogeneous summand of A:

$$A_0 = \langle 1\rangle; \qquad A_1 = \langle x, y\rangle; \qquad A_2 = \langle xy, y^2\rangle; \qquad A_3 = \langle y^3\rangle; \qquad A_4 = \langle y^5\rangle.$$

When we consider $\ell_1 = x + y$, the multiplication map m_{ℓ_1} sends 1 to $x + y$, then it sends $x + y$ to

$$(x + y)^2 = x^2 + 2xy + y^2 = 2xy + y^2.$$

and so on, giving the following string:

$$1 \quad \mapsto \quad x + y \quad \mapsto \quad 2xy + y^2 \quad \mapsto \quad y^3 \quad \mapsto \quad y^4 \quad \mapsto \quad 0.$$

This string has five elements composing a linearly independent subset of A. Since A has dimension 7 as a vector space over k, we may still consider $x \in A_1$, for instance. We see that the multiplication map m_{ℓ_1} sends x to xy, and sends xy to 0. So we get two strings:

$$\begin{array}{c} 1 \longmapsto x+y \longmapsto 2xy+y^2 \longmapsto y^3 \longmapsto y^4 \longmapsto 0 \\ x \longmapsto xy \longmapsto 0 \end{array} \tag{1}$$

Using these strings, we may consider the basis

$$\mathcal{B} = \{1,\, x+y,\, 2xy+y^2,\, y^3,\, y^4,\, x,\, xy\},$$

and reorder its elements in the sequence

$$(y^4,\, y^3,\, 2xy+y^2,\, x+y,\, 1,\, xy,\, x),$$

to obtain the following matrix representing m_{ℓ_1} with respect to $\mathcal{B}$:

$$\begin{array}{c|ccccc|cc|} & y^4 & y^3 & 2xy+y^2 & x+y & 1 & xy & x \\ \hline y^4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ y^3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2xy+y^2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ x+y & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline xy & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

This matrix is in the Jordan canonical form, having two Jordan blocks, one of size 5, the other of size 2, so we get that the Jordan type of $x+y$ is $P_{x+y,A} = (5,2)$. We call the basis $\mathcal{B}$ a Jordan basis for m_{ℓ_1}. Note that even in the case $\operatorname{char} \mathsf{k} = 2$, $\mathcal{B}$ remains a basis of A. The way we got this basis is not the standard way of computing a Jordan basis, but in the case of Artinian algebras this approach works. For a good explanation of how to find a Jordan basis of a vector space endomorphism (not necessarily nilpotent), see sections 7.7 and 7.8 in [**Mey23**], or the proof of Proposition 4.7.1 in [**Art93**, Section 4.7], where the convention for the Jordan canonical form is lower triangular.

Now if we look back at the strings (1), we see that the first string has 5 non-zero elements, and the second has 2, so we can conclude directly from the strings that the Jordan type is $(5, 2)$.

We can now consider $\ell_2 = x$, and we get the strings

$$
\begin{tikzcd}
1 \arrow[r, mapsto] & x \arrow[r, mapsto] & 0 \\
 & y \arrow[r, mapsto] & xy \arrow[r, mapsto] & 0 \\
 & & y^2 \arrow[r, mapsto] & 0 \\
 & & & y^3 \arrow[r, mapsto] & 0 \\
 & & & & y^4 \arrow[r, mapsto] & 0
\end{tikzcd} \tag{2}
$$

Therefore the Jordan type of x is $P_{x,A} = (2, 2, 1, 1, 1)$.

If we consider $\ell_3 = x + y^2$, a non-homogeneous element, still using this naïve approach to find the Jordan type, we get the strings

$$
\begin{tikzcd}
1 \arrow[r, mapsto] & x+y^2 \arrow[rrr, mapsto] & & & y^4 \arrow[r, mapsto] & 0 \\
 & y \arrow[r, mapsto] & xy+y^3 \arrow[r, mapsto] & 0 \\
 & & y^2 \arrow[rr, mapsto] & & y^4 \\
 & & & y^3 \arrow[r, mapsto] & 0
\end{tikzcd} \tag{3}
$$

Note that this time we did not obtain a Jordan basis, because y^2 is not sent to zero, it is sent to an element in a previous string. But we can modify that string, if we observe that $\ell_3(x + y^2) = \ell_3 y^2$, and therefore

$$\ell_3(x) = \ell_3(x + y^2) - \ell_3 y^2 = 0$$

So we may replace the third string and obtain

$$
\begin{tikzcd}
1 \arrow[r, mapsto] & x+y^2 \arrow[rrr, mapsto] & & & y^4 \arrow[r, mapsto] & 0 \\
 & y \arrow[r, mapsto] & xy+y^3 \arrow[r, mapsto] & 0 \\
 & x \arrow[r, mapsto] & 0 \\
 & & & y^3 \arrow[r, mapsto] & 0
\end{tikzcd} \tag{4}
$$

Therefore the Jordan type of $x + y^2$ is $P_{x+y^2,A} = (3, 2, 1, 1)$.

In this example, both Jordan bases coming from strings in (1) and (2) are homogeneous (and the strings are presented with each column corresponding to a degree). In a graded module M over a graded algebra A, if $\ell \in \mathsf{m}$ is a homogenous element, it is always possible to find a homogeneous Jordan basis for m_ℓ, see [**IMMM22**, Lemma 2.2], for the case of a linear element. However, if ℓ is not homogeneous, or M is not graded, any Jordan basis will be also non-homogeneous, but we have arranged the strings by order. Here is a definition:

Definition 2.4. The *order* of a non-zero element a in an Artinian algebra A is the unique integer i such that $a \in \mathsf{m}^i \setminus \mathsf{m}^{i+1}$.

The basis $\{1,\ x + y^2,\ y^4,\ y,\ xy + y^3,\ y^2,\ y^3\}$ coming from strings in (3) is what we will call a pre-Jordan basis. Here is Definition 1.2 in [**AIMM24**]:

Definition 2.5 (Jordan basis, pre-Jordan basis). With the notation of Definition 2.2, a *pre-Jordan basis* for ℓ is a basis of M as a vector space over k of the form

$$\mathcal{B} = \{\ell^i z_k : 1 \leq k \leq s,\ 0 \leq i \leq p_k - 1\}, \tag{5}$$

where $P_{\ell,M} = (p_1, \dots, p_s)$ is the Jordan type of ℓ. We call the subsets

$$S_k = \{z_k, \ell z_k, \dots, \ell^{p_k - 1} z_k\}$$

strings, or *ℓ-strings*, of the basis $\mathcal{B}$, and each element $\ell^i z_k$ a *bead* of the string. The Jordan blocks of the multiplication m_ℓ are determined by the strings S_k, and M is the direct sum

$$M = \langle S_1 \rangle \oplus \cdots \oplus \langle S_s \rangle. \tag{6}$$

If the elements $z_1, \dots, z_s \in M$ satisfy $\ell^{p_k} z_k = 0$ for each k, we call $\mathcal{B}$ a *Jordan basis* for ℓ, recovering the usual definition in linear algebra, since a matrix representing the multiplication by ℓ with respect to $\mathcal{B}$, ordering elements as

$$(\ell^{p_1-1} z_1, \dots, z_1, \ell^{p_2-1} z_2, \dots, z_2, \dots, \ell^{p_s-1} z_s, \dots, z_s),$$

is a canonical Jordan form. In that case the $\langle S_k \rangle$ are cyclic $\mathsf{k}[\ell]$-submodules of M.

The maximal ideal m is a vector subspace of A, so if we look at A as an affine algebraic set, m is an irreducible algebraic subset. If A is graded, the same holds for A_1. Therefore, it makes sense to consider a generic element in m or in A_1. This motivates the following definition (see Definition 1.2 in [**AIMM24**]).

Definition 2.6 (Generic Jordan type of an Artinian algebra). Suppose k is infinite. The *generic Jordan type* of A, denoted P_A, is the Jordan type $P_{\ell,A}$ for a generic element ℓ of A_1 (when A is graded), or of m (when A is local).

In the next example, we will see that the Jordan type may depend on the characteristic of the field k.

Example 2.7. Let $A = \mathsf{k}[x, y]/(x^2, y^2)$. Then an element ℓ in its maximal ideal can be written as $\ell = ax + by + cxy$.

Suppose $\operatorname{char} \mathsf{k} \neq 2$. It is easy to check that if $ab \neq 0$, $P_{\ell,A} = (3, 1)$. In fact, if the pair (a', b') satisfies $ab' - a'b \neq 0$, we see that the strings

$$\begin{array}{c} 1 \longmapsto \ell \longmapsto 2abxy \longmapsto 0 \\ a'x + b'y \longmapsto (ab' + a'b)xy \end{array} \tag{7}$$

give the pre-Jordan basis $\{1,\ \ell,\ 2abxy,\ a'x + b'y\}$. We may omit the zeros and the elements that are in the span of previous strings and get the simpler diagram

$$\begin{array}{c} 1 \longmapsto \ell \longmapsto 2abxy \\ a'x + b'y \end{array} \tag{8}$$

If k is an infinite field, since the set

$$\{ax + by + cxy \in \mathsf{m} : ab \neq 0\}$$

is open and dense in m, we get that the generic Jordan type of A is $P_A = (3, 1)$. Outside this set, we can consider $b = 0$, and assume $a = 1$, making $\ell = x + cxy$, so we get the strings

$$\begin{array}{c} 1 \longmapsto x + cxy \\ y \longmapsto xy \end{array} \tag{9}$$

so $P_{x+cxy,A} = (2, 2)$. By symmetry on the variables, also $P_{y+cxy,A} = (2, 2)$. Finally, we can easily check that A admits two further Jordan types, namely $P_{xy,A} = (2, 1, 1)$ and the Jordan type of the zero map $P_{0,A} = (1, 1, 1, 1)$.

Now suppose $\operatorname{char} \mathsf{k} = 2$. Then $\ell^2 = 0$, so if $(a, b) \neq (0, 0)$, we can check that $P_{\ell,A} = (2, 2)$. Taking a pair (a', b') satisfying $ab' - a'b \neq 0 \neq ab' + a'b$, we get the strings

$$\begin{array}{r}1 \longmapsto \ell \\ a'x + b'y \longmapsto (ab' + a'b)xy\end{array} \tag{10}$$

Again, if the field k is infinite, this is the generic Jordan type, as the set

$$\{ax + by + cxy \in \mathsf{m} : (a, b) \neq (0, 0)\}$$

is open and dense in m. The two other possible Jordan types for A are again $P_{xy,A} = (2, 1, 1)$ and $P_{0,A} = (1, 1, 1, 1)$.

EXAMPLE 2.8. Let $A = \mathsf{k}[x, y, z]/(yz - x^3,\ y^3,\ z^2)$, a non-graded complete intersection. A monomial basis for A is

$$\begin{array}{ccccccc} 1 & x & x^2 & x^3 & x^4 & x^5 & x^5y \\ & y & xy & x^2y & x^3y & x^4y & \\ & z & xz & x^2z & x^2y^2 & & \\ & & y^2 & xy^2 & & & \end{array}$$

Note that yz has order 3, because in A, $yz = x^3 \in \mathsf{m}^3$. The basis above is organised by order, and we have chosen representatives of each element that make their order apparent (that is why, instead of having yz in the second column, we have x^3 in the third).

Consider $\ell_1 = x + y + z$. Staring with 1 and then taking lower-order elements for the next strings, as in the previous examples, we get:

$$\begin{array}{l} 1 \longmapsto \ell_1 \longmapsto \ell_1^2 \longmapsto \ell_1^3 \longmapsto \ell_1^4 \longmapsto \ell_1^5 \longmapsto \ell_1^6 \\ \quad y \longmapsto \ell_1 y \longmapsto \ell_1^2 y \longmapsto \ell_1^3 y \longmapsto \ell_1^4 y \\ \quad z \longmapsto \ell_1 z \longmapsto \ell_1^2 z \\ \qquad y^2 \longmapsto \ell_1 y^2 \longmapsto \ell_1^2 y^2 \end{array} \tag{11}$$

where

$$\begin{aligned}
\ell_1^2 &= x^2 + 2xy + 2xz + y^2 + 2x^3 \\
\ell_1^3 &= x^3 + 3x^2y + 3x^2z + 3xy^2 + 6x^4 + 3x^3y \\
\ell_1^4 &= x^4 + 4x^3y + 6x^2y^2 + 12x^5 + 12x^4y \\
\ell_1^4 &= x^4 + 4x^3y + 6x^2y^2 + 12x^5 + 12x^4y \\
\ell_1^5 &= x^5 + 5x^4y + 30x^5y \\
\ell_1^6 &= 6x^5y
\end{aligned}$$

$$\begin{aligned}
\ell_1 y &= xy + y^2 + x^3 & \ell_1 z &= xz + x^3 \\
\ell_1^2 y &= x^2y + 2xy^2 + 2x^4 + 2x^3y & \ell_1^2 z &= x^2z + 2x^4 + x^3y \\
\ell_1^3 y &= x^3y + 3x^2y^2 + 3x^5 + 6x^4y & \ell_1 y^2 &= xy^2 + x^3y \\
\ell_1^4 y &= x^4y + 12x^5y & \ell_1^2 z &= x^2y^2 + 2x^4y
\end{aligned}$$

Let us assume that $\operatorname{char} \mathsf{k} \notin \{2, 3, 5\}$, so that all coefficients of these elements are non-zero. In particular $\ell_1^6 \neq 0$. Do the strings in (11) give us a pre-Jordan basis?

Can we conclude from here that the Jordan type of ℓ_1 is $(7,5,3,3)$? Let us look closer at how we built the strings.

After getting the first string, which we know cannot be longer, since $\ell_1^7 = 0$, we chose two elements $y, z \in \mathsf{m} \setminus \mathsf{m}^2$. These elements were chosen because the classes of ℓ_1, y, z in m/m^2 form a linear independent set. Let us look at the next string, whose first bead is y. Could we have chosen another starting bead and get a longer string? If we can, this will be true for a general element in A. Let us suppose we choose a bead b outside the maximal ideal. We can assume $b = 1 + c$, with $c \in \mathsf{m}$. Then the first two strings would be

$$\begin{array}{l} 1 \longmapsto \ell_1 \longmapsto \ell_1^2 \longmapsto \ell_1^3 \longmapsto \ell_1^4 \longmapsto \ell_1^5 \longmapsto \ell_1^6 \\ 1+c \longmapsto \ell_1 + \ell_1 c \longmapsto \ell_1^2 + \ell_1^2 c \longmapsto \ell_1^3 + \ell_1^3 c \longmapsto \ell_1^4 + \ell_1^4 c \end{array}$$

Note that $\ell_1(\ell_1^4 + \ell_1^4 c) = \ell_1^5 + \ell_1^5 c$, and since $\ell_1^5 c \in \mathsf{m}^6 = \langle x^5 y\rangle = \langle \ell_1^6\rangle$, we get that $\ell_1(\ell_1^4 + \ell_1^4 c)$ is a linear combination of elements in the first string. So the second string cannot have length greater than 5. with some patience and looking carefully at the details, we can show that the third string cannot have a larger length that 3, and conclude that $P_{\ell_1,A} = (7,5,3,3)$. However, we will see in Lemma 2.11 below that there is a simpler way of computing the Jordan type of an element.

The following result is well known and is an easy consequence of the existence and construction of the Jordan canonical form of a nilpotent endomorphism (see [**Art93**, Section 4.7], or [**Wei22**]).

Lemma 2.9. *If M has a pre-Jordan basis $\mathcal{B}$ as in* (5)*, then for each k, we have*

$$\ell^{p_k} z_k \in \langle \ell^a z_i : a \geq p_k,\ i < k\rangle.$$

There is a Jordan basis of M derived from the pre-Jordan basis, and having the same partition invariant $P_{\ell,M}$ giving the lengths of strings.

Proof. See Remark A.3. □

Definition 2.10 (Conjugate partition). Let $P = (p_1, \ldots, p_s)$, with $p_1 \geq \cdots \geq p_s$, be a partition of an integer $n > 0$. The *conjugate partition* $P^\vee$ of P is the partition $P^\vee = (p_1^\vee, \ldots, p_t^\vee)$ defined by

$$p_i^\vee = \#\{j : p_j \geq i\}.$$

This corresponds to swapping rows and columns in the Ferrers diagram.

The following well-known result allows us to compute the Jordan type, see [**HMM$^+$13**, Lemma 3.60].

Lemma 2.11. *Let A be an Artinian graded or local algebra with maximum ideal m and socle degree j, and let $\ell \in \mathsf{m}$. Let M be a finite length A-module. The increasing dimension sequence*

$$d_\ell\colon (0 = d_0, d_1, \ldots, d_j, d_{j+1}),\ \text{where } d_i = \dim_{\mathsf{k}} M/\ell^i M, \tag{12}$$

has first difference $\Delta(d_\ell) = (\delta_{d_\ell,1}, \delta_{d_\ell,2}, \ldots, \delta_{d_\ell,j+1})$, which satisfies

$$P_\ell = \Delta(d_\ell)^\vee. \tag{13}$$

The (*decreasing*) *rank sequence*

$$r_\ell\colon (r_0, r_1, \ldots, r_j, 0),\ \text{where } r_i = \dim_{\mathsf{k}}(\ell^i \cdot M) = \text{ rank } m_{\ell^i} \text{ on } M, \tag{14}$$

has first difference $\Delta(r_\ell) = (\delta_{r_\ell,1}, \delta_{r_\ell,2}, \dots, \delta_{r_\ell,j})$ *which satisfies*

$$P_\ell^\vee = \Delta(r_\ell) = \Delta(d_\ell). \tag{15}$$

CHAPTER 3

Lefschetz properties

Lefschetz properties are defined for graded algebras. Here we will see that Jordan type gives us a way of extending these definitions for the non-graded case. We start by giving a summary of the graded case.

Definition 3.1 (Hilbert function). Let A be a graded k-algebra, with $A_0 = \mathsf{k}$, and let M be a graded module over A. The *Hilbert function* of M is the function

$$H(M) : \mathbb{N} \to \mathbb{N} \quad \text{given by} \quad i \mapsto \dim_{\mathsf{k}} M_i.$$

We say that $H(M)$ is *unimodal* if there is an integer k such that for all $i < k$,

$$H(M)_i \leq H(M)_{i+1},$$

and for all $i \geq k$,

$$H(M)_i \geq H(M)_{i+1},$$

If A is a local algebra, with maximal ideal m, we can consider its associated graded algebra

$$A^* = \bigoplus_{i \geq 0} \frac{\mathsf{m}^i}{\mathsf{m}^{i+1}}.$$

We define the Hilbert function of A as that ofA^*, i.e. $H(A) = H(A^*)$, so

$$H(A)_i = \dim_{\mathsf{k}} \frac{\mathsf{m}^i}{\mathsf{m}^{i+1}}.$$

More generally, if M is a module over a local algebra $(A, \mathsf{m}, \mathsf{k})$, we consider the *Hilbert function with respect to the* m*-adic filtration* as

$$H_{\mathsf{m}}(M) = \dim_{\mathsf{k}} \frac{\mathsf{m}^i M}{\mathsf{m}^{i+1} M}.$$

Definition 3.2 (Sperner number). Let A be a graded Artinian k-algebra and let M be a finite-length graded module over A. The Sperner number of M is the maximum value of the Hilbert function of M:

$$\operatorname{Sperner} M = \max\{H(M)_i : i \geq 0\}.$$

The following definitions are standard. Here we are adopting Definitions 3.1 and 3.8 in [**HMM$^+$13**] (see also, for instance, Definition 2.4 in [**MN13**]).

Definition 3.3 (Weak Lefschetz property). Let A be a graded Artinian algebra and consider a linear form $\ell \in A_1$. We say that ℓ is a *weak Lefschetz* (WL) element if for each integer $i \geq 0$, the map

$$\times \ell : A_i \to A_{i+1}, \quad x \mapsto \ell x,$$

has maximal rank. We say that A satisfies the *weak Lefschetz property* (WLP) if it has a weak Lefschetz element.

Definition 3.4 (Strong Lefschetz property). Let A be a graded Artinian algebra and consider a linear form $\ell \in A_1$. We say that ℓ is a *strong Lefschetz* (SL) element if for each pair of integers $i, d \geq 0$, the map

$$\times \ell^d : A_i \to A_{i+d}, \quad x \mapsto \ell^d x,$$

has maximal rank. We say that A satisfies the *strong Lefschetz property* (SLP) if it has a strong Lefschetz element.

Note 3.5. If you read or hear *Lefschetz element*, in these notes or elsewhere, you should understand it as "strong Lefschetz element".

Example 3.6. Consider the Artinian algebra $A = \mathsf{k}[x, y]/(x^2, xy^2, y^5)$, from Example 2.3. It is easy to check that $x + y$ is a strong Lefschetz element, and, furthermore that any linear form $\ell = ax + by$, with $b \neq 0$ is also SL.

Example 3.7. Consider the Artinian algebra $A = \mathsf{k}[x, y]/(x^2, y^2)$, from Example 2.7. It is easy to check that any non-zero linear form $\ell = ax + by$ is a weak Lefschetz element. Furthermore, if $ab \neq 0$ and $\operatorname{char} \mathsf{k} \neq 2$ then ℓ is strong Lefschetz, because $\ell^2 \neq 0$, so the map $\times \ell^2 : A_0 \to A_2$ is an isomorphism.

The following result is well known (see [**HW03**, Remark 3] or [**HMM$^+$13**, Proposition 3.2]). We include the proof from [**HMM$^+$13**] for completeness:

Proposition 3.8. *Let A be a standard-graded Artinian algebra and suppose that A satisfies the weak Lefschetz property. Then the Hibert funtion of A is unimodal.*

Proof. (See proof of Proposition 3.2 in [**HMM$^+$13**].) Since A satisfies the WLP, there is a WL element $\ell \in A_1$. Let $\mathsf{m} = \bigoplus_{i \geq 1} A_i$, and let

$$k = \min\{i \geq 0 : H(A)_i > H(A)_{i+1}\}.$$

Since ℓ is WL, the map $\times \ell : A_k \to A_{k+1}$ is surjective. Since A has the standard grading, for any degree i, the ideal m^i is generated by the homogeneous summand A_i. In particular, we get $\mathsf{m}^{k+1} = \ell \mathsf{m}^k$. By induction on i, we can check that for all $i \geq k$, $\mathsf{m}^{i+1} = \ell \mathsf{m}^i$. In particular, for any $i \geq k$, the map $\times \ell : A_k \to A_{k+1}$ is surjective, and therefore $H(A)_i \geq H(A)_{i+1}$. Hence $H(A)$ is unimodal. $\square$

Harima and Watanabe showed the following result in [**HW03**]:

Proposition 3.9. (*See* [**HW03**, Remark 3 and Proposition 14].) *Let A be a standard-graded Artinian algebra. Then a linear form $\ell \in A_1$ is a weak Lefschetz element if and only if the number of parts in the Jordan type $P_{\ell,A}$ equals the Sperner number of A.*

Proof. Let $\ell \in A_1$ be any linear form. Note that the number of parts in $P_{\ell,A}$ is the number of blocks in the canonical Jordan form of $\times \ell : A \to A$, and equals the dimension of the kernel of this map. If ℓ is WL then $H(A)$ is unimodal, by Proposition 3.8. Let

$$k = \min\{i \geq 0 : H(A)_i > H(A)_{i+1}\}.$$

Then, since ℓ is homogeneous, the dimension of the kernel of $\times \ell : A \to A$ is the sum

$$\begin{aligned} &\dim \ker(\times \ell : A_k \to A_{k+1}) + \dim \ker(\times \ell : A_{k+1} \to A_{k+2}) + \cdots \\ &\qquad = (\dim A_k - \dim A_{k+1}) + (\dim A_{k+1} - \dim A_{k+2}) + \cdots \\ &\qquad\qquad = \dim A_k = \operatorname{Sperner} A. \end{aligned}$$

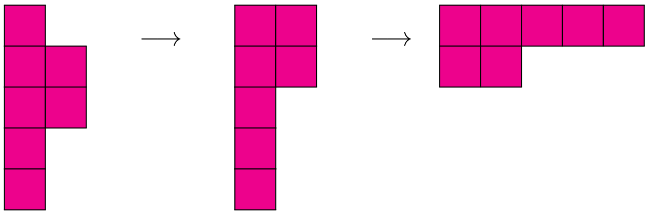
The converse is easy to check, since this last computation shows that $\mathrm{Sperner}\, A$ is a lower bound for the dimension of the kernel of $\times\ell : A \to A$. So if the number of parts in $P_{\ell,A}$ equals $\mathrm{Sperner}\, A$ then $H(A)$ must be unimodal and ℓ must be a WL element. □

Definition 3.10 (Conjugate partition of a Hilbert function). If H is the Hilbert function of an Artinian algebra, or a finite-length module over an Artinian algebra, we consider the partition whose parts are the values of H, after reordering them to become non-increasing; we call the conjugate of this partition the *conjugate partition* of H.

Example 3.11. Consider the Artinian algebra $A = \mathsf{k}[x, y]/(x^2, xy^2, y^5)$, from Example 2.3. The Hilbert function of A is $H(A) = (1, 2, 2, 1, 1)$, and its conjugate is $H(A)^\vee = (5, 2)$.

$H(A)$ $H(A)^\vee$

$(1, 2, 2, 1, 1)$ $(2, 2, 1, 1, 1)$ $(5, 2)$

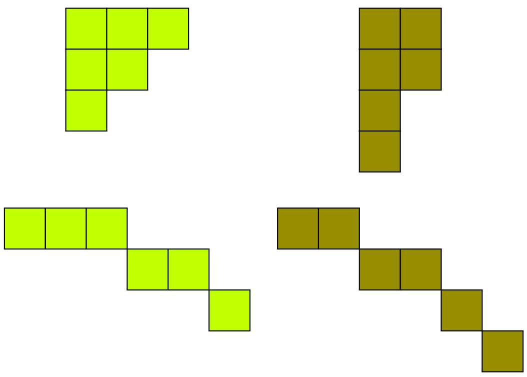


Definition 3.12 (Dominance). Let $P = (p_1, \dots, p_s)$ and $Q = (q_1, \dots, q_r)$, with $p_1 \geq \cdots \geq p_s$, and $q_1 \geq \cdots \geq q_r$, be two partitions of a positive integer n. We say that P *dominates* Q (written $P \geq Q$), if for each $k \in \{1, \dots, \min\{s, r\}\}$, we have

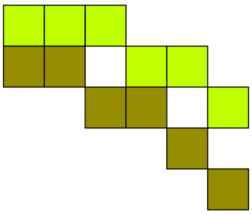
$$\sum_{i=1}^{k} p_i \geq \sum_{i=1}^{k} q_i.$$

One way of viewing dominance partial order graphically is to take a partition and redraw it, putting the beginning of each new row at the point where the previous one ends. Here is the case of two partitions of 6:

$(3, 2, 1)$ $(2, 2, 1, 1)$

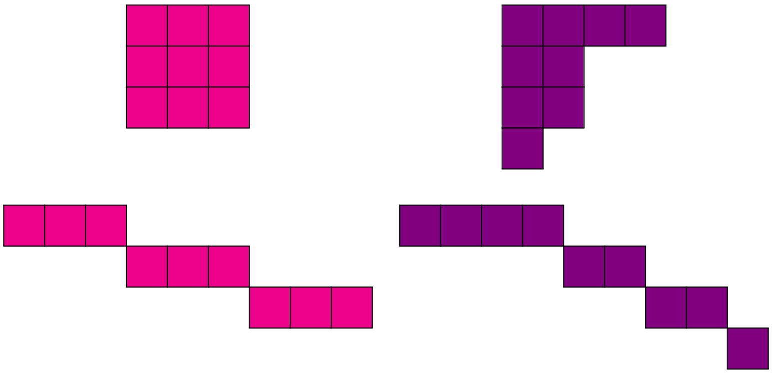


Then, if it is possible to draw one of the partitions right below the other, without a space between first rows, and without overlapping, as in the next picture, we see that the first partition dominates the second.

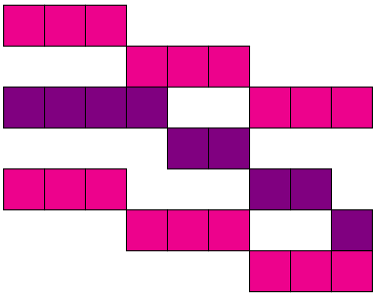


In this case, $(2,2,1,1) < (3,2,1)$. Let us look at the case of the next two partitions of 9:

$(3,3,3)$ $(4,2,2,1)$



We see that we cannot fit one below the other, without space between first rows, or without overlapping:



So $(3,3,3)$ and $(4,2,2,1)$ are incomparable.

Using these two definitions, Harima et al. [**HMM$^+$13**] obtained the following result, giving an upper bound for the Jordan type of a homogeneous element, in the case of a standard graded Artinian algebra whose Hilbert function is unimodal, and a characterisation of strong Lefschetz elements in terms of their Jordan type.

PROPOSITION 3.13. [**HMM$^+$13**, Proposition 3.64] *Let A be a standard graded Artinian algebra, and let $\ell \in \mathsf{m}$ be a homogeneous element.*

(1) *If $H(A)$ is unimodal, then*

$$P_{\ell,A} \leq H(A)^{\vee}.$$

(2) *If $\ell \in A_1$ is a linear form then ℓ is a strong Lefschetz element if and only if*

$$P_{\ell,A} = H(A)^{\vee}.$$

Iarrobino, McDaniel, and the author [**IMMM22**] were able to generalise the upper bound in the first statement of this result to local algebras, or modules over algebras having a non-standard grading, also dropping the hypothesis that the Hilbert function is unimodal.

THEOREM 3.14. [**IMMM22**, Theorem 2.5] *Let $(A, \mathsf{m}, \mathsf{k})$ be a local Artinian algebra over k and let $M \subset A^k$ be an A-module, with Hilbert function $H(M)$. For any $\ell \in \mathsf{m}$, its Jordan type satisfies*

$$P_{\ell,M} \leq H(M)^\vee. \tag{16}$$

If A has weight function w, for which $A_0 = \mathsf{k}$, and if M is a graded module over A with w-Hilbert function $H_\mathsf{w}(M)$, then for any w-homogeneous element $\ell \in \mathsf{m}$ its Jordan type also satisfies

$$P_{\ell,M} \leq H_\mathsf{w}(M)^\vee. \tag{17}$$

Furthermore, in the same paper, the second statement was generalised to non--standard graded Artinian algebras.

PROPOSITION 3.15. [**IMMM22**, Proposition 2.10] *Let A be a (possibly non-standard) graded Artinian algebra and $\ell \in A_1$. Then the following statements are equivalent:*

(1) *For each integer b, the multiplication maps $\times \ell^b : A_i \to A_{i+b}$ have maximal rank in each degree i. (That is, ℓ is a strong Lefschetz element.)*
(2) *The Jordan type of ℓ is equal to the conjugate partition of the Hilbert function, i.e.*
$$P_\ell = H(A)^\vee.$$
(3) *There is a set of strings $S_1, \ldots, S_s$ as in Equation (6), composed of homogeneous elements, for the multiplication map $\times \ell^b : A \to A$ such that for each degree u and each integer i we have the equivalence*

$$\dim_\mathsf{k} A_u \geq i \quad \textit{if and only if} \quad A_u \cap S_a \neq \emptyset,\ \forall a \leq i. \tag{18}$$

EXAMPLE 3.16. Consider the Artinian algebra $A = \mathsf{k}[x, y]/(x^2, y^2)$, from Example 2.7, and suppose $\operatorname{char} \mathsf{k} \neq 2$. Its Hilbert function is $H(A) = (1, 2, 1)$, so we can see that the number of parts in both Jordan types $(3, 1)$ and $(2, 2)$ is the Sperner number of A. Therefore, since $P_{ax+by,A} = (3, 1)$, if $ab \neq 0$, and $P_{x,A} = P_{y,A} = (2, 2)$, we have a different way of checking that $ax + by$, x, and y are all weak Lefschetz. Furthermore, since $H(A)^\vee = (3, 1)$, we have that $ax + by$ is a strong Lefschetz element, if $ab \neq 0$.

The previous results motivated the extension of the definitions of Lefschetz properties to non-graded algebras (see [**IMMM22**, Definition 2.12]):

DEFINITION 3.17. Let $(A, \mathsf{m}, \mathsf{k})$ be a local Artinian algebra over k with Hilbert function $H(A)$. We say that an element $\ell \in \mathsf{m}$ is

(1) a *strong Lefschetz element* if $P_{\ell,A} = H(A)^\vee$.
(2) a *weak Lefschetz element* if the number of parts in $P_{\ell,A}$ equals the Sperner number os A.

Additionally if A is graded via a weight function w with w-Hilbert function $H_\mathsf{w}(A)$, then we say that a w-homogeneous element $\ell \in \mathsf{m}$ has

(3) w-*strong Lefschetz Jordan type* (w-SLJT) if $P_\ell = H_\mathsf{w}(A)^\vee$.

We say that A is strong Lefschetz, respectively weak Lefschetz, respectively w-SLJT if it has an element $\ell \in \mathsf{m}$ of that type.

In light of this extended definition, we can recover a well-known result on Lefschetz properties (see [**IMMM22**, Lemma 2.15]), shown by Briançon in [**Bri77**] in characteristic zero, and extended by Basili and Iarrobino in [**BI08**] for a large enough characteristic. In the case of graded Artinian algebras, this result has also been proved in [**HMNW03**, Proposition 4.4], for characteristic zero, in [**Coo14**, Theorem 4.11], for monomial ideals in any characteristic.

LEMMA 3.18 (Height two Artinian algebras are strong Lefschetz). [**Bri77**], [**BI08**, Theorem 2.16] *Let* $A = \mathsf{k}[x, y]/I$ *be a standard graded Artinian algebra, or a local Artinian algebra of socle degree* j*, and suppose* $\operatorname{char} \mathsf{k} = 0$ *or* $\operatorname{char} \mathsf{k} \geq j$*. Let* ℓ *be a general element of* m*. Then* ℓ *is a strong Lefschetz element and* A *has the strong Lefschetz property.*

CHAPTER 4

Finer invariants, Jordan type, and their behaiviour under deformations

One nice feature of Jordan type is its behaviour along flat families:

LEMMA 4.1 (Generic Jordan type of a module). [**IMMM22**, Lemma 2.54] *Given an A-module M, there is an open dense subset $U_M \subset \mathsf{m}$ for which $\ell \in U_M$ implies that the partition $P_{\ell,M}$ satisfies $P_{\ell,M} \geq P_{\ell',M}$ for any other element $\ell' \in \mathsf{m}$.*

Likewise, if A admits a weight function w, then for each weight i, there is a dense open set $U_{i,M} \subset A_i(\mathsf{w})$ for which $\ell \in U_{i,M}$ implies that $P_{\ell,M} \geq P_{\ell',M}$ for any other $\ell' \in A_i(\mathsf{w})$.

PROPOSITION 4.2 (Semicontinuity of Jordan type). [**IMMM22**, Corollary 2.44]

(1) *Let M_t, for $t \in T$, be a family of constant-length modules over a parameter space T. Then for a neighbourhood $U \subset T$ of t_0, we have that the generic Jordan types satisfy*
$$t \in U \Rightarrow P_{M_t} \geq P_{M_{t_0}}.$$
(2) *Let A_t, $t \in T$ be a constant-length family of local or graded Artinian algebras. Then for a neighbourhood $U \subset T$ of t_0, we have*
$$t \in U \Rightarrow P_{A_t} \geq P_{A_{t_0}}.$$
(3) *Let $\ell_t \in \mathcal{M}_n(\mathsf{k})$ for $t \in T$ be a family of $n \times n$ nilpotent matrices, and let P_t be their Jordan type. Then there is a neighbourhood $U \subset T$ of t_0 such that*
$$P_t \geq P_{t_0} \text{ for all } t \in U.$$

We have seen in the previous chapter that Jordan type is a finer invariant than the Lefschetz properties. We will see now some refinements of Jordan type, starting with Jordan degree type, for the graded case.

EXAMPLE 4.3. Consider the Artinian algebra $A = \mathsf{k}[x,y]/(x^2,y^2)$, from Example 2.7, and suppose $\operatorname{char} \mathsf{k} \neq 2$. The generic Jordan type of A, $P_A = (3,1)$ dominates all other Jordan types. In fact we have a complete chain
$$(3,1) > (2,2) > (2,1,1) > (1,1,1,1).$$

DEFINITION 4.4 (Jordan degree-type). Let A be a graded Artinian algebra, let M be a finite graded A-module, and let $\ell \in A_1$ be any linear element. Let $\mathcal{B}$ be a homogeneous Jordan basis for ℓ, as in Definition 2.5, and consider the decomposition of M as a direct sum
$$M = \langle S_1 \rangle \oplus \cdots \oplus \langle S_s \rangle$$
of cyclic $\mathsf{k}[\ell]$-modules generated by ℓ-strings, with homogeneous beads, of the form
$$S_k = \{z_k,\, \ell z_k, \ldots, \ell^{p_k-1} z_k\}.$$
The *Jordan degree type of ℓ in M* is the sequence of pairs of integers
$$\mathcal{S}_{\ell,M} = \big((p_1,\nu_1), \ldots, (p_s,\nu_s)\big), \tag{19}$$

where $P_{\ell,M} = (p_1, \ldots, p_s)$ is the Jordan type of ℓ in M and for each k, ν_k is the degree of the initial bead z_k. For any $k, k' \in \{1, \ldots, s\}$ if $k < k'$ and $p_k = p_{k'}$, we assume $\nu_k \leq \nu_{k'}$.

Remark 4.5. By Lemma 2.2 in [**IMMM22**], a homogeneous Jordan basis always exists, and the sequence of pairs (19) is an invariant of (M, ℓ), so the Jordan degree type is always well defined.

In the non-graded local case, the closest we have to degree is the order of an element, (see Definition 2.4). The following invariants are introduced and studied in the unpublished work [**IMMS**]:

Definition 4.6 (Sequential Jordan type and Loewy sequential Jordan type). Let A be an Artinian local algebra of socle degree j, let m be its maximal ideal, and let $\ell \in \mathsf{m}$.

The *sequential Jordan type* (SJT) of ℓ in A is given by the sequence

$$(P_{\ell, A/\mathsf{m}^i}),\ i \in \{1, \ldots, j\}$$

of Jordan types of successive quotients of A by powers of the maximal ideal.

The *Loewy sequential Jordan type* (LJT) of ℓ in A is given by the sequence

$$\big(P_{\ell, A/(0:\mathsf{m}^{j-k})}\big),\ k \in \{1, \ldots, j\}$$

of Jordan types of successive quotients of A by the Loewy ideals.

The *double sequential Jordan type* (DSJT) is given by the table whose (a, i) entry is the partition

$$P_{\ell, B_{a,i}},\ \text{where } B_{a,i} = A/\big(\mathsf{m}^i \cap (0 : \mathsf{m}^{j+1-a-i})\big),\ 0 \leq a \leq j,\ 0 \leq i \leq j + 1 - a$$

giving the Jordan type of the quotient of A by intersections of a Loewy ideal with a power of the maximal ideal.

Proposition 4.7. [**IMMS**] *Let $\mathcal{R} = \mathsf{k}\{x_1, \ldots, x_r\}$ be the local regular ring, and consider an element ℓ in the maximal ideal $(x_1, \ldots, x_r)$ of $\mathcal{R}$. Let $\mathcal{A} = \{A_w \mid w \in W\}$ be a family of Artinian algebras, quotients of R, and denote m_w the maximal ideal of each A_w. Let $w_0 \in W$.*

(1) *If the Hilbert function $H(A_w)$ is constant along the family, then there is a neighbourhood U of w_0 such that the sequential Jordan type satisfies*

$$P_{\ell, A_w/\mathsf{m}_w^i} \geq P_{\ell, A_{w_0}/\mathsf{m}_{w_0}^i},\ \textit{for all } i.$$

(2) *If the dimensions of the Loewy ideals $(0 : \mathsf{m}_w^i)$ are constant along the family, then there is a neighbourhood U of w_0 such that the sequential Loewy Jordan type satisfies*

$$P_{\ell, A_w/(0:\mathsf{m}_w^i)} \geq P_{\ell, A_{w_0}/(0:\mathsf{m}_{w_0}^i)},\ \textit{for all } i.$$

(3) *If the dimensions of the ideals $\mathsf{m}_w^i \cap (0 : \mathsf{m}_w^k)$ are constant along the family, then there is a neighbourhood U of w_0 such that the double sequential Jordan type satisfies*

$$P_{\ell, A_w/(\mathsf{m}_w^i \cap (0:\mathsf{m}_w^k))} \geq P_{\ell, A_{w_0}/(\mathsf{m}_{w_0}^i \cap (0:\mathsf{m}_{w_0}^k))},\ \textit{for all } i \textit{ and } k.$$

CHAPTER 5

Artinian Gorenstein algebras

We will now focus on an important class of Artininan algebras, namely those that are Gorenstein. An Artinian algebra A as in Setting 2.1 is Gorentein if and only if its socle $(0 : \mathsf{m})$ is a one-dimensional vector space over k. In this case, we will have $(0 : \mathsf{m}) = \mathsf{m}^j$, where j is the socle degree of A.

Iarrobino showed in [**Iar86, Iar89, Iar94**] that the associated graded algebra $A^* = \bigoplus_{i\geq 0} \frac{\mathsf{m}^i}{\mathsf{m}^{i+1}}$ of an Artinian Gorenstein algebra A has a canonical stratification by ideals $C(a)$ whose successive quotients $Q(a) = C(a)/C(a+1)$ yield an exact pairing:

$$Q(a)_i \times Q(a)_{j_A-a-i} \to \mathsf{k}. \tag{20}$$

Each graded piece of the module $Q(a)$ admits a presentation

$$Q(a)_i \cong \frac{\mathsf{m}^{\,i} \cap (0 : \mathsf{m}^{\,j+1-a-i})}{\mathsf{m}^{\,i} \cap (0 : \mathsf{m}^{\,j-a-i}) + \mathsf{m}^{\,i+1} \cap (0 : \mathsf{m}^{\,j+1-a-i})} \tag{21}$$

(see [**Iar89**, page 350] or [**Iar86**, Section 3]). To better understand this quotient, we may observe that A admits the m-adic filtration

$$A \supset \mathsf{m} \supset \mathsf{m}^2 \supset \cdots \supset \mathsf{m}^j \supset \mathsf{m}^{j+1} = 0$$

and the Loewy filtration

$$A \supset (0 : \mathsf{m}^j) \supset (0 : \mathsf{m}^{j-1}) \supset \cdots \supset (0 : \mathsf{m}) \supset 0.$$

The graded piece $Q(a)_i$ is the quotient of an intersection of a power of m and a Loewy ideal, modded out by the next two such intersections contained in it.

A very useful tool when studying AG algebras is their dual generator. Consider the local regular ring $\mathcal{R} = \mathsf{k}\{x_1, \ldots, x_r\}$, as in Setting 2.1, and the divided-power ring $\mathfrak{D} = \mathsf{k}_{DP}[X_1, \ldots, X_r]$ (for details see [**IK99**, Appendix A]). The ring $\mathcal{R}$ acts on $\mathfrak{D}$ by contraction:

$$x_i^k \circ X_i^K = \begin{cases} X_i^{K-k} & \text{if } K \geq k, \\ 0 & \text{if } K < k. \end{cases} \tag{22}$$

We have ([**Mac94**], [**Iar94**, Lemma 1.1]):

Lemma 5.1 (AG algebras and k-linear maps of $\mathcal{R}$). *There is a one-to-one isomorphism of sets*

$\{$*AG quotients A of $\mathcal{R}$ having socle degree j*$\} \leftrightarrow$

$\{\mathsf{k}$-*linear homomorphisms $\phi : \mathcal{R} \to \mathsf{k}$, with $\phi|_{\mathsf{m}^{j+1}} = 0$ but $\phi|_{\mathsf{m}^j} \neq 0$*$\}$.

Here $A = \mathcal{R}/I$ with $I = \{h : \phi(\mathcal{R} \cdot h) = 0\}$.

Definition 5.2. Let A be an AG algebra, quotient of $\mathcal{R}$. We call an element $F \in \mathfrak{D}$ such that $A = \mathcal{R}/\operatorname{Ann} F$ a *dual generator* for A.

EXAMPLE 5.3. Let $A = \mathsf{k}[x, y, z]/(yz - x^3,\ y^3,\ z^2)$, from Example 2.8. Being a complete intersection, we know that A is an AG algebra, and we can check that its dual generator is $F = X^5Y + X^2Y^2Z$.

APPENDIX A

Jordan basis *à la carte*

The material of this appendix was taken mainly from work in preparation with Tony Iarrobino and Johanna Steinmeyer [**IMMS**].

The following definition is the usual one for a Jordan basis, in the case of nilpotent endomrophisms.

Definition A.1. Let V be a finite-dimensional vector space over a field k, let $\phi : V \to V$ be a nilpotent endomorphism, and let $(p_1, \dots, p_k)$ be its Jordan type, with $p_1 \geq \cdots \geq p_k$. We say that a basis

$$(v_{1,1}, \dots, v_{1,p_1}, v_{2,1}, \dots, v_{2,p_2}, \dots, v_{k,1}, \dots, v_{k,p_k})$$

of V is a *Jordan basis* for ϕ if for every $s \in \{1, \dots, k\}$,

$$\phi(v_{s,m}) = \begin{cases} v_{s,m+1}, & \text{if } m < p_s \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

Definition A.2. Let V be a finite-dimensional vector space over a field k, let $\phi : V \to V$ be a nilpotent endomorphism, and let $(p_1, \dots, p_k)$ be its Jordan type, with $p_1 \geq \cdots \geq p_k$. A *pre-Jordan basis* for ϕ is a basis

$$(v_{1,1}, \dots, v_{1,p_1}, v_{2,1}, \dots, v_{2,p_2}, \dots, v_{k,1}, \dots, v_{k,p_k})$$

of V such that for every $s \in \{1, \dots, k\}$,

$$\phi(v_{s,m}) = v_{s,m+1}, \quad \text{if } m < p_s. \tag{24}$$

Remark A.3. We may construct a Jordan basis $(v'_{s,m})_{1 \leq s \leq k,\, 1 \leq m \leq p_s}$ by induction, from a given pre-Joradn basis $(v_{s,m})_{1 \leq s \leq k,\, 1 \leq m \leq p_s}$. First note that since ϕ is nilpotent, its only eigenvalue is zero. Also, we must have $\phi^{p_1} = 0$, for p_1 is the maximum size of a Jordan block for ϕ. Therefore $\phi(v_{1,p_1}) = \phi^{p_1}(v_{1,1}) = 0$. Write $v'_{1,m} = v_{1,m}$, for $1 \leq m \leq p_1$.

For the induction step, suppose that for some s, we have a list of vectors

$$v'_{1,1}, \dots, v'_{1,p_1}, \dots, v'_{s,1}, \dots, v'_{s,p_s}$$

satisfying (23) and such that

$$v'_{1,1}, \dots, v'_{1,p_1}, \dots, v'_{s,1}, \dots, v'_{s,p_s}, v_{s+1,1}, \dots, v_{s+1,p_{s+1}}, \dots, v_{k,1}, \dots, v_{k,p_k}$$

are linearly independent. Now, we know that the rank of $\phi^{p_{s+1}}$ is

$$(p_1 - p_{s+1}) + \cdots + (p_s - p_{s+1}) = p_1 + \cdots + p_s - sp_{s+1},$$

so its image is spanned by $(v'_{l,m})_{1 \leq l \leq s,\, p_{s+1} < m \leq p_l}$. Since for each $l \in \{1, \dots, s\}$ and each $m \in \{p_{s+1} + 1, \dots, p_l\}$ we have $v'_{l,m} = \phi^{p_{s+1}}(v'_{l,m-p_{s+1}})$, we obtain

$$\operatorname{im} \phi^{p_{s+1}} = \langle \phi^{p_{s+1}}(v'_{l,m}) : 1 \leq l \leq s,\, 1 \leq m \leq p_l - p_{s+1} \rangle.$$

Therefore there is $u \in \langle v'_{l,m} : 1 \le l \le s,\ 1 \le m \le p_l - p_{s+1} \rangle$ such that

$$\phi(v_{s+1,p_{s+1}}) = \phi^{p_{s+1}}(v_{s+1,1}) = \phi^{p_{s+1}}(u).$$

Let us write $v'_{s+1,1} = v_{s+1,1} - u$, and $v'_{s+1,m+1} = \phi(v'_{s+1,m})$, for $1 \le m < p_{s+1}$. It is now straighforward to check that $v'_{1,1}, \ldots, v'_{1,p_1}, \ldots, v'_{s+1,1}, \ldots, v'_{s+1,p_{s+1}}$ satisfy (23) and

$$v'_{1,1}, \ldots, v'_{1,p_1}, \ldots, v'_{s+1,1}, \ldots, v'_{s+1,p_{s+1}}, v_{s+2,1}, \ldots, v_{s+2,p_{s+2}}, \ldots, v_{k,1}, \ldots, v_{k,p_k}$$

are linearly independent, which concludes the induction step.

Lemma A.4. *Let V be a finite-dimensional vector space over an infinite field k, let $\phi : V \to V$ be a nilpotent endomorphism, and let $(p_1, \ldots, p_k)$ be its Jordan type, with $p_1 \ge \cdots \ge p_k$. Let $V = V_0 \supset \cdots \supset V_j$ be a strictly-decreasing filtration of vector subspaces of V such that $\phi(V_i) \subseteq V_{i+1}$, for $0 \le i < j$. Then there is a pre-Jordan basis B for ϕ such that the cardinality of $B \cap V_i$ equals $\dim V_i$, for each $i \in \{0, \ldots, j\}$.*

Proof. If $p_1 > 1$, then $\phi^{p_1-1} \neq 0$, and therefore there is $v_{1,1} \in V$ such that $\phi^{p_1-1}(v_{1,1}) \neq 0$. Since k is infinite, we can ask that $v_{1,1} \in V_0 \setminus V_1$. For each $m < p_1$, define $v_{1,m+1} = \phi(v_{1,m})$, and let $W_1 = \langle v_{1,1}, \ldots, v_{1,p_1} \rangle$.

Now, ϕ induces an endomorphism $\phi_1 : \frac{V}{W_1} \to \frac{V}{W_1}$, whose Jordan type is $(p_2, \ldots, p_k)$. Let e_1 be the lowest integer such that

$$\frac{V_{e_1}}{(V_{e_1} \cap W_1) + V_{e_1+1}} \neq 0.$$

We can then choose $v_{2,1} \in V_{e_1} \setminus V_{e_1+1}$ such that $\phi_1^{p_2-1}(v_{2,1} + W_1) \neq 0$, for this is an open condition on V_{e_1}. Define $v_{2,m+1} = \phi(v_{2,m})$, for each $m < p_2$. Continuing in this manner, suppose for some $s < k$ we have vectors

$$v_{1,1}, \ldots, v_{1,p_1}, \ldots, v_{s,1}, \ldots, v_{s,p_s}$$

satisfying condition (24). In addition, suppose that, for each $l \in \{1, \ldots, s-1\}$, $\phi_l^{p_{l+1}-1}(v_{l+1,1} + W_l) \neq 0$, where $W_l = \langle v_{1,1}, \ldots, v_{1,p_1}, \ldots, v_{l,1}, \ldots, v_{l,p_l} \rangle$ and

$$\phi_l : \frac{V}{W_l} \to \frac{V}{W_l}$$

is the endomorphism induced by ϕ, and $v_{l+1,1} \in V_{e_l} \setminus V_{e_l+1}$, where e_l is the lowest integer such that

$$\frac{V_{e_l}}{(V_{e_l} \cap W_l) + V_{e_l+1}} \neq 0.$$

By construction, we see that we get a basis for V

$$B = \{v_{1,1}, \ldots, v_{1,p_1}, \ldots, v_{k,1}, \ldots, v_{k,p_k}\}$$

satisfying (24) and $\#(B \cap V_i) = \dim V_i$, for each $i \in \{0, \ldots, j\}$. □

Definition A.5. Let A be an Artinian local algebra over a field k and let m be its maximal ideal. Let B be a basis of A as a k-vector space. We say that B is *compatible with the Hilbert function* of A if for every $i \ge 0$, the cardinality of $B \cap \mathsf{m}^i$ equals $\dim \mathsf{m}^i$.

A simple linear algebra argument shows that any Artinian local algebra A over a field k always admits a basis as a k-vector space that is compatible with its Hilbert function, but the following example by Chris McDaniel shows that it is not always possible to find such a basis that is also a Jordan basis for the multiplication of an element in the maximal ideal.

Example A.6 (AG algebra A having no Jordan basis for the multiplication $\times x$ consistent with $H(A)$ - C. McDaniel, private communication). Let

$$F = XY^3 + X^2Y \in \mathsf{k}_{DP}[X, Y],$$

$A = \mathsf{k}\{x, y\}/I$, $I = \operatorname{Ann} F = (x^2 - xy^2,\ y^4)$, having Hilbert function

$$H(A) = (1, 2, 2, 2, 1).$$

Considering $\ell = x$, we easily find a pre-Jordan basis for the multiplication $\times x$ compatible with the Hilbert function:

$$\begin{tikzcd}
1 \arrow[r, mapsto] & x \arrow[rr, mapsto] & & xy^2 & \\
& y \arrow[r, mapsto] & xy \arrow[rr, mapsto] & & xy^3 \\
& & y^2 & & \\
& & & y^3 &
\end{tikzcd} \tag{25}$$

Note that this is not a Jordan basis, as y^2 and y^3 are not kernel elements. We have here, that multiplication by x on A does not admit a Jordan basis that is compatible with the Hilbert function of A. See [**IMM**, Example 2.14].

Lemma A.7. *Let A be an Artinian local algebra over a field k, let m be its maximal ideal, and let $\ell \in \mathsf{m}$. Then the multiplication by ℓ admits a pre-Jordan basis compatible with the Hilbert function of A.*

Proof. Apply Lemma A.4 to the filtration $A \supset \mathsf{m} \supset \mathsf{m}^2 \supset \cdots \supset \mathsf{m}^j$, where j is the socle degree of A. $\square$

Bibliography

[AIMM24] Nasrin Altafi, Anthony Iarrobino, and Pedro Macias Marques, *Jordan type of an Artinian algebra, a survey*, arXiv:2307.00957, to appear in "Lefschetz Properties: Current and New Directions", Springer INdAM Series, 2024.

[Art93] Michael Artin, *Algebra*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 1993.

[BI08] Roberta Basili and Anthony Iarrobino, *Pairs of commuting nilpotent matrices, and Hilbert function*, Journal of Algebra **320** (2008), no. 3, 1235–1254.

[Bri77] Joël Briançon, *Description de* $H\mathrm{ilb}^n C\{x, y\}$, Inventiones Mathematicae **41** (1977), no. 1, 45–89.

[Coo14] David Cook, II, *The strong Lefschetz property in codimension two*, Journal of Commutative Algebra **6** (2014), no. 3, 323–345.

[HMM$^+$13] Tadahito Harima, Toshiaki Maeno, Hideaki Morita, Yasuhide Numata, Akihito Wachi, and Junzo Watanabe, *The Lefschetz properties*, Lecture Notes in Mathematics, vol. 2080, Springer, Heidelberg, 2013.

[HMNW03] Tadahito Harima, Juan Carlos Migliore, Uwe Nagel, and Junzo Watanabe, *The weak and strong Lefschetz properties for Artinian K-algebras*, Journal of Algebra **262** (2003), no. 1, 99–126.

[HW03] Tadahito Harima and Junzo Watanabe, *The finite free extension of Artinian K-algebras with the strong Lefschetz property*, Rendiconti del Seminario Matematico della Università di Padova **110** (2003), 119–146.

[Iar86] Anthony Iarrobino, *Tangent cone of a Gorenstein singularity*, Proceedings of the conference on algebraic geometry (Berlin, 1985), Teubner-Texte Math., vol. 92, Teubner, Leipzig, 1986, pp. 163–176.

[Iar89] ———, *The Hilbert function of a Gorenstein Artin algebra*, Commutative algebra (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 15, Springer, New York, 1989, pp. 347–364.

[Iar94] ———, *Associated graded algebra of a Gorenstein Artin algebra*, Memoirs of the American Mathematical Society **107** (1994), no. 514, viii+115.

[IK99] Anthony Iarrobino and Vassil Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, Lecture Notes in Mathematics, vol. 1721, Springer-Verlag, 1999, Appendix C by Iarrobino and Steven L. Kleiman.

[IMM] Anthony Iarrobino and Pedro Macias Marques, *Reducibility of a family of local artinian gorenstein algebras*, arXiv:2112.14664.

[IMMM22] Anthony Iarrobino, Pedro Macias Marques, and Chris McDaniel, *Artinian algebras and Jordan type*, Journal of Commutative Algebra **14** (2022), no. 3, 365–414.

[IMMS] Anthony Iarrobino, Pedro Macias Marques, and Johanna Steinmeyer, *Jordan type and symmetric decomposition of an Artinian Gorenstein algebra*, in preparation.

[Mac94] Francis Sowerby Macaulay, *The algebraic theory of modular systems*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1994, Revised reprint of the 1916 original, With an introduction by Paul Roberts.

[Mey23] Carl Dean Meyer, *Matrix analysis and applied linear algebra*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2023.

[MN13] Juan Migliore and Uwe Nagel, *Survey article: a tour of the weak and strong Lefschetz properties*, Journal of Commutative Algebra **5** (2013), no. 3, 329–358.

[Wei22] Steven H. Weintraub, *Jordan canonical form—theory and practice*, Synthesis Lectures on Mathematics and Statistics, vol. 6, Springer, Cham, [2022] ©2022, Reprint of the 2009 original.

APPENDIX B

Paper “Artinian algebras and Jordan type”, by Anthony Iarrobino, Pedro Macias Marques, and Chris McDaniel

JOURNAL OF
COMMUTATIVE ALGEBRA
Volume 14 (2022), No. 3, 365–414

ARTINIAN ALGEBRAS AND JORDAN TYPE

ANTHONY IARROBINO, PEDRO MACIAS MARQUES AND CHRIS MCDANIEL

There has been much work on strong and weak Lefschetz conditions for graded Artinian algebras A, especially those that are Artinian Gorenstein. A more general invariant of an Artinian algebra A or finite A-module M that we consider here is the set of Jordan types of elements of the maximal ideal $\mathfrak{m}$ of A, acting on M. Here, the Jordan type of $\ell \in \mathfrak{m}_A$ is the partition giving the Jordan blocks of the multiplication map $m_\ell : M \to M$. In particular, we consider the Jordan type of a generic linear element ℓ in A_1, or in the case of a local ring $\mathcal{A}$, that of a generic element $\ell \in \mathfrak{m}_{\mathcal{A}}$, the maximum ideal.

We often take $M = A$, the graded algebra, or $M = \mathcal{A}$ a local algebra. The strong Lefschetz property of an element, as well as the weak Lefschetz property can be expressed simply in terms of its Jordan type and the Hilbert function of M. However, there has not been until recently a systematic study of the set of possible Jordan types for a given Artinian algebra A or A-module M, except, importantly, in modular invariant theory, or in the study of commuting Jordan types.

We first show some basic properties of the Jordan type. In a main result we show an inequality between the Jordan type of $\ell \in \mathfrak{m}_{\mathcal{A}}$ and a certain local Hilbert function. In our last sections we give an overview of topics such as the Jordan types for Nagata idealizations, for modular tensor products, and for free extensions, including examples and some new results. We also propose open problems.

2020 AMS *Mathematics subject classification:* primary 13E10; secondary 13D40, 13H10, 14B07, 14C05.
Keywords and phrases: Artinian algebra, Hilbert function, Jordan type, Lefschetz property, tensor product.
Received by the editors on December 23, 2019, and in revised form on June 25, 2020.

DOI: 10.1216/jca.2022.14.365

1. Introduction

The strong and weak Lefschetz properties of graded Artinian algebras A have been extensively studied, especially in the last twenty years. These properties are determined by the ranks of the multiplication operators $m_{\ell^i} : A \to A$ by powers ℓ^i of generic linear elements of $\ell \in A$. Our first main goal is to generalize these properties in several ways. First, given any linear element ℓ in A_1, we will consider the set of all such ranks of m_{ℓ^i}, which determine a *Jordan type* P_ℓ, which is a partition of $n = \dim_{\mathsf{k}} A$. The partition P_ℓ compared to the Hilbert function of A determines whether the pair (ℓ, A) is strong Lefschetz, or weak Lefschetz; however, in general there are many other possible Jordan types, and we explore these possibilities. Second, we will consider not only graded (standard or nonstandard) Artinian algebras A, but also extend the notions of Lefschetz properties and Jordan type to local Artinian algebras $\mathcal{A}$ with maximal ideal $\mathfrak{m}_{\mathcal{A}}$: for these we will consider the Jordan type for either arbitrary or generic elements $\ell \in \mathfrak{m}_{\mathcal{A}}$.

Our second goal is to give a systematic account of the basic facts about Jordan type for multiplication maps in Artinian local algebras, or modules over them. We state in broad terms what is known, prove new results, and give many examples. One of our main new results, Theorem 2.5, establishes inequalities between the Jordan type $P_{\ell,M}$ of any element $\ell \in \mathfrak{m}_{\mathcal{A}}$, acting by the map $m_\ell : M \to M$ on an $\mathcal{A}$-module M and its Hilbert function $H(M)$; if $\mathcal{A}$ admits a weight function w, i.e., if it is graded, then the inequalities are between $P_{\ell,M}$ and its w-Hilbert function $H_{\mathsf{w}}(M)$, where ℓ is w-homogeneous. We explore subtleties of the definition of Jordan type and its connection with Lefschetz properties in Section 2C. We also introduce a refinement, the Jordan degree type, which specifies not only the Jordan type, but the initial degrees of the strings — the maximal cyclic $\mathsf{k}[\ell]$ submodules of A under the action of m_ℓ into which we may decompose A, in Section 2F. For a graded Artinian Gorenstein algebra A the Jordan degree type is symmetric: this greatly restricts the possible Jordan types for A. This invariant has been in effect studied by T. Harima and J. Watanabe [53], and by B. Costa and R. Gondim [39], the latter introducing a colorful notion of string diagrams.

Our third main goal, in Section 3, is to outline several other approaches to the study of Jordan type, adding our own comments and results. For example, there have been studies in representation theory of modules having a constant Jordan type, and of the generic Jordan type of a module, and, as well, of the connection of Jordan type loci to vector bundles [17; 30; 42; 104; 105]. In addition there has been progress in the study of the modular case of tensor products of Jordan types, generalizing the Clebsch–Gordan formula in characteristic zero (Section 3B, Remark 3.16). These studies have been by several groups, some apparently unaware of the related work of others: we give an overview in Section 3B.

Free extensions of an Artinian algebra A with fiber B were introduced by Harima and Watanabe [52]; in [66, Theorem 2.1] we showed that, geometrically, a free extension is a flat deformation of $A \otimes_{\mathsf{k}} B$. Our Theorem 3.23 gives criteria for finding free extensions that are complete intersections.

There are subtle conditions, investigated by P. Oblak, T. Košir and others, on which pairs of Jordan types P_ℓ, $P_{\ell'}$, partitions of n, "commute", i.e., they can simultaneously occur for a single Artinian graded algebra A or local algebra $\mathcal{A}$ of length n [13; 65; 71; 72; 74; 98; 99; 102] (Section 3D). We hope that our discussion and results in Section 3 will suggest new connections that will be useful to the reader.

Detailed overview. In Section 2 we state and prove the basic properties for Jordan types for elements of Artinian algebras. In Section 2A we present well known equivalent definitions of the Jordan type of the multiplication map m_ℓ for an element $\ell \in \mathfrak{m}$, the maximal ideal, and we present some properties of

Artinian algebras that we will need. In Section 2B we prove a main result, Theorem 2.5, bounding above the Jordan type of a nonunit element ℓ of a local algebra $\mathcal{A}$, acting via m_ℓ on an $\mathcal{A}$-module M, by the conjugate of the Hilbert function of M; and we also show the analogue for graded algebras. Here we use the dominance partial order on partitions (Definition 2.4).

In Section 2C we connect Jordan type with Lefschetz properties. We will also in Section 2C consider Jordan type for more general elements $\ell \in \mathfrak{m}_{\mathcal{A}}$ or for nonhomogeneous elements of $\mathfrak{m}_A$ when A is graded. We say that an element ℓ of the maximal ideal $\mathfrak{m}_A$ has "strong Lefschetz Jordan type" (SLJT) if $P_\ell = H(A)^\vee$, the conjugate partition of the Hilbert function $H(A)$ (Definition 2.12) — this gives a fine tuning of the concept of strong Lefschetz, and as well provides an additional invariant (namely, SLJT) for distinguishing Artinian nonstandard graded algebras.

In Section 2D we define an invariant, the contiguous partition $P_c(H)$ associated to a Hilbert function H (Definition 2.17) and show that for a graded A-module M, its Jordan type is bounded above in the dominance order by the contiguous partition $P_c(H)$ (Theorem 2.20).

We introduce the Macaulay duality in Section 2E, which we use to define examples. In Section 2F we introduce a finer invariant, the Jordan degree type (JDT) of a graded module over a graded algebra (Definition 2.28) and relate it to the central simple modules of Harima and Watanabe. There is a natural partial order on JDT related to specialization (Lemma 2.29). We use this to show that a Jordan type locus in G_T may have several irreducible components (Example 2.31). We show also that the Jordan degree type is bounded in the concatenation order by a degree-invariant associated to the Hilbert function (Proposition 2.32). Finally, we give in Section 2G several results and examples highlighting the deformation properties of Jordan type; in particular we discuss Jordan type and initial ideal, and we compare the generic Jordan types of a local algebra $\mathcal{A}$ and its associated graded algebra.

Section 3 outlines further work by many groups on several aspects of Jordan types. In Section 3A we discuss the generic Jordan types for algebras constructed by idealization or by partial idealization: these include some well-known non-WL examples of H. Sekiguchi (H. Ikeda), idealization examples of Watanabe and R. Stanley, and, as well, partial idealization examples of Gondim and collaborators. These Jordan types can be weak Lefschetz (number of parts equal to the Sperner number of H) but not strong Lefschetz. In Section 3B we report briefly on the Jordan type of tensor products, in both nonmodular (char k is zero or char $\mathsf{k} = p$ is large) and modular cases. Our contribution is to introduce here the Jordan degree type, and prove for it an analogue of the Clebsch–Gordan formula for the tensor product.

Harima and Watanabe introduced free extensions C of a base algebra A with fiber B [52], which we have shown elsewhere are deformations of the tensor product [66]. We here in Section 3C show a new result that is useful in deciding when C is a free extension and also in determining complete intersection extensions (Theorem 3.23). In Section 3D we discuss which pairs of Jordan types are compatible for different elements of the same Artinian algebra A. In Section 3E we propose further problems and possible directions of study.

Related notes are [66], which focuses on free extensions; and [83], joint with S. Chen, which studies Jordan type and rings of relative coinvariants. This paper is our introduction to Jordan type and Jordan degree type for Artinian algebras A, and to some of the subtleties that arise, particularly in comparing these invariants with the Hilbert function of A.

1A. Notation. Throughout the paper k will be an arbitrary field unless otherwise specified — except that we will assume k is infinite when we discuss "generic" Jordan type or parametrization.

Local Artinian algebras. We will on the one hand consider a local Artinian algebra $\mathcal{A}$ containing the field $\mathsf{k} = \mathcal{A}/\mathfrak{m}_{\mathcal{A}}$, where $\mathfrak{m}_{\mathcal{A}}$ is the maximal ideal. The Jordan type P_ℓ of an element $\ell \in \mathfrak{m}_{\mathcal{A}}$ is the partition of $n = \dim_{\mathsf{k}}(\mathcal{A})$ giving the sizes of the blocks in the Jordan block decomposition of the multiplication operator $m_\ell : \mathcal{A} \to \mathcal{A}$; since the homomorphism m_ℓ is nilpotent, it is determined up to similarity by the partition P_ℓ. Recall that the *socle* of $\mathcal{A}$ is the ideal $\mathrm{Soc}(\mathcal{A}) = 0 : \mathfrak{m}_{\mathcal{A}}$. We denote by $j_{\mathcal{A}}$, the *maximal socle degree* of $\mathcal{A}$, the highest integer j such that $\mathfrak{m}_{\mathcal{A}}^j \neq 0$. There is a natural $\mathfrak{m}_{\mathcal{A}}$-adic filtration

$$\mathcal{A} \supset \mathfrak{m}_{\mathcal{A}} \supset \mathfrak{m}_{\mathcal{A}}^2 \supset \cdots \supset \mathfrak{m}_{\mathcal{A}}^{j_{\mathcal{A}}} \supset 0 \tag{1-1}$$

of $\mathcal{A}$; the *order* $\nu(a)$ of a nonzero element $a \in \mathcal{A}$ is the largest integer i such that $a \in \mathfrak{m}_{\mathcal{A}}^i$. The *associated graded algebra* $\mathcal{A}^*$ of $\mathcal{A}$ is $\mathcal{A}^* = \mathrm{Gr}_{\mathfrak{m}_{\mathcal{A}}}(\mathcal{A}) = \bigoplus_{i=0}^{j} \mathcal{A}_i$, where $\mathcal{A}_i = \mathfrak{m}_{\mathcal{A}}^i/\mathfrak{m}_{\mathcal{A}}^{i+1}$. Here $\mathcal{A}^*$ is standard graded, in the sense that $\mathcal{A}^*$ is generated over $\mathcal{A}_0 = \mathsf{k}$ by $\mathcal{A}_1$. The *Hilbert function* of the local algebra $\mathcal{A}$ is $H(\mathcal{A}) = (h_0, h_1, \dots, h_j)$, where $h_i = \dim_{\mathsf{k}} \mathcal{A}_i$.

Graded Artinian algebras. By an $\mathbb{N}$-*graded Artinian algebra* over k we mean a graded ring $A = \bigoplus_{i \geq 0} A_i$ with $A_0 = \mathsf{k}$ which has finite dimension as a k-vector space. In this graded case we denote by j_A the largest integer j for which $A_j \neq 0$. We can write $A = R/I$, where R is the polynomial ring $R = \mathsf{k}[x_1, \dots, x_r]$, and I an ideal. A local Artinian algebra $\mathcal{A}$ (with a single maximal ideal, as we will always assume) can be written $\mathcal{A} = \mathcal{R}/I$, where $\mathcal{R}$ is the regular local ring $\mathcal{R} = \mathsf{k}\{x_1, \dots, x_r\}$.

The socle of A is $\mathrm{Soc}(A) = 0 : \mathfrak{m}_A \subset A$, an ideal of A that includes A_{j_A}. The Hilbert function of the graded algebra A is the sequence of nonnegative integers $H(A) = (h_0, \dots, h_j)$, where $h_i = \dim_{\mathsf{k}} A_i$. We say that A is *standard graded* if $A = \mathsf{k}[A_1]$, the algebra generated by degree one (linear) forms over $A_0 = \mathsf{k}$; otherwise it is nonstandard graded.

We observe that a graded Artinian algebra A over k may be regarded as a local Artinian algebra $\mathcal{A}$ over k, with maximal ideal $\mathfrak{m}_{\mathcal{A}} = \bigoplus_{i=1}^{j_A} A_i$, and a *weight function* w specifying the grading. That is, there is an algebra homomorphism $\mathsf{w} : \mathcal{A} \to \mathcal{A}[t]$, and the i*-th graded component* of $\mathcal{A}$ (with respect to w) is given by $\mathcal{A}_{i(\mathsf{w})} = \mathsf{w}^{-1}(\mathcal{A} \cdot t^i)$. Then $\mathcal{A}(\mathsf{w}) = \bigoplus_{i \geq 0} \mathcal{A}_{i(\mathsf{w})}$ is a graded Artinian algebra in the above sense. We will sometimes use the notation $\mathcal{A}(\mathsf{w})$ when we want to stress that the Artinian algebra $\mathcal{A}$ is endowed with the weight function w. We define the w-socle degree of $\mathcal{A}$ to be the largest integer $j = j(\mathsf{w})$ for which $\mathcal{A}_{j(\mathsf{w})} \neq 0$, and the w-Hilbert function $H_{\mathsf{w}}(\mathcal{A}) = (h_0^{\mathsf{w}}, \dots, h_j^{\mathsf{w}})$ where $h_i^{\mathsf{w}} = \dim_{\mathsf{k}} \mathcal{A}_{i(\mathsf{w})}$. Recall that for a local ring $\mathcal{A}$ the Hilbert function $H(\mathcal{A})$ is defined as that of the associated graded algebra $\mathrm{Gr}_{\mathfrak{m}_{\mathcal{A}}}(\mathcal{A})$; thus, $\mathcal{A}(\mathsf{w})$ is standard-graded if and only if $H(\mathcal{A}) = H_{\mathsf{w}}(\mathcal{A})$.

For either R or the regular local ring $\mathcal{R} = \mathsf{k}\{x_1, \dots, x_r\}$ we may specify a weight function w, hence a grading, on suitable quotients of either, using the shorthand notation $\mathsf{w}(x_1, \dots, x_r) = (d_1, \dots, d_r)$, meaning that $\mathsf{w}(x_i) = t^{d_i}$ for each i: a quotient $A = R/I$ is *suitable* for w if I is a homogeneous ideal in the weighting w.

We fix a finite $\mathcal{A}$-module M, of $\dim_{\mathsf{k}} M = n$ and we consider multiplication maps $m_\ell : M \to M$ by an element $\ell \in \mathfrak{m}_{\mathcal{A}}$. The *Jordan type* $P_\ell = P_{\ell,M}$ (to make M explicit) is the partition of n specifying the block sizes in the Jordan canonical form of m_ℓ. When k is infinite, the *generic Jordan type* is $P_M = P_{\ell,M}$ for a sufficiently general element $\ell \in \mathfrak{m}_{\mathcal{A}}$ (Definition 2.55). We can make similar definitions for $\mathbb{N}$-graded algebras A with $A_0 = \mathsf{k}$ by localizing at the maximum ideal, so $\mathcal{A} = A_{\mathfrak{m}_A}$ where $\mathfrak{m}_A = \bigoplus_{i \geq 1} A_i$. Evidently, the Jordan type of $a \in A$, the graded algebra, is the same as that of the corresponding element $a \in \mathcal{A} = A_{\mathfrak{m}_A}$. However, for a standard graded algebra (generated by A_1 over $A_0 \cong \mathsf{k}$), a special role is played by linear elements $a \in A_1$. We explore this in Sections 2B and 2C.

Question 1.1. For a standard graded local algebra $\mathcal{A}$ are the two Jordan types $J_{\mathcal{A}}$ defined by a generic element $a \in \mathcal{A}_1$ and the local algebra Jordan type defined by a generic element $a \in \mathfrak{m}_{\mathcal{A}}$ the same? Of course, the latter is greater or equal J_A in the dominance order (Definition 2.4). We will show these Jordan types are the same when $\mathcal{A}$ is standard graded and the Hilbert function $H(\mathcal{A})$ is unimodal (Proposition 2.14). In the nonstandard graded context, the Jordan type of $\mathcal{A}$ over $\ell \in \mathfrak{m}$ can be strictly greater than J_A [83, Proposition 3.9].

2. Basic properties of Jordan type

2A. Jordan type of a multiplication map. Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathsf{k})$ be an Artinian local algebra over k and let M be a finite $\mathcal{A}$-module; in particular M is a finite-dimensional vector space over k. For any element $\ell \in \mathfrak{m}_{\mathcal{A}}$, let $m_\ell : M \to M$ denote the multiplication map $m_\ell(x) = \ell \cdot x$. Then m_ℓ is a nilpotent k-linear transformation. We will sometimes denote m_ℓ by $\times\ell$.

Definition 2.1 (Jordan type). (See also [58, Section 3.5]) For any element $\ell \in \mathfrak{m}$ its Jordan type is the partition of $\dim_{\mathsf{k}} M$, denoted $P_\ell = P_{\ell,M} = (p_1, \dots, p_s)$, where $p_1 \geq \cdots \geq p_s$, whose parts p_i are the block sizes in the Jordan canonical form matrix of the multiplication map m_ℓ.

If $P_{\ell,M} = (p_1, \dots, p_s)$ is the Jordan type of ℓ, then there are elements $z_1, \dots, z_s \in M$ (depending on ℓ) such that $\{\ell^i z_k \mid 1 \leq k \leq s,\ 0 \leq i \leq p_k - 1\}$ is a k-basis for M: we will term this set a *pre-Jordan basis* for M. The Jordan blocks of the multiplication m_ℓ are determined by the *strings* $S_k = \{z_k, \ell z_k, \dots, \ell^{p_k-1} z_k\}$, and M is the direct sum

$$M = \langle S_1 \rangle \oplus \cdots \oplus \langle S_s \rangle. \tag{2-1}$$

We say that the same set is a *Jordan basis* or *ℓ-basis* for M if also, for each k, $\ell^{p_k} z_k = 0$. In that case the $\langle S_k \rangle$ are cyclic $\mathsf{k}[\ell]$-submodules.

If A is a graded Artinian algebra and M is a graded A-module, we say that the string S_k is *homogeneous* if each element $\ell^i z_k$ is. For simplicity, we state the following lemma for $\ell \in A_1$, but it has a natural generalization to $\ell \in A_d$, $d \geq 1$.

Lemma 2.2. *Let A be a graded Artinian algebra, M a finite-length graded A-module, and let $\ell \in A_1$. Then we may choose strings $S_1, \dots, S_s$ as in Definition 2.1 defining the unique Jordan type $P_{\ell,M}$ such that*:

(i) *Each z_k is homogeneous of a degree ν_k, and $\ell^i z_k$ has degree $\nu_k + i$ for $0 \leq i < p_k$, but $\ell^{p_k} z_k = 0$.*

(ii) *We have*
$$\dim_{\mathsf{k}} M_d = \#\{k \mid \nu_k \leq d < \nu_k + p_k\}.$$

(iii) *Any pre-Jordan basis for M as in Definition 2.1 may be refined to a set satisfying* (i) *and* (ii) *by $\mathsf{k}[\ell]$-linear operations.*

(iv) *Given $\ell \in A_1$, the set of pairs of integers $\mathcal{S}_{\ell,M} = \{(p_k, \nu_k), k = 1, \dots, s\}$ is independent of the choice of the set of strings $\{S_k\}$ decomposing M, and satisfying* (i).

Proof. This is the result of applying the standard method of finding a good basis of a vector space $V = A$ in which a transformation $T = m_\ell = \times\ell$ will have Jordan normal form [44, §VII.7]. We briefly sketch the proof. For (i), choose an element z_1 such that $\ell^{p_1-1} z_1 \neq 0$. Since ℓ is homogeneous, we can assume that z_1 is also homogeneous and make $S_1 = \{z_1, \ell z_1, \dots, \ell^{p_1-1} z_1\}$. We can construct the remaining strings inductively, choosing in each step a homogeneous element $z_{k+1} \in \ker(\times\ell^{p_{k+1}})$ such that $\ell^{p_{k+1}-1} z_{k+1} \notin \langle S_1\rangle \oplus \cdots \oplus \langle S_k\rangle$ — this is always possible, given that, by construction, we have $\operatorname{im}(\times\ell^{p_{k+1}}) \subseteq \langle S_1\rangle \oplus \cdots \oplus \langle S_k\rangle$, but $\operatorname{im}(\times\ell^{p_{k+1}-1}) \nsubseteq \langle S_1\rangle \oplus \cdots \oplus \langle S_k\rangle$, so we can take z'_{k+1} such that $\ell^{p_{k+1}-1} z'_{k+1} \notin \langle S_1\rangle \oplus \cdots \oplus \langle S_k\rangle$, and consider $u \in \langle S_1\rangle \oplus \cdots \oplus \langle S_k\rangle$ such that $\ell^{p_{k+1}} z'_{k+1} = \ell^{p_{k+1}} u$.

Then $z'_{k+1} - u \in \ker(\times\ell^{p_{k+1}})$, and we may take z_{k+1} to be a homogeneous component of $z'_{k+1} - u$ satisfying $\ell^{p_{k+1}-1} z_{k+1} \notin \langle S_1\rangle \oplus \cdots \oplus \langle S_k\rangle$.

Alternatively, we could start with a Jordan basis $z'_1, \ldots, \ell^{p_1-1} z'_1, \ldots, z'_s, \ldots, \ell^{p_s-1} z'_s$, and replace each Jordan chain $z'_k, \ldots, \ell^{p_1-1} z'_k$ by $z_k, \ldots, \ell^{p_k-1} z_k$, where z_k is a homogeneous summand of z'_k satisfying $\ell^{p_1-1} z_k \neq 0$.

We easily see that (ii) is a direct consequence of (i).

To check (iii) we may also construct a new set of strings inductively. The key point here is that if $S_1, \ldots, S_s$ is a pre-Jordan basis of M as in Definition 2.1 then for each k, the generator z_k is not in $\operatorname{im}(\times\ell)$, otherwise the lengths of the strings would not match the Jordan type. If $S_1, \ldots, S_k$ are already modified into strings $\{S'_1, \ldots, S'_k\}$, with generators $z'_1, \ldots, z'_k$, satisfying (i), we can observe that

$$\ell^{p_{k+1}} z_{k+1} \in \big(\langle S'_1\rangle \oplus \cdots \oplus \langle S'_k\rangle\big) \cap \operatorname{im}(\times\ell^{p_{k+1}}) = \langle l^i z'_b : i \geq p_{k+1}, 1 \leq b \leq k\rangle.$$

In particular there exists $u \in \langle S'_1\rangle \oplus \cdots \oplus \langle S'_k\rangle$ such that $\ell^{p_{k+1}} z_{k+1} = \ell^{p_{k+1}} u$. We now set $z'_{k+1} = z_{k+1} - u$ and $S'_{k+1} = \{z'_{k+1}, \ldots, \ell^{p_{k+1}-1} z'_{k+1}\}$.

For the proof of (iv), let $1 \leq n_1 < \cdots < n_t = s$ be the integers defined by the conditions

$$p_1 = p_{n_1} > p_{n_1+1} = p_{n_2} > p_{n_2+1} = p_{n_3} > \cdots,$$

capturing the places where the partition $P_{\ell,M} = (p_1, \ldots, p_s)$ drops. Let $S_1, \ldots, S_s$ be any set of strings of a Jordan basis satisfying (i). Then the classes of $z_1, \ldots, z_{n_1}$ form a homogeneous basis of the graded module

$$\frac{M}{\ker(\times\ell^{p_1-1})} = \frac{\ker(\times\ell^{p_1})}{\ker(\times\ell^{p_1-1})},$$

so their degrees $\nu_1, \ldots, \nu_{n_1}$ are determined by the dimensions of its graded pieces, up to permutation, which shows that the pairs $(p_1, \nu_1), \ldots, (p_{n_1}, \nu_{n_1})$ are uniquely determined. For induction, suppose that the pairs $(p_1, \nu_1), \ldots, (p_{n_i}, \nu_{n_i})$ are uniquely determined. Then the classes of

$$\ell^{p_1-p_{n_i+1}} z_1, \ \ldots, \ \ell^{p_1-p_{n_i+1}} z_{n_1}, \ \ell^{p_{n_1+1}-p_{n_i+1}} z_{n_1+1}, \ \ldots, \ \ell^{p_{n_1+1}-p_{n_i+1}} z_{n_2}, \ \ldots, \ z_{n_i+1}, \ \ldots, \ z_{n_{i+1}}$$

form a homogeneous basis of the graded module $\ker(\times\ell^{p_{n_i+1}})/\ker(\times\ell^{p_{n_i+1}-1})$. Since the first pairs are determined, the degrees $\nu_{n_i+1}, \ldots, \nu_{n_{i+1}}$ are also uniquely determined. □

The set of pairs $S_{\ell,M} = \{(p_k, \nu_k)\}$ in Lemma 2.2(iv) we will later term the *Jordan degree type* of (M, ℓ) (Definition 2.28).

For an increasing sequence $d_\ell = (d_0 \leq d_1 \leq \cdots \leq d_j)$ we let $\delta_{d_\ell,i} = d_i - d_{i-1}$ for $0 \leq i$ with $d_{-1} = 0$. For a decreasing sequence $r_\ell = (r_0 \geq r_1 \geq \cdots \geq r_j)$ we let $\delta_{r_\ell,i} = r_i - r_{i+1}$.

The following result is well-known; see [58, Lemma 3.60].

Lemma 2.3. (i) *Let A be an Artinian graded or local algebra with maximum ideal $\mathfrak{m}_A$ and highest socle degree $j = j_A$ (so $A_j \neq 0$ but $A_i = 0$ for $i > j$); and assume $\ell \in \mathfrak{m}_A$. Let M be a finite length A-module. The increasing dimension sequence*

$$d_\ell : (0 = d_0, d_1, \dots, d_j, d_{j+1}), \qquad \textit{where } d_i = \dim_{\mathsf{k}} M/\ell^i M, \tag{2-2}$$

has first difference $\Delta(d_\ell) = (\delta_{d_\ell,1}, \delta_{d_\ell,2}, \dots, \delta_{d_\ell,j+1})$, which satisfies

$$P_\ell = \Delta(d_\ell)^\vee. \tag{2-3}$$

Here $\Delta(d_\ell)^\vee$ is the conjugate (exchange rows and columns in the Ferrers diagram) of $\Delta(d_\ell)$.

(ii) *The (decreasing) rank sequence*

$$r_\ell : (r_0, r_1, \dots, r_j, 0), \qquad \textit{where } r_i = \dim_{\mathsf{k}}(\ell^i \cdot M) = \textit{ rank } m_{\ell^i} \textit{ on } M, \tag{2-4}$$

has first difference $\Delta(r_\ell) = (\delta_{r_\ell,1}, \delta_{r_\ell,2}, \dots, \delta_{r_\ell,j})$ which satisfies

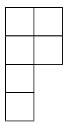
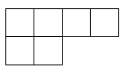
$$P_\ell^\vee = \Delta(r_\ell) = \Delta(d_\ell). \tag{2-5}$$

Note. The Jordan type partition P_ℓ has sometimes been called the Segre characteristic of ℓ [112]. The Weyr canonical form of a multiplication map is a block decomposition "dual" to the Jordan canonical form [101, §2.4]; the Weyr characteristic is the partition giving the sizes of the blocks in the Weyr form, and is just the conjugate $P_\ell^\vee$ of P_ℓ. For further discussion of the Weyr form, which may have advantages for some problems, see [112; 81; 101; 100].

It is readily seen that the Jordan type of (A, ℓ) may depend on the characteristic of k. For example A. Wiebe notes that $\mathsf{k}[x, y, z]/(x^2, y^2, z^2)$ is strong Lefschetz when char $\mathsf{k} \neq 2$, but is not even weak Lefschetz when char $\mathsf{k} = 2$ [115, Example 2.10]. The same holds for $\mathsf{k}[x, y]/(x^p, y^p)$, which is strong Lefschetz for char $\mathsf{k} = 0$ or char $\mathsf{k} > p$ but is not SL when char $\mathsf{k} = p$. This dependence is studied in particular by D. Cook, II for monomial ideals, or in codimension two [35; 36]. We discuss it for tensor products in Section 3B.

2B. Jordan type and Hilbert function for a local algebra.

Partitions and the dominance order. By a *partition* we mean a weakly decreasing sequence of nonnegative integers $P = (p_1, \dots, p_s)$, $p_1 \geq \dots \geq p_s$. The p_i are called the *parts* of P, the *length* of P is the number of its parts $\ell(P) = s$, and the *size* of P is the sum of its parts $|P| = p_1 + \dots + p_s$. We can represent the partition P as a *Ferrers diagram* by which we mean a left justified array of boxes with p_i boxes in the i-th row. The *conjugate partition* $P^\vee = (p_1^\vee, \dots, p_t^\vee)$ is the partition with parts $p_i^\vee = \#\{j \mid p_j \geq i\}$; its Ferrers diagram can be obtained from that of P by reflection about the main diagonal, that is, swapping the rows and columns. For example $P = (2, 2, 1, 1)$ and the conjugate $P^\vee = (4, 2)$ have Ferrers diagrams

 and 

respectively. Note that $\left(P^\vee\right)^\vee = P$.

Definition 2.4 (Dominance order). Given two partitions $P = (p_1, \dots, p_s)$ and $P' = (p'_1, \dots, p'_t)$ with $p_1 \geq \cdots \geq p_s$ and $p'_1 \geq \cdots \geq p'_t$, we write

$$P \leq P' \quad \text{if} \sum_{k=1}^{i} p_k \leq \sum_{k=1}^{i} p'_k \text{ for all } i. \tag{2-6}$$

Thus, $(2, 2, 1, 1) < (3, 2, 1)$ but $(3, 3, 3)$ and $(4, 2, 2, 1)$ are incomparable.

In the dominance partial order we have for two partitions P, P' of n (see [33, Lemma 6.3.1])

$$P \leq P' \Leftrightarrow P^\vee \geq P'^\vee. \tag{2-7}$$

Any sequence $H = (h_0, \dots, h_j)$ of nonnegative integers can be made into a partition by simply rearranging its parts so that they are in nonincreasing order. In particular if $H = H(N)$, the Hilbert function of an Artinian module N over an Artinian algebra A and $\max\{h_k\} = r$ (a special case is $N = \mathcal{A}$, a local Artinian algebra) then we form the *conjugate partition of the Hilbert function*

$$H(N)^\vee = (h_0^\vee, \dots, h_r^\vee), \qquad \text{where } h_i^\vee = \#\{k \mid h_k \geq i\} \text{ in } H(N). \tag{2-8}$$

The following key result says that the Jordan type is always bounded above in the dominance partial order by the conjugate partition of the appropriate Hilbert function. We will consider modules $M \subset \mathcal{A}^k$ that are subsets of free modules $\mathcal{A}^k$, in particular their Hilbert functions satisfy $H(M) = \big(h_0(M), h_1(M), \dots\big)$ with entries only in nonnegative degrees.

Theorem 2.5 (Jordan type and Hilbert function). *Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathsf{k})$ be a local Artinian algebra over k and let $M \subset \mathcal{A}^k$ be an $\mathcal{A}$-module, with Hilbert function $H(M)$. For any $\ell \in \mathfrak{m}_{\mathcal{A}}$, its Jordan type satisfies*

$$P_{\ell,M} \leq H(M)^\vee. \tag{2-9}$$

If $\mathcal{A}$ has weight function w, for which $\mathcal{A}_0 = \mathsf{k}$, and if M is a graded module over $\mathcal{A}(\mathsf{w})$ with w-Hilbert function $H_{\mathsf{w}}(M)$, then for any w-homogeneous element $\ell \in \mathfrak{m}_{\mathcal{A}}$ its Jordan type also satisfies

$$P_{\ell,M} \leq H_{\mathsf{w}}(M)^\vee. \tag{2-10}$$

Proof. To prove (2-9), let $\mathcal{A}$ and M be as above, fix any element ℓ of $\mathfrak{m}_{\mathcal{A}}$ and let $\mathsf{k}[x]$ act on the Artinian $\mathcal{A}$-module M via $x = m_\ell$, multiplication by ℓ. Recall that the integer $j(N)$ is the socle degree of an $\mathcal{A}$-module N. For $a \in \mathcal{A}$ the order $\nu(a)$ is the largest integer ν such that $a \in (\mathfrak{m}_{\mathcal{A}})^\nu$, and we extend the definition to elements of modules $M \subset \mathcal{A}^k$: the order of $m = (a_1, \dots, a_k)$ is the minimum order $\min\{\nu(a_1), \dots, \nu(a_k)\}$. We will denote by $\{\{H(N)\}\}$ the multiset of integers in the sequence $H(N)$, with their multiplicities specified. We will show:

Claim. For any $T = \mathsf{k}[x]$ submodule N of the $\mathcal{A}$-module M where $\dim_{\mathsf{k}} M = m$, $\dim_{\mathsf{k}} N = n$, we have $P_\ell(N) \leq H(N)^\vee$.

Proof of claim. We proceed by induction on the pairs (m, n), where we let $(m, n) < (m', n')$ if $m < m'$ or if $m = m'$ and $n < n'$. The claim is true for all pairs (m, n) with $m = 1$ or $n = 1$. Fix (m', n') and suppose the claim is true for all pairs $(m, n) < (m', n')$, let M be an $\mathcal{A}$-module of length m' and N a T submodule of length n'. Let $S = (a, \ell a, \ell^2 a, \dots) \subset N$ be a longest string (k-basis of a cyclic T-submodule) in N. Then S has length $p_{1,\ell}$ no greater than $j(N)+1$, the largest part of $H(N)^\vee$, and $\langle S\rangle$ is a direct T-summand of N (as it has maximum length). Consider a complementary T submodule $N' \subset N$ with $N' \cong N/\langle S\rangle$ and

$N' \oplus_T \langle S\rangle = N$ and choose N' of maximum possible order. Denote by $\{\{H(N)\}\}$ the multiset of integers from $H(N)$. Then $\{\{H(N')\}\}$ is obtained from $\{\{H(N)\}\}$ by decreasing $p = p_{1,\ell}$ entries of $\{\{H(N)\}\}$ by one. No entry $H(N)_i$ is decreased by 2 in $H(N')_i$ as the orders of $a, \ell a, \ell^2 a, \ldots$ are strictly increasing. Evidently, $P_{\ell,N} = (p, P_{\ell,N'})$ — we simply adjoin a largest part p to the Jordan partition $P_{\ell,N'}$. Since $P_{\ell,N'} \le H(N')^\vee$ by the induction assumption, we have by (2-7)

$$P^\vee_{\ell,N'} \ge \bigl(H(N')^\vee\bigr)^\vee. \tag{2-11}$$

The partition $P^\vee_{\ell,N}$ is obtained from $P^\vee_{\ell,N'}$ by increasing the first p entries by one. The multiset $\{\{H(N)\}\}$ is obtained from $\{\{H(N')\}\}$ by increasing some subset of p entries by one. Thus we have (here $\bigl(H(N)^\vee\bigr)^\vee$ is just the integers in the multiset $\{\{H(N)\}\}$ rearranged in nonincreasing order to form a partition)

$$P^\vee_{\ell,N'} \ge \bigl(H(N')^\vee\bigr)^\vee \Rightarrow P^\vee_{\ell,N} \ge \bigl(H(N)^\vee\bigr)^\vee \tag{2-12}$$

in the dominance partial order, since the sum of the first k entries for $P^\vee_{\ell,N}$ remains greater than the analogous sum of the first k entries of $H(N)$, for each $k = 1, \ldots, j(N)+1$. We are using here that the difference $H(N)_i - H(N')_i \le 1$ for each i. By conjugating the partitions in (2-12) and applying (2-7) we have shown $P_{\ell,N} \le H(N)^\vee$. This completes the induction step.

For (2-10), i.e., the graded version, let w be any weight function on $\mathcal{A}$ as above, so that $A = \mathcal{A}(\mathsf{w})$ is an $\mathbb{N}$-graded Artinian algebra over k, and let M be any finite $\mathcal{A}$-module $M \subset \mathcal{A}^k$ with highest socle degree j_M.

Fix a w-homogeneous element $\ell \in \mathfrak{m}_A$, and let $M = \bigoplus_i \langle S_i\rangle$ as in (2-1), where each S_i is a w-homogeneous Jordan basis of the string $\langle S_i\rangle$ of M, $p_i = |S_i|$ and $p_1 \ge \cdots \ge p_s$. The Jordan type of ℓ acting on M is $P_{\ell,M} = (p_1, \ldots, p_s)$. For each $1 \le i \le s$ and each integer u, define the new integer $m(i,u)$ to be the number of elements of degree u in the disjoint union of strings $S_1 \sqcup \cdots \sqcup S_i$; clearly we have

$$\sum_{u\ge 0} m(i,u) = |S_1 \sqcup \cdots \sqcup S_i| = p_1 + \cdots + p_i.$$

Recall that $H(M)^\vee = (q_1, \ldots, q_t)$, where

$$q_i = \#\underbrace{\{u \mid \dim_{\mathsf{k}} M_u \ge i\}}_{T_i} = |T_i|.$$

For each index $1 \le i \le t$, and each integer u, define the new integer $n(i,u)$ to be the number of times the index u appears in the multi-set $T_1 \cup \cdots \cup T_i$; clearly we have

$$\sum_{u\ge 0} n(i,u) = |T_1| + \cdots + |T_i| = q_1 + \cdots + q_i.$$

Since no two elements of the same string have the same degree, we see that $0 \le m(i,u) \le i$. Since $\dim_{\mathsf{k}}(M_u) \ge m(i,u)$, the index u must appear in $T_{m(i,u)}$, as well as in $T_{m(i,u)-1}, \ldots, T_1$. Thus we see that

$$m(i,u) \le n(i,u). \tag{2-13}$$

Summing (2-13) over all u, gives (2-10), and completes the proof of Theorem 2.5. □

In the graded case we will see that those linear forms $\ell \in A_1$ which achieve the bounds in Theorem 2.5 are exactly those with strong Lefschetz property (Proposition 2.10).

Example 2.6. We thank Lorenzo Robbiano for pointing out that we needed to make explicit our assumption that $A_0 = \mathsf{k}$ in Theorem 2.5. He provided the following example when $A_0 \neq \mathsf{k}$. Let $\mathsf{k} = \mathbb{Q}$ be the rationals, set $P = \mathbb{Q}[x]$, let M be the maximal ideal $M = (x^2+1) \subset P$ and denote by $K = P/M$ the quotient field. Consider the Artinian ring $A = P/M^2$ over $\mathbb{Q}$. Let $\mathfrak{m}_A$ be the maximal ideal of A, and consider the associated graded algebra $G = \mathrm{Gr}_{\mathfrak{m}_A}(A)$. It satisfies $\dim_K G = 2$, with Hilbert function $H_K(G) = (1, 1)$; the Jordan type of the multiplication m_x is $P_{x,G} = (2) = H_K(G)^\vee$. However, over $\mathbb{Q}$ we have $\dim_{\mathbb{Q}}(A) = 4$, $\dim_{\mathbb{Q}} \mathfrak{m}_A = 2$, so the Hilbert function $H_{\mathbb{Q}}(A) = (2, 2)$. The multiplication m_x on A has the string $1 \to x \to x^2 \to x^3$, so the Jordan type $P_{x,A} = (4) > H_{\mathbb{Q}}(A)^\vee = (2, 2)^\vee = (2, 2)$.

Remark 2.7. The Hilbert function $H = (1, 2, \ldots)$ of a codimension two algebra is unimodal, and has no strict increases; it follows that H is determined by the partition $P(H)$ (a reordering of H). This is no longer true in height three.

2C. Lefschetz properties and Jordan type. We first recall in Definition 2.8 various traditional notions of Lefschetz properties for graded Artinian algebras; see for example [58, Definition 3.8ff], or [89, Definition 2.4]. It is well known that if A is a standard graded Artinian algebra with Hilbert function $H = H(A)$ then $\ell \in A_1$ is a strong Lefschetz element for A if and only if $P_\ell = H(A)^\vee$, the conjugate partition (exchange rows and columns in the Ferrers diagram) of $H(A)$ regarded as a partition (Proposition 2.10 refining [58, Proposition 3.64]). The element ℓ is weak Lefschetz for A if the number of parts of P_ℓ is the Sperner number of A, the maximum value of the Hilbert function $H(A)$ (Lemma 2.11). We then define for a local algebra $\mathcal{A}$ the notion of its having an element of *strong Lefschetz Jordan type* (SLJT) (Definition 2.12); we give examples and show that if $H(A)$ is unimodal then A has SLJT implies that the algebra is strong Lefschetz (Proposition 2.14).

Definition 2.8. Let A be a graded Artinian algebra of socle degree j (not necessarily standard graded), and let $\ell \in A_1$ be a linear form. We say that ℓ is

(i) (WL) weak Lefschetz if the multiplication maps $\times\ell : A_i \to A_{i+1}$ have maximal rank for each degree $0 \leq i < j$;

(ii) (SL) strong Lefschetz if the multiplication maps $\times\ell^b : A_i \to A_{i+b}$ have maximal rank for each degree $0 \leq i < j$ and each integer $b \geq 0$.

A is said to have the weak (resp. strong) Lefschetz property if it has a weak (resp. strong) Lefschetz element $\ell \in A_1$.

For a survey (as of 2013) of the Lefschetz properties for graded Artinian algebras, see [89]. For more recent discussion see relevant portions of [58].

Remark 2.9. Harima and Watanabe refer to the strong Lefschetz property in the *narrow sense* to mean that for every degree $0 \leq i \leq \lfloor \frac{j}{2} \rfloor$ the multiplication maps $\times\ell^{j-2i} : A_i \to A_{j-i}$ are isomorphisms (see [58, Definition 3.18]). If the Hilbert function $H(A)$ is symmetric, i.e., $h_i = h_{j-i}$ for each i, then SL in the narrow sense is equivalent to SL in Definition 2.8.

Note that a necessary condition for A to have SL is that its Hilbert function $H(A)$ is unimodal. The following result, which is a generalization of Proposition 3.64 in [58], relates the strong Lefschetz property and Jordan type.

Proposition 2.10. *Let A be a (possibly nonstandard) graded Artinian algebra and $\ell \in A_1$. Then the following statements are equivalent:*

(i) *For each integer b, the multiplication maps $\times\ell^b : A_i \to A_{i+b}$ have maximal rank in each degree i. (That is, ℓ is SL.)*

(ii) *The Jordan type of ℓ is equal to the conjugate partition of the Hilbert function, i.e.,*

$$P_\ell = H(A)^\vee.$$

(iii) *There is a set of strings $S_1, \dots, S_s$ as in (2-1), composed of homogeneous elements, for the multiplication map $\times\ell^b : A \to A$ such that for each degree u and each integer i we have the equivalence*

$$\dim_{\mathsf{k}} A_u \geq i \iff A_u \cap S_a \neq \varnothing \ \ \forall a \leq i. \tag{2-14}$$

Proof. For each integer $i \in [1, j_A]$ define the set of indices $T_i = \{u \mid \dim A_u \geq i\}$, and let $n_i = \#T_i$. We denote by t the maximum i such that T_i is nonempty. Let $S_1, \dots, S_s$ as in (2-1) be strings for the action of ℓ on A, arranged so that their lengths $p_i = \#S_i$ are nonincreasing, and with S_i having generator z_i. Note that since the map $\times\ell : A \to A$ respects the grading of A, its kernel is a homogeneous ideal, and therefore we can construct a Jordan basis composed of homogeneous elements, and we may assume that the elements in each string S_i are homogeneous. To these strings we associate their degree sets $\deg(S_1), \dots, \deg(S_s)$, where

$$\deg(S_i) = \{\deg z_i, \dots, \deg \ell^{p_i - 1} z_i\}.$$

(i) $\Rightarrow$ (ii): Assume $\times\ell^b : A_i \to A_{i+b}$ has maximal rank for each i, b. We want to show that $t = s$ and $p_i = n_i$ for $1 \leq i \leq s$.

Claim. There is an injective function $\sigma : \{1, \dots, t\} \to \{1, \dots, s\}$ such that for each index $i \in [1, t]$, we have

$$T_i \subseteq \deg(S_{\sigma(i)}).$$

Note that the claim implies that $n_i \leq p_{\sigma(i)}$ for $1 \leq i \leq t \leq s$. Then we have

$$\dim_{\mathsf{k}} A = \sum_{i=1}^{t} n_i \leq \sum_{i=1}^{t} p_{\sigma(i)} \leq \sum_{i=1}^{r} p_i = \dim_{\mathsf{k}} A$$

which implies that $t = s$ and $n_i = p_i$ for all $1 \leq i \leq t$, as desired.

Proof of claim. We proceed by induction on i. We have $T_1 = \{j \mid \dim_{\mathsf{k}}(A_j) \geq 1\} = \{0, \dots, d\}$. Since $\times\ell^d : A_0 \to A_d$ has full rank, we conclude that T_1 must belong to the degree sequence of some Jordan string, say $S_{\sigma(1)}$. Inductively, assume that we have defined an injective function $\sigma : \{1, \dots, i-1\} \to \{1, \dots, s\}$ for which

$$T_1 \subseteq \deg(S_{\sigma(1)}), \quad \dots, \quad T_{i-1} \subseteq \deg(S_{\sigma(i-1)}). \tag{2-15}$$

Write $T_i = \{u \mid \dim_{\mathsf{k}} A_u \geq i\} = \{u_1 < \cdots < u_{n_i}\}$. By our assumption, the multiplication map

$$\times\ell^{u_{n_i} - u_1} : A_{u_1} \to A_{u_{n_i}}$$

has rank at least i, hence there are at least i distinct Jordan strings which meet both A_{u_1} and $A_{u_{n_i}}$. Since there are only $(i-1)$ strings appearing in (2-15), there must be one not listed, call it $S_{\sigma(i)}$ for which $T_i \subseteq \deg(S_{\sigma(i)})$. This completes the induction step and proves the claim. □

(ii) $\Rightarrow$ (iii): Assume that $P_\ell = H(A)^\vee$. Then $s = t$ and $p_i = n_i = \#\{u \mid \dim_{\mathsf{k}} A_u \geq i\}$. Clearly, if $A_u \cap S_a \neq \varnothing$ for all $a \leq i$ then $\dim_{\mathsf{k}} A_u \geq i$. We prove the other implication by downward induction on the integers $i \leq s$. For the base case, if $\dim_{\mathsf{k}} A_u \geq s$, then A_u must contain exactly one element from each Jordan string, hence $A_u \cap S_a \neq \varnothing$ for all $a \leq s$. For the inductive step, assume the implication $\Rightarrow$ of (iii) for indices greater than i, and suppose that $\dim_{\mathsf{k}} A_u \geq i$. If $\dim_{\mathsf{k}} A_u \geq i+1$ then by the induction hypothesis $A_u \cap S_a \neq \varnothing$ for all $a \leq i+1$. On the other hand if $\dim_{\mathsf{k}} A_u = i$, then $A_u \cap S_a = \varnothing$ for all $a \geq i+1$. Indeed, for each index $m \geq i+1$,

$$\dim_{\mathsf{k}} A_u \geq m \;\Rightarrow\; A_u \cap S_m \neq \varnothing.$$

By our assumption there are exactly p_m such indices u, hence if $\dim_{\mathsf{k}} A_u = i < m$ then $A_u \cap S_m = \varnothing$. So if $\dim_{\mathsf{k}} A_u = i$ we must have $A_u \cap S_a \neq \varnothing$ for all $a \leq i$. This completes the induction step.

(iii) $\Rightarrow$ (i): Assume that for each $1 \leq i \leq t$ we have the equivalence

$$\dim_{\mathsf{k}} A_u \geq i \;\Leftrightarrow\; A_u \cap S_a \neq \varnothing \;\; \forall a \leq i.$$

Fix an index $i \in [1, t]$ and an integer b, and consider the multiplication map $\times\ell^b : A_i \to A_{i+b}$. Let $m = \min\{\dim_{\mathsf{k}} A_i,\, \dim_{\mathsf{k}} A_{i+b}\}$. Then

$$\dim_{\mathsf{k}} A_i,\, \dim_{\mathsf{k}} A_{i+b} \geq m$$

implies that the m Jordan strings $S_1, \dots, S_m$ each intersect both A_i and A_{i+b}, which in turn implies that $\times\ell^b : A_i \to A_{i+b}$ has rank m. □

Recall that for A graded Artinian the *Sperner number* satisfies $\operatorname{Sperner}(A) = \max\{H(A)_i \mid i \in [0, j]\}$ [58, §2.3.4]. For a local ring $\mathcal{A}$, we have $\operatorname{Sperner}(\mathcal{A}) = \max\{\mu(\mathfrak{m}_{\mathcal{A}}{}^i) \mid i \in [0, j]\}$, where $\mu(I) = \#$ minimal generators of I.

Lemma 2.11 [58, Proposition 3.5]. *When the Hilbert function $H(A)$ for a standard graded Artinian algebra A is unimodal and symmetric then $\ell \in A_1$ is weak Lefschetz for A if and only if* $\dim_{\mathsf{k}} A/\ell A = \operatorname{Sperner}(A)$ *or*, *equivalently*, *if P_ℓ has* $\operatorname{Sperner}(A)$ *parts.*

Lefschetz properties for local algebras. Here we use Jordan type to extend the strong and weak Lefschetz properties to local Artinian algebras.

Definition 2.12. Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathsf{k})$ be a local Artinian algebra over k with Hilbert function $H(\mathcal{A})$. We say that an element $\ell \in \mathfrak{m}$ has

(i) (SLJT) strong Lefschetz Jordan type if $P_\ell = H(\mathcal{A})^\vee$.

(ii) (WLJT) weak Lefschetz Jordan type if P_ℓ has $\operatorname{Sperner}(\mathcal{A})$ parts.

Additionally if $\mathcal{A}$ is graded via a weight function w with w-Hilbert function $H_{\mathsf{w}}(\mathcal{A})$, then we say that a w-homogeneous element $\ell \in \mathfrak{m}$ has

(iii) w-*strong Lefschetz Jordan type* (w-SLJT) if $P_\ell = H_{\mathsf{w}}(\mathcal{A})^\vee$.

We say that $\mathcal{A}$ has SLJT, WLJT, or w-SLJT if it has an element $\ell \in \mathfrak{m}$ of that type.

Remark 2.13. Note that if $\mathcal{A}$ is graded with weight function w, and $H(\mathcal{A})^\vee < H_{\mathsf{w}}(\mathcal{A})^\vee$, then $\mathcal{A}$ cannot possibly be w-SLJT and $\mathcal{A}(\mathsf{w})$ cannot possibly be SL, while SLJT may or may not hold. On the other hand in the standard graded case we have $H(\mathcal{A}) = H_{\mathsf{w}}(\mathcal{A})$, and hence w-SLJT and SLJT are equivalent conditions on $\mathcal{A}$. Evidently, in the standard graded case the SL condition on $\mathcal{A}(\mathsf{w})$ implies the SLJT condition on $\mathcal{A}$. The next result shows that the converse holds under the additional assumption that the Hilbert function $H(\mathcal{A})$ is unimodal.

Proposition 2.14. *Assume that A is a standard graded Artinian algebra and $H(A)$ is unimodal. Then A has an element of strong Lefschetz Jordan type if and only if A has a strong Lefschetz element.*

Proof. Assume that A has an element ℓ (possibly nonhomogeneous) of strong Lefschetz Jordan type, so $P_\ell = H(A)^\vee = (p_1, p_2, \dots, p_s)$ with $p_1 \ge p_2 \ge \cdots \ge p_s$. All that is needed for the forward direction is to show that there is a linear element ℓ' that is of strong Lefschetz Jordan type. Consider Jordan strings $S_1, \dots, S_s$ for ℓ as in Definition 2.1, where $S_k = (z_k, \ell z_k, \dots, \ell^{p_k-1} z_k)$. The orders of elements in a single string are distinct. Let ℓ' be the initial form of ℓ, which, as we will see, must be linear. We will modify the strings, if needed, to a set of Jordan strings $S'_1, \dots, S'_s$ for ℓ whose initial forms are Jordan strings for ℓ': this will show $P_{\ell'} = P_\ell = H(A)^\vee$, and prove that A is strong Lefschetz.

Given that $H(A)$ is unimodal, we claim that we may choose the strings so that

(i) the first t strings together contain $\min\{H(A)_i,\ t\}$ elements of order i for each $i \in [0, j_A]$; and the initial forms of these elements are linearly independent;

(ii) the order $\nu(\ell^i z_t) = \nu(z_t) + i$ for each pair (t, i) satisfying $1 \le t \le s$ and $0 \le i \le p_t - 1$.

We prove (i) and (ii) by complete induction on t. Considering $t = 1$, the longest string S_1 is equal to $(z_1, \ell z_1, \dots, \ell^{p_1-1} z_1)$, where $p_1 - 1 = j_A$, the socle degree of A; and we may choose, after scaling by a nonzero constant, $z_1 = 1 + \alpha$, $\alpha \in \mathfrak{m}_A$. It follows (since A has standard grading) that the initial term ℓ' of ℓ is linear, and that the elements in the string $S'_1 = \big(1, \ell', \dots, (\ell')^{p_1-1}\big)$ satisfy $S'_1 = \pi(S_1)$, where π is the projection of the elements of S_1 onto their initial forms, and form a string of length p_1 for ℓ'.

For the induction step we will need several facts. Denote by $m(t, H)$ the smallest integer i such that $H(A)_i \ge t$, and $n(t, H)$ the largest integer i such that $H(A)_i \ge t$.

Fact 1. That $H(A)$ is unimodal is equivalent to the inequalities

$$m(1, H) \le m(2, H) \le \cdots \le m(s, H) \le n(s, H) \le n(s-1, H) \le \cdots \le n(1, H). \tag{2-16}$$

Also, we have $p_u = 1 + n(u, H) - m(u, H)$.

Fact 2. Given $t \in [1, s]$ the condition (i) above implies:

(iii) Let $i < m(t, H)$ then the initial forms of all elements of $S_1 \cup \cdots \cup S_t$ having order no greater than i are a basis for $A/\mathfrak{m}_A^{i+1} \cong \bigoplus_{k=0}^{i} A_k$.

(iv) Let $i > n(t, H)$. The union $\bigcup_{k=1}^{t} (\mathfrak{m}_A^i \cap S_k)$ is a basis for $\mathfrak{m}_A^i = \bigoplus_{k=i}^{j_A} A_k$.

Induction step: Fix $u \in [1, s-1]$ and assume that a set $S_1, \dots, S_s$ of Jordan strings for m_ℓ has been chosen satisfying (i) and (ii) for all integers $t \le u$. We will keep the strings $S_1, S_2, \dots, S_u$ fixed and will modify the chain S_{u+1} to obtain a set of s Jordan chains for ℓ so that the conditions (i), (ii) will be satisfied for all $t \in [1, u+1]$.

Consider the next string S_{u+1}, of length p_{u+1}; by assumption, its elements are linearly independent of the span of those from $S_1 \cup \cdots \cup S_u$. Using that (i) and (ii), hence (iii), (iv) are satisfied for $t \le u$, we may adjust the generator z_{u+1} for the string S_{u+1} by linear combinations of elements from the previous strings to obtain a possibly new generator within the span of $S_1, \ldots, S_{u+1}$ having order $m(u+1, H)$, and whose initial form $z'_{u+1} = \pi(z_{u+1})$ is linearly independent of the degree $m(u+1, H)$ initial forms from elements of the strings $S_1, \ldots, S_u$. Using (iv), we may adjust the generator z_{u+1} further by suitable elements of order at least $m(u+1, H)$ from the previous u strings so that $\ell^{p_{u+1}} \cdot z_{u+1} = 0$. It follows that $\ell'^{p_{u+1}} \cdot z'_{u+1} = 0$, and z'_{u+1} is generator of an ℓ' string of length p_{u+1}, linearly independent from the ℓ' strings $S'_1, \ldots, S'_u$ determined by the initial elements from $S_1, \ldots, S_u$. It follows that (i) and (ii) are satisfied for $t \in [1, u+1]$. This completes the induction step.

We have shown (i) and (ii) for S_1 and the induction step. It follows that $P_{\ell'} = P_\ell = H(A)^\vee$, as claimed.

The converse, that A has a strong Lefschetz element implies it has an element of strong Lefschetz Jordan type, is obvious from the definitions. □

The following result is well known (and has been reproved several times).

Lemma 2.15 (height two Artinian algebras are strong Lefschetz). *Let $A = \mathsf{k}[x, y]/I$ be Artinian standard graded of socle degree j, or $\mathcal{A} = \mathsf{k}\{x, y\}/I$ be local Artinian, and suppose* char $\mathsf{k} = 0$ *or* char $\mathsf{k} > j$. *Let ℓ be a general element of $\mathfrak{m}_A$ in the first case, or of $\mathfrak{m}_{\mathcal{A}}$ in the second. Then ℓ has strong Lefschetz Jordan type and A is strong Lefschetz, or $\mathcal{A}$ is of strong Lefschetz Jordan type, in the second.*

Proof. These statements follow readily from J. Briançon's standard basis theorem for ideals in $\mathbb{C}[x, y]$ [26], that extends to the case char $\mathsf{k} = p > j$ (see [14, Theorem 2.16]).[1] □

Example 2.16. Let $\mathcal{A} = \mathsf{k}\{x, y\}/(xy - x^3, y^2)$ with weight function $\mathsf{w}(x, y) = (1, 2)$. Then the Hilbert function and w-Hilbert function are, respectively $H(\mathcal{A}) = (1, 2, 1, 1, 1)$ and $H_{\mathsf{w}}(\mathcal{A}) = (1, 1, 2, 1, 1)$, with conjugate partitions $H(\mathcal{A})^\vee = H_{\mathsf{w}}(\mathcal{A})^\vee = (5, 1)$.

A *standard basis* B for the local algebra $\mathcal{A}$ is one such that the elements of $\mathsf{B} \cap \mathfrak{m}^i_{\mathcal{A}}$ are a basis for $\mathfrak{m}^i_{\mathcal{A}}$. Such a basis for $\mathcal{A}$ is $\{1, x; y, x^2; x^3, x^4\}$. The multiplication $\times x$ in this basis has Jordan strings

$$1 \to x \to x^2 \to x^3 \to x^4 \to 0 \quad \text{and} \quad (y - x^2) \to 0. \tag{2-17}$$

Thus, $P_{x,\mathcal{A}} = (5, 1)$.

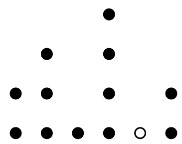
A basis for the graded ring $\mathcal{A}(\mathsf{w})$ are the classes of $\{1, x, y, x^2, xy, x^2y\}$ and the only linear element is x: the strings of m_x on this basis for A are $(1 \to x \to x^2 \to xy = x^3 \to x^2y = x^4 \to 0)$ and $\big((y - x^2) \to 0\big)$ so here we have $P_{x,\mathcal{A}} = (5, 1) = H_{\mathsf{w}}(\mathcal{A})^\vee$.

However, the Example 2.57 shows that a generic linear element ℓ of the graded algebra A, and a generic element $\ell' \in \mathfrak{m}_{\mathcal{A}}$ of the related local algebra $\mathcal{A}$, may satisfy $P_{\ell,A} = (7, 1, 1) < (7, 2) = P_{\ell',\mathcal{A}}$.

2D. Jordan types consistent with a Hilbert function. In Section 2B we proved an inequality for Jordan types of elements of $\mathcal{A}$ and the Hilbert function (Theorem 2.5). Here we show a similar inequality for graded algebras A, that is sharper when the Hilbert function is non-unimodal. It depends on a certain refinement $P_c(H)$ of the conjugate partition $H^\vee$, that we now define.

[1] Proofs in the case A is graded occur also in [57, Proposition 4.4] and [36, Theorem 4.11]; see [89, Theorem 2.27].

The contiguous partition $P_c(H)$ of a Hilbert function. In general, given any finite sequence of nonnegative integers $H = (h_0, \ldots, h_j)$ we consider its *bar graph* as an array of dots (or boxes) arranged into columns with h_i dots (or boxes) in the i-th column. For example if $H = (2, 3, 1, 4, 0, 2)$ then its bar graph is



Definition 2.17 (contiguous partition, relative Lefschetz property). (i) The *contiguous partition $P_c(H)$* of the Hilbert function H is the partition whose parts are the lengths of the maximal contiguous row segments of the bar graph of H.

(ii) We say a linear form $\ell \in A_1$ in a graded Artinian algebra A has the *Lefschetz property relative to H* if its Jordan type is equal to the contiguous partition of H:

$$P_\ell = P_c(H).$$

The following result is immediate.

Lemma 2.18 (Hilbert function $H(M)$ of a finite-length module over R and P_ℓ). *Let $M = M_0 \oplus M_1 \oplus \cdots \oplus M_j$ be an Artinian graded module over the polynomial ring $R = \mathsf{k}[x_1, \ldots, x_r]$, satisfying $H(M) = (h_0, \ldots, h_j)$, and let $\ell \in R_1$. Then for $1 \le k \le j$*

$$\operatorname{rk} m_\ell^k \le \sum_{i=0}^{j-k} \min\{h_i, h_{i+1}, \ldots h_{i+k}\}. \tag{2-18}$$

Also, $P_\ell = P_c(H)$ if and only if there is equality for every k in (2-18).

Proof. We give the proof for $M = A$. We observe that for any $\ell \in A_1$, the map $m_\ell^k : A_i \to A_{i+k}$ has rank at most $\min\{h_i, \ldots, h_{i+k}\}$. Summing over all i we get the inequality of (2-18).

Recall that the conjugate Jordan type $P_\ell^\vee = (q_1, \ldots, q_j)$ is the first difference of the rank sequence of m_ℓ, i.e.,

$$q_k = \operatorname{rk}(m_\ell^k) - \operatorname{rk}(m_\ell^{k+1}).$$

Hence if ℓ has the Lefschetz property relative to H, then it follows that (2-18) is actually an equality. In particular, a Lefschetz element $\ell \in A_1$ relative to H is one whose multiplication maps $m_\ell^k : A \to A$ have the maximal possible rank, given the Hilbert function, for each integer k. $\square$

The following result pertains to the conjugate of $P_c(H)$: when H is unimodal then $P_c(H)^\vee = \{H\}$, that is, the Hilbert function viewed as a partition.

Lemma 2.19. *Given any finite sequence of nonnegative integers $H = (h_0, \ldots, h_j)$ if $P_c(H)^\vee = (p_1, \ldots, p_s)$, then the parts are given by*

$$p_i = \sum_{k=0}^{j+1-i} \min\{h_k, \ldots, h_{k+i-1}\} - \sum_{k=0}^{j-i} \min\{h_k, \ldots, h_{k+i}\}. \tag{2-19}$$

Proof. The i-th part of $P_c(H)^\vee$ is

$$p_i = \#\text{ maximal contiguous row segments of length} \geq i \text{ in the bar graph of } H.$$

Note that the sum

$$\sum_{k=0}^{j+1-i} \min\{h_k, \dots, h_{k+i-1}\} \tag{2-20}$$

counts the maximal contiguous row segments of length greater or equal to i with a multiplicity equal to the number of length i-intervals it contains; in particular it counts a contiguous row segment of length i exactly once. On the other hand, the sum

$$\sum_{k=0}^{j-i} \min\{h_k, \dots, h_{k+i}\} \tag{2-21}$$

counts maximal contiguous row segments of length $\geq i+1$ with multiplicity one less than they are counted in (2-20). Therefore the difference of the sums in (2-20) and (2-21) must count every maximal contiguous row segment of length $\geq i$ exactly once. □

Theorem 2.20. *For a finite graded module M over a graded Artinian algebra A with Hilbert function $H(M)$, we have for any linear form $\ell \in A_1$*

$$P_{\ell,M} \leq P_c\big(H(M)\big).$$

Proof. Given $\ell \in A_1$ we may choose a Jordan basis for $m_\ell : M \to M$ with strings $S_1, \dots, S_s$ as in (2-1), each of cardinality $|S_i| = p_i$ with $P_{\ell,M} = (p_1, \dots, p_s)$ with $p_1 \geq \cdots \geq p_s$. Since ℓ is linear, each string must belong to some maximal contiguous row segment of the Hilbert function $H(M)$, which implies the desired inequality. □

Example 2.21. Let $\mathcal{A} = \mathsf{k}\{y, z\}/(yz, z^3, y^7)$ with weights $\mathsf{w}(y, z) = (1, 2)$, and w-Hilbert function $H_{\mathsf{w}}(\mathcal{A}) = (1, 1, 2, 1, 2, 1, 1)$. Then $y \in \mathcal{A}_1$ is a generic linear form with Jordan type $P_y = P_c(H) = (7, 1, 1)$, the maximum possible by Theorem 2.20; in particular y has the Lefschetz Property relative to $H_{\mathsf{w}}(\mathcal{A})$ (Definition 2.17). On the other hand the conjugate partition of the w-Hilbert function is $H_{\mathsf{w}}(\mathcal{A})^\vee = (7, 2)$. The Hilbert function for the related localization $\mathcal{A}$ at $\mathfrak{m}_{\mathcal{A}} = \sum_{i\geq 1} \mathcal{A}_i$ is $H(\mathcal{A}) = (1, 2, 2, 1, 1, 1, 1)$; the conjugate partition of this Hilbert function is also $H(\mathcal{A})^\vee = (7, 2)$. Thus y does not have the strong Lefschetz Jordan type for $\mathcal{A}$, nor is it strong Lefschetz for $\mathcal{A}(\mathsf{w})$ (or even WL). But the non-w-homogeneous element $\ell = (y + z) \in \mathfrak{m}$ has Jordan type $P_\ell = (7, 2)$, hence $\mathcal{A}$ has strong Lefschetz Jordan type (SLJT).

Note. Recall that we have adopted in Definition 2.8(ii) the convention of Harima and Watanabe that a nonstandard graded algebra A is strong Lefschetz if and only if there is a *linear* form $\ell \in A_1$ that has SLJT (see [54; 58]). Thus, the rings A of Examples 2.21, 2.57, 3.14 and 3.26 are not strong Lefschetz, even though in each Example the corresponding local ring $\mathcal{A}$ has an element of strong Lefschetz Jordan type: as $y + z$ in Example 2.21 is an element in $\mathcal{A}$ having SLJT — so $\mathcal{A}$ has SLJT by our Definition 2.12.

Remark 2.22. Instead of using $(\dim_{\mathsf{k}} M, \dim_{\mathsf{k}} M/\ell M, \dim_{\mathsf{k}} M/\ell^2 M, \dots)$ as in (2-2) to define $P_{\ell,M}$ we may replace each ℓ^k by $(\ell_1 \cdot \ell_2 \cdots \ell_k)$, a product of different — generic — linear forms, yielding a partition $Q(M)$. It can be shown similarly to the proof of (2-9) that

$$P_{\ell,M} \leq Q(M) \leq H(M)^{\vee}. \tag{2-22}$$

We can ask similar questions for $Q(M)$ to those we ask about $P_{\ell,M}$. When is $P_{\ell,M} = Q(M)$? Also, the partition $Q(M)$ appears to be related to the concepts of "k-Lefschetz" [58, §6.1] and "mixed Lefschetz" [31]. What is the relation of these concepts to Jordan type?

We note that Jordan type and Hilbert function has been in particular studied for codimension two complete intersections in [3]; also Costa and Gondim have used mixed Hessians to study other examples of Jordan type in higher codimension [39].

2E. Artinian Gorenstein algebras and Macaulay dual. The polynomial ring $R = \mathsf{k}[x_1, \dots, x_r]$, acts on its dual $\mathfrak{D} = \mathsf{k}_{DP}[X_1, \dots, X_r]$ by contraction: $x_i^k \circ X_j^{[k']} = \delta_{i,j} X_j^{[k'-k]}$ for $k' \geq k$, extended multilinearly.[2] We say that a graded Artinian algebra is *standard graded* if A is generated by A_1. We next define a homogeneous element of $\mathfrak{R}$, the Macaulay dual generator for a graded Artinian quotient $A = R/I$.

Likewise, the regular local ring $\mathcal{R} = \mathsf{k}\{x_1, \dots, x_r\}$ acts on $\mathfrak{D}$, also by contraction and we will define similarly a dual generator in $\mathfrak{R}$ for a local Artin algebra $A = \mathcal{R}/I$.

Definition 2.23 (Macaulay dual generator). An Artinian Gorenstein (AG) algebra quotient $A = R/I$ (resp. $\mathcal{A} = \mathcal{R}/I$) satisfies $A = R/\operatorname{Ann} f$ (resp. $\mathcal{A} = \mathcal{R}/\operatorname{Ann} f$), where $f \in \mathfrak{D} = \mathsf{k}_{DP}[X_1, \dots, X_r]$ is called the *dual generator* of $\mathcal{A}$. The module $\hat{A} = R \circ f$ in the graded case, or $\hat{\mathcal{A}} = \mathcal{R} \circ f$ in the local case is the Macaulay dual of $\mathcal{A}$, equivalent to the Macaulay *inverse system* of the ideal I. The *socle* of A (resp. of $\mathcal{A}$) is $\operatorname{Soc}(A) = (0 : \mathfrak{m}_A) \subset A$ (resp. $\operatorname{Soc}(\mathcal{A}) = (0 : \mathfrak{m}_{\mathcal{A}}) \subset \mathcal{A}$), is the unique minimal nonzero ideal of A or of $\mathcal{A}$, and $\dim_{\mathsf{k}} \operatorname{Soc}(\mathcal{A}) = 1$.

For a more general Artinian algebra $A = R/I$ (graded) or $\mathcal{A} = \mathcal{R}/I$ (local), a set of *Macaulay dual generators* of A are a minimal set of A (or $\mathcal{A}$) module generators in $\mathfrak{D}$ of $I^{\perp} = \{h \in \mathfrak{D} \mid I \circ h = 0\}$.

Example 2.24 (Artinian Gorenstein). (i) Let $R = \mathsf{k}[x, y]$, $A = R/I$, $I = \operatorname{Ann} f$ with $f = XY \in \mathfrak{D} = \mathsf{k}_{DP}[X, Y]$. Then $I = (x^2, y^2)$ and $A = R/(x^2, y^2)$ of Hilbert function $H(A) = (1, 2, 1)$. Here $x^2 \circ XY = 0$ is the contraction analogue of $\partial^2(XY)/(\partial X)^2 = 0$ and the dualizing module satisfies $\hat{A} = R \circ f = \langle 1, X, Y, XY \rangle$.

(ii) Let $\mathcal{R} = \mathsf{k}\{x, y\}$, the regular local ring, and take $f = X^{[4]} + X^{[2]}Y$. Then $\mathcal{A} = \mathcal{R}/I$, $I = \operatorname{Ann} f = (xy - x^3, y^2)$, and the Hilbert function $H(\mathcal{A}) = (1, 2, 1, 1, 1)$. The dualizing module satisfies $\hat{\mathcal{A}} = \mathcal{R} \circ f = \langle 1,\ X,\ Y,\ X^{[2]},\ X^{[3]} + XY,\ f \rangle$.

Letting $\mathfrak{m}_{\mathcal{A}}$ be the maximal ideal of the Artinian Gorenstein local algebra $\mathcal{A}$, we have $\operatorname{Soc}(\mathcal{A}) = \mathfrak{m}_{\mathcal{A}}^j$, where $\mathfrak{m}_{\mathcal{A}}^j \neq 0$ and $\mathfrak{m}_{\mathcal{A}}^{j+1} = 0$. Then we have the following result ([77, §60–63], [60, Lemma 1.1], or, in the graded case, [85, Lemma 1.1.1]):

[2] F.H.S. Macaulay used the notation x_i^{-s} for the element we term $X_i^{[s]}$ in $\mathfrak{D}$.

Lemma 2.25 (dual generator for AG algebra). (i) *Assume that $\mathcal{A} = \mathcal{R}/I$ is Artinian Gorenstein of socle degree j. Then there is a degree-j element $f \in \mathfrak{D}$ such that $I = I_f = \operatorname{Ann} f$. Furthermore f is uniquely determined up to action of a differential unit $u \in \mathcal{R}$: that is*

$$\operatorname{Ann} f = \operatorname{Ann}(u \circ f). \quad \textit{Also}, \ \operatorname{Ann} f = \operatorname{Ann} g \Leftrightarrow g = u \circ f \textit{ for some unit } u \in \mathcal{R}. \tag{2-23}$$

The $\mathcal{R}$-module $(\operatorname{Ann} f)^{\perp} = \{h \in \mathfrak{D} \mid (\operatorname{Ann} f) \circ h = 0\}$ satisfies $(\operatorname{Ann} f)^{\perp} = R \circ f$. When f is homogeneous, it is uniquely determined by $\operatorname{Ann} f$ up to nonzero constant multiple.

(ii) *Denote by $\phi : \operatorname{Soc}(\mathcal{A}) \to \mathsf{k}$ a fixed nontrivial isomorphism, and define the pairing $\langle \cdot, \cdot \rangle_\phi$ on $\mathcal{A} \times \mathcal{A}$ by $\langle (a, b) \rangle_\phi = \phi(ab)$. Then the pairing $\langle (\cdot, \cdot) \rangle_\phi$ is an exact pairing on $\mathcal{A}$, for which $(\mathfrak{m}_{\mathcal{A}}^i)^{\perp} = (0 : \mathfrak{m}_{\mathcal{A}}^i)$. We also have $0 : \mathfrak{m}_{\mathcal{A}}^i = \operatorname{Ann}(\mathfrak{m}_{\mathcal{A}}^i \circ f)$ and $\operatorname{Ann}(\ell^i \circ f) = I_f : \ell^i$.*

(iii) *When $A = \bigoplus_0^j A_i$ is graded (not necessarily standard-graded) of socle degree j (largest integer for which $A_j \neq 0$), analogously to (ii), we choose an isomorphism $\phi : A_j \to \mathsf{k}$, and then define the bilinear map $\langle \cdot, \cdot \rangle_\phi$.*

We note that in the above pairing $(A_{\geq i})^{\perp} = (0 : A_{\geq i}) = A_{\geq j+1-i}$. Passing to quotients $A_i = A_{\geq i}/A_{\geq i+1}$ we conclude that each $A_i \cdot A_{j-i} \to A_j$ is an exact pairing, and the Hilbert function $H(A)$ is symmetric about $j/2$. When A is standard graded, we also have, taking $\mathfrak{m}_A = \bigoplus_{k=1}^j A_k$, that $(\mathfrak{m}_A^i) = A_{\geq i}$.

When the AG algebra $\mathcal{A} = \mathcal{R}/\mathcal{I}$ is a local ring then in general the dual generator f is not homogeneous, and the Hilbert function $H(\mathcal{A})$ is not in general symmetric; however the associated graded algebra $\mathcal{A}^*$ has a filtration whose successive quotients are reflexive k modules, and $H(\mathcal{A})$ has a corresponding "symmetric decomposition" [60; 63]. When the AG algebra A is (perhaps nonstandard) graded, then the dual generator $f \in \mathfrak{D}$ may be taken homogeneous in a suitable grading of $\mathfrak{D}$, and (iii) implies that the Hilbert function $H(A)$ is symmetric about $j/2$, where j is the socle degree of A.

Example 2.26. We let $R = \mathsf{k}[x, y]$, with weights $\mathsf{w}(x, y) = (3, 1)$, and consider the complete intersection algebra $A = R/(x^2 - y^6, xy)$, then $I^{\perp} = R \circ f$, where $f = X^2 + Y^6$, which is homogeneous in the analogous grading of $\mathfrak{D}$. We have $H(A) = (1, 1, 1, 2, 1, 1, 1)$.

Lemma 2.27. *Let $\mathcal{A} = \mathcal{R}/I$ be local Artinian Gorenstein of socle degree j with Macaulay dual generator $F \in \mathfrak{D}$ and let $\ell \in \mathfrak{m}_{\mathcal{A}}$. The conjugate $(P_\ell)^\vee$ to the Jordan type P_ℓ satisfies*

$$(P_\ell)^\vee = \Delta\big(\dim_{\mathsf{k}} \mathcal{A}, \dim_{\mathsf{k}} \mathcal{A}(1), \dots, \dim_{\mathsf{k}} \mathcal{A}(i), \dots, \dim_{\mathsf{k}} \mathcal{A}(j)\big) \tag{2-24}$$

where $\mathcal{A}(i) = \mathcal{R}/(I : \ell^i) = \mathcal{A}/(0 : \ell^i) = \mathcal{R}/\operatorname{Ann}(\ell^i \circ F)$.

Proof. This result is standard and follows from Lemma 2.3. See also [58, Lemma 3.60]. □

For a generalization of Macaulay dual over a field k (as here) to Macaulay dual over any base, see S. Kleiman and J. O. Kleppe [73]. Several authors have studied the Artinian Gorenstein algebras arising from polynomials attached to combinatorial objects such as a matroid [78; 95].

2F. Jordan degree type and Hilbert function.

We next introduce a finer invariant than Jordan type of an A-module M, the Jordan degree type. We will define the Jordan degree type for graded modules M over a graded ring A.

Definition 2.28 (Jordan degree type, contiguous degree type of H, order on JDT).

(i) *Jordan degree-type of M.* We give several equivalent notations for Jordan degree type. Let A be a graded Artinian algebra, let M be a finite graded A-module, and let $\ell \in A_1$ be any linear element.

(a) Suppose $P_{\ell,M} = (p_1, \dots, p_s)$, and write M as a direct sum $M = \langle S_1 \rangle \oplus \cdots \oplus \langle S_s \rangle$ of cyclic $\mathsf{k}[\ell]$-modules generated by ℓ-strings of the form $S_k = \{z_k, \ell z_k, \dots, \ell^{p_k-1} z_k\}$ satisfying $\ell^{p_k} z_k = 0$, as in Definition 2.1, and let ν_k be the order of z_k. For any $k, k' \in \{1, \dots, s\}$ if $k < k'$ and $p_k = p_{k'}$, we assume $\nu_k \leq \nu_{k'}$. By Lemma 2.2(iv) the sequence of pairs of integers

$$\mathcal{S}_{\ell,M} = \big((p_1, \nu_1), \dots, (p_s, \nu_s)\big) \tag{2-25}$$

is an invariant of (M, ℓ), that we term the *Jordan degree type* of M with respect to ℓ.

Notation. With ℓ understood, we will denote the pair (n, ν) by

$$\mathsf{n}_\nu = \text{ a string — a cyclic } \mathsf{k}[\ell]\text{-module — of length } n \text{ beginning in degree } \nu. \tag{2-26}$$

Therefore, $\mathcal{S} = (5_0, 3_1, 3_1, 1_2)$ denotes a Jordan degree type consistent with the Hilbert function $H = (1, 3, 4, 3, 1)$.

(b) Denote by $P_{\ell,i}$ (or $P_{\ell,i,M}$) the partition giving the lengths of those strings of m_ℓ acting on M that begin in degree i; that is $P_{\ell,i} = (p_k \mid \nu_k = i)$. We denote by $\mathcal{P} = \mathcal{P}_{\deg,\ell}$ or by $\mathcal{P}_{\ell,M} = \mathcal{P}_{\deg,\ell,M}$ (to specify the module M) the sequence

$$\mathcal{P}_{\deg,\ell} = (P_{\ell,0}, \dots, P_{\ell,j-1}), \tag{2-27}$$

which we also term the *Jordan degree type* (JDT) of ℓ. For example the JDT $\mathcal{S} = (5_0, 3_1, 3_1, 1_2)$ can be written $\mathcal{P} = \big(P_0 = (5),\ P_1 = (3, 3),\ P_2 = (1)\big)$. Given such a JDT sequence $\mathcal{S}_{\ell,M}$ or $\mathcal{P}_{\deg,\ell,M}$ as in (2-25) or (2-27) we denote by $H(\mathcal{S})$ or $H(\mathcal{P})$ the sequence $H = (h_0, \dots, h_j)$, where h_i counts the number of beads (basis elements) of the strings of $\mathcal{S}$ having degree i; it is the Hilbert function of any module having JDT $\mathcal{S}$ or $\mathcal{P}$.
Given an A-module M, we will denote by $Q_{\ell,i}$ (or $Q_{\ell,i,M}$) the partition giving the lengths of those strings of m_ℓ acting on M that *end* in degree i. We analogously define $\mathcal{Q}_{\ell,M}$ the *end Jordan degree type* to how we defined $\mathcal{P}_{\ell,M}$.

(c) We give an alternate notation for Jordan degree type, closer to Harima and Watanabe's central simple modules (Definition 2.33 below, see [53] and also Costa and Gondim's [39, Definition 4.1]). Recall from Lemma 2.2(i) that, given an element $\ell \in \mathfrak{m}_A = \bigoplus_{i=1}^{j} A_i$, we may write $M = \bigoplus_{k=1}^{s} \langle S_u \rangle$ where each S_k is an *ℓ-string* of M, a cyclic $\mathsf{k}[t]/(t^{p_k})$-submodule with generator z_k, where t acts as m_ℓ; the Jordan type $P_{\ell,M} = (p_1, \dots, p_s)$ where $p_k = |S_k|$. We define

$$\mathcal{E}_{\ell,M} = \{e_i^n(M, \ell), i \in [0, j], n \in [p_s, p_1]\},$$

where

$$\begin{aligned} e_i^n(\ell) = e_i^n(M, \ell) &= \#\{\text{length-}n \text{ strings } S_k,\ \langle S_k \rangle \cong \mathsf{k}[\ell]/(\ell^n), \text{ in } M \text{ beginning in degree } i\} \\ &= \#\{(p_k, \nu_k) \in \mathcal{S}_{\ell,M} \mid p_k = n, \nu_k = i\}. \end{aligned} \tag{2-28}$$

By Lemma 2.2(iv) the integers $e_i^n(\ell)$ are an invariant of the pair (M, ℓ) and do not depend on the particular decomposition.

(ii) *Contiguous degree type of H.* Given a Hilbert function H of an Artinian algebra, we define the *contiguous Jordan degree type* $\mathcal{P}_{c,\deg}(H)$ to be the degree-type obtained from the bar graph of H (similar construction to the continguous Jordan type $P_c(H)$ of Definition 2.17). More precisely, let $H = (h_0, h_1, \dots, h_j)$ be a sequence of nonnegative integers (the Hilbert function). We denote by $P_{\deg,i}(H)$ the partition having $[h_i - h_{i-1}]^+$ parts, each of which is the length of a contiguous string of the bar graph of H beginning in degree i. The *degree-type* of the sequence H is the sequence $\mathcal{P}_{\deg}(H)$ of partitions

$$\mathcal{P}_{c,\deg}(H) = \big(P_{\deg,0}(H), P_{\deg,1}(H)\dots, P_{\deg,j}(H)\big). \tag{2-29}$$

It is the stratification of the contiguous partition $P_c(H)$ by the initial degree of the bars. We may also write $\mathcal{S}_{c,\deg}(H)$ as the JDT associated to H in the sense of (2-26).

(iii) We will say for Jordan degree types $\mathcal{P}, \mathcal{P}'$ with the same Hilbert function $H(\mathcal{P}) = H(\mathcal{P}')$ that

$$\mathcal{P} \leq_c \mathcal{P}' \tag{2-30}$$

if the strings of $\mathcal{P}$ can be concatenated — that is, combined — so as to form $\mathcal{P}'$. For example, $\mathcal{S} = (3_0, 2_3) \leq_c \mathcal{S}' = (5_0)$ (notation of (2-25), (2-26)).

(iv) We say that ℓ has the *relative degree-Lefschetz property* with respect to H if $\mathcal{P}_{\deg,\ell} = P_{c,\deg}(H)$.

(v) A *truncation* $\mathcal{S}_{\ell,A,\leq k}$ of the Jordan degree type $\mathcal{S}_{\ell,A}$ of a graded algebra A to degree less or equal k is its projection to $A/\mathfrak{m}_A^{k+1}$. That is, each pair $(p_i, \nu_i) \in \mathcal{S}_{\ell,A}$ is replaced by

$$(\min\{p_i, k+1-\nu_i\}, \nu_i) \in \mathcal{S}_{\ell,A,\leq k} \quad \text{(or is omitted if } \nu_i \geq k+1). \tag{2-31}$$

(vi) Given two standard-graded algebras $A, B \in \mathrm{B}_T$, $A = R/I$, $B = R/J$ of the same Jordan type $P_{\ell,A} = P_{\ell,B}$ with respect to a fixed element $\ell \in R_1$ we say that the Jordan degree type $\mathcal{S}_{\ell,A} \geq \mathcal{S}_{\ell,B}$ if for each k, the partition associated to $\mathcal{S}_{\ell,A,\leq k}$ is greater or equal to that associated to $\mathcal{S}_{\ell,B,\leq k}$ in the dominance partial order (Definition 2.4).

For an example of (v), the Jordan degree type $\mathcal{S} = \big((3,0),(3,1),(3,2),(3,3)\big)$ has truncation $\mathcal{S}_{\ell,\leq 4} = \big((3,0),(3,1),(3,2),(2,3)\big)$. The JDT $\mathcal{S}' = \big((3,0),(3,1),(3,1),(3,2)\big)$ with $S'_{\leq 4} = \mathcal{S}'$ satisfies $\mathcal{S}' > \mathcal{S}$. See Example 2.31.

Lemma 2.29 (specialization of Jordan degree type). *Fix $\ell \in R_1$ and let $A(w)$, $w \in W \backslash w_0$ be a family of graded Artinian algebras in G_T of constant Jordan type P_ℓ, and constant Jordan degree type $\mathcal{S}_{\ell,A(w)} = S_\ell$ for $w \neq w_0$. Assume that the limit algebra $A(w_0)$ has the same Jordan type P_ℓ. Then $\mathcal{S}_\ell \geq \mathcal{S}_{\ell,A(w_0)}$.*

Proof. The projection from G_T to $G_{T\leq k}$, forgetting the portion of the algebra in degrees $k+1$ and higher, is an algebraic morphism. If there is a specialization of Jordan types, it needs to extend to a specialization of the Jordan types projected to $A(w)_{\leq k}$. Now the condition is obtained by reading the Jordan types of the projections from the Jordan degree types of $A(w)$, then applying Corollary 2.44 below about the semicontinuity of Jordan types in the dominance order. □

The following result is a consequence of Briançon's "vertical strata" analysis of ideals in $\mathsf{k}[x, y]$ [26]. See also [51; 116, §2; 4, pp. 6–7].

Lemma 2.30. *When A is a standard graded algebra of codimension two, and has Jordan type $P_{\ell,A} = (p_1, p_2, \dots, p_s)$ with respect to an element $\ell \in A_1$ and char $k = 0$ or char $k > j$, the socle degree of A, the Jordan degree type satisfies*

$$\mathcal{S}_{\ell,A} = \big((p_1, 0), (p_2, 1), \dots, (p_i, i-1), \dots, (p_s, s-1)\big). \tag{2-32}$$

Proof. Let $A = R/I$, I graded. Supposing $P_{\ell,A} = P$, then replacing ℓ by x (change of basis), and using degree-lex order $1 < x < y < \cdots < x^i < x^{i-1}y < \cdots < y^i < \cdots$ we may project I to its initial monomial ideal E_P, which satisfies

$$E_P = (x^{p_1}, yx^{p_2-1}, \dots, y^{i-1}x^{p_i-1}, \dots, y^{s-1}x^{p_s-1}, y^s). \tag{2-33}$$

The Jordan degree type of I and E_P are the same, as the projection to initial form fixes degree. Also, the JDT of E_P is $\mathcal{S}_{\ell,A}$ of (2-32). □

The beginning idea of the next example is that when $I \subset R$ is a monomial ideal defining $A = R/I$, and $z \in R_1$ there is a kind of "order ideal" of z-strings: that is, if μ is a monomial generator of a length-t z-string of A and if ν divides μ then A has a z-string with generator ν whose length is at least t. For the first algebra A we begin with a length-3 string xyW, where $W = \{\langle 1, z, z^2\rangle\}$. For the second algebra B we begin with a length-3 string x^3W. Our subsequent discussion of structure/components involves more general AG algebras — where I is non-monomial — and shows that for $P = (3^4, 1^4)$ the locus $G_{T,P} \subset G_T \times \mathbb{P}(R_1)$, $T = (1, 3, 5, 4, 2, 1)$ of pairs (A, ℓ), $A \in G_T$, $\ell \in \mathbb{P}(R_1)$ (linear forms up to constant multiple) for which the Jordan type $P_{\ell,A} = P$, has several irreducible components.

Example 2.31 (JDT not equivalent to JT in codimension three). Let $R = \mathsf{k}[x, y, z]$ and set $T = (1, 3, 5, 4, 2, 1)$. We will define $A = R/I_A$, $B = R/I_B$, each with Jordan type $P_{z,A} = P_{z,B} = (3^4, 1^4)$ for the linear form z, where

$$\mathcal{S}_A = \mathcal{S}_{z,A} = \big((3, 0), (3, 1)^2, (3, 2); (1, 2), (1, 3), (1, 4), (1, 5)\big),$$

and

$$\mathcal{S}_B = \mathcal{S}_{z,B} = \big((3, 0), (3, 1), (3, 2), (3, 3); (1, 1), (1, 2)^2, (1, 3)\big);$$

after (2-26), $\mathcal{S}_A = (3_0, 3_1, 3_1, 3_2, 1_2, 1_3, 1_4, 1_5)$ and $\mathcal{S}_B = (3_0, 3_1, 3_2, 3_3, 1_1, 1_2, 1_2, 1_3)$. Let

$$A = \langle W, xW, yW, xyW, \{x^i, 2 \le i \le 5\}\rangle,$$

where $W = \langle 1, z, z^2\rangle$; it is defined by the ideal $I_A = (y^2, x^2z, x^2y, z^3, x^6)$. Let

$$B = \langle W, xW, x^2W, x^3W, y, y^2, xy, y^3\rangle,$$

defined by the ideal $I_B = (yz, x^2y, xy^2, z^3, x^4, y^4)$. Note that both A and B are strong Lefschetz, as the Jordan type of $x + y + z$ is $(6, 4, 3, 2, 1) = T^\vee$.

Specialization of JDT, structure of $G_{T,P} \subset G_T$. From Lemma 2.29 one concludes that a family of graded AG algebras having JDT $\mathcal{S}_B$ cannot specialize to an algebra having JDT $\mathcal{S}_A$. We now show the converse. Let A' have JDT $\mathcal{S}_A$ and B' have JDT $\mathcal{S}_B$ with respect to z. Evidently, there is an element yz (for a suitable choice of y) in $I_2(B')$. We will now show that either

(i) $I_2(A')$ is a perfect square. Then a family $I_2(A'(w)) = x_w^2$ cannot specialize to an I_2 which is composite, or

(ii) $I_2(A') = \langle xy\rangle$ and $A_5'^\vee = (aY - bX)^{[5]}$, a pure power. But we will show in (iii) that $B_5^\vee$ is composite, and again this subfamily with JDT $\mathcal{S}_A$ cannot specialize to an algebra with JDT $\mathcal{S}_B$.

Proof that A' satisfies (i) or (ii). The elements of A'_2 must include zR_1: we may assume then that $I_2(A') = xy$ (up to change of basis x, y), or, case (i), that $I_2(A') = \langle u^2\rangle$ for some $u \in \langle x, y\rangle$.

(ii) Let us assume that $I_2(A') = xy$. Then $A'_2 \supset x^2, y^2$, and the string beginning in degree two has generator $\alpha = x^2 + cy^2$, and last element $z^2(x^2 + cy^2)$. Then we claim $I_5(A') \supset (z, xy) \cap R_5$: this is so, since $(zx^k, zy^k) \in \langle I, \alpha z, \alpha z^2\rangle$ for all $k \geq 2$ because of the JT $(3^4, 1^4)$ and JDT $\mathcal{S}_A$. For example $zx^4 \in I$ since there is no string $\{x^4, zx^4\}$ as 2_4 does not occur. So we may assume that (after possible base change) $I_5 = \langle (z, xy)_5, (ax+by)^5\rangle$. Then the dual $A'^\vee{}_5$ can be written as a pure 5-th power, $(aY - bX)^{[5]}$, showing (ii).

Other ingredients:

(iii) $B'^\vee{}_5$ is composite. It needs to have a mixed $Z^{[2]}\beta$ term where $\beta \in \mathsf{k}_{DP}[X, Y]$. This cannot be part of a perfect power as $Z^{[2]}$ is the highest power of Z that can occur.

This completes the proof that families of AG algebras with the Jordan degree type $\mathcal{S}_A$ and Hilbert function T cannot specialize to an algebra in G_T having Jordan degree type $\mathcal{S}_B$. We have also shown that those algebras in G_T of JDT $\mathcal{S}_A$ have two irreducible components, corresponding to whether I_2 is a perfect square or is composite.

(iv) There are no further JDT associated to $P = (3^4, 1^4)$ for T. Let $(A, z) \in G_{T,P}$. First there must be 3_0 and 3_1 (else $I_2 \supset xz, yz$, but $\dim_{\mathsf{k}} I_2 = 1$). We must rule out $(3_0, 3_1, 3_1, 3_3)$ and $(3_0, 3_1, 3_2, 3_3)$ as the 3^4 part of JDT. The former requires $I_3 \supset \langle zx^2, zxy, zy^2\rangle$ but then $z^2A_3 \subset I$ and there is no room for a string 3_3. The latter requires $\langle yz\rangle = I_2$ (for suitable y); and $A_2 = \langle y^2, x^2, xy, z^2, zx\rangle$. It follows that one of the two strings $3_2, 3_2$ must begin with $\alpha = ay^2 + byx$, but then $z\alpha \in (zy) \in I$, a contradiction.

We have shown that there are exactly two JDT $\mathcal{S}_A, \mathcal{S}_B$ associated to the pair (P, T), $P = (3^4, 1^4)$, $T = (1, 3, 5, 4, 2, 1)$, and that a family of algebras having one of the JDT cannot specialize to an algebra having the other JDT. We have also shown that the JDT locus $\mathcal{S}_A$ has two components. This implies that the locus of pairs $(A, \ell \in A_1) \subset G_{T,P}$ has three irreducible components. □

Comments. When $H(A)$ is not unimodal, the relative degree-Lefschetz property is the closest one can get to strong Lefschetz (Proposition 2.32).

In the Jordan type P_ℓ the part n occurs with multiplicity $\sum_i e_i^n$ where e_i^n is from (2-28) so

(2-34) $$P_{\ell,i} = (\dots, n^{e_i^n}, \dots) \quad \text{and} \quad P_\ell = (\dots, n^{\sum_i e_i^n}, \dots).$$

Evidently, the degree-type $P_{c,\deg}(H)$ of the Hilbert function determines H, so it is equivalent to H — in contrast to $P(H)$ or even $P_c(H)$ which, when H is non-unimodal, may not determine H (see Example 2.35 below). Here are two more examples of concatenation: first, using the $\mathcal{S}_{c,\deg}(H)$ notation, $(2_2, 2_4) \leq_c (4_2)$ of Hilbert function $H = (0, 0, 1, 1, 1, 1)$; and, second, $(2_2, 2_2, 3_4) \leq_c (5_2, 2_2)$ of Hilbert function $H' = (0, 0, 2, 2, 1, 1, 1)$.

Note that $P_{\deg,\ell} \leq_c P_{c,\deg}(H)$ implies that $P_\ell \leq P_c(H)$, but not vice versa. Note also that the contiguous Jordan degree type $P_{c,\deg}(H)$ determines H, so is equivalent in information content to giving H. Recall that Theorem 2.20 bounded the Jordan type by the contiguous Hilbert function. We prove a refinement to JDT in the special case A is standard-graded.

Proposition 2.32 (Jordan degree type bound). *Let M be a finite-length graded module over a standard graded Artinian algebra A. Let $\ell \in A_1$ be any linear form and let $\mathcal{P}_{\deg,\ell} = \mathcal{P}_{\deg,\ell,M}$ be its Jordan degree type as in (2-27). Then in the concatenation partial order,*

(2-35) $$\mathcal{P}_{\deg,\ell,M} \leq_c \mathcal{P}_{c,\deg}\big(H(M)\big).$$

Let M be a fixed finite-length graded A-module. Then there is a generic linear Jordan degree type $\mathcal{P}_{\deg}(M) = \mathcal{P}_{\deg,\ell}(M)$ for $\ell \in U$, a dense open set of A_1.

Proof. The first statement is evident. For the second, begin with the generic Jordan type $P(M)$, consider the set of highest length parts of $P(M)$, and their initial degrees: that these initial degrees are minimal is an open condition on ℓ. Now fix this open set U_1 and go to the set of next highest-length parts for $\ell \in U_1$, forming an open U_2. In a finite number of steps one shows that $P_{deg}(M)$ is achieved for an open dense set U of $\ell \in A_1$. □

For a graded Artinian algebra A, knowing the Jordan degree type $\mathcal{P}_{\deg,\ell}$ is equivalent to knowing the Hilbert functions with respect to $\mathfrak{m}_A$ of the central simple modules (CSM) defined by Harima and Watanabe in [54]. We now explain this.

Definition 2.33 (central simple module). Let A be a graded Artinian algebra. Suppose that $\ell \in A$ satisfies $\ell^c \neq 0$, $\ell^{c+1} = 0$. The central simple modules defined by Harima and Watanabe in [53] are the nonzero factors in the series,

$$A = (0 : \ell^{c+1}) + (\ell) \supset (0 : \ell^c) + (\ell) \supset \cdots \supset (0 : \ell) + (\ell). \tag{2-36}$$

Let s_ℓ be the number of distinct parts of P_ℓ. We denote by $V_{i,\ell}$ for $1 \le i \le s_\ell$ the i-th central simple module: the vector space span of the initial elements of length-f_i strings of the multiplication m_ℓ on M:

$$V_{i,\ell} \cong \langle (0 : \ell^{f_i}) + (\ell) \rangle \mod \langle (0 : \ell^{f_i - 1}) + (\ell) \rangle. \tag{2-37}$$

Let $W_i = \bigoplus_{k=0}^{f_i-1} \ell^k V_{i\ell}$, a direct sum of those cyclic submodules $\langle S_u \rangle$ from (2-1) corresponding to length-f_i strings. Then, evidently $M = \bigoplus_i W_i$.

We have, for the dimension of the degree-u component of $V_{i,\ell}$,

$$\dim_{\mathsf{k}} (V_{i,\ell})_u = e_u^{f_i}(\ell) \quad \text{and} \quad \dim_{\mathsf{k}} V_{i,\ell} = \sum_u e_u^{f_i}(\ell). \tag{2-38}$$

This definition of CSM is perfectly general, and does not require A to be graded nor $\ell \in \mathfrak{m}_A$ to be special. See [58, §3.1], [54, §5.1]; the latter treats nonstandard grading.

The following lemma connects the Jordan degree type and the central simple modules. The proof is straightforward. Besides [54] and [58, §4.1] see also [61, Corollary 2.7] for an approach using the symmetric decomposition with respect to the principal ideal (ℓ).

Lemma 2.34. *The set $\mathcal{H}_\ell = \big(H(V_{1,\ell}), \dots, H(V_{s_\ell,\ell})\big)$ of Hilbert functions of the central simple modules $V_{i,\ell}$ is equivalent to the Jordan degree type of ℓ, rearranged according to the lengths $f_1 > f_2 > \cdots > f_{s_\ell}$. In particular $H(V_{i,\ell})_u = (\dots, e_u^{f_i}(\ell), \dots)$.*

Example 2.35 (degree types of Hilbert functions). We illustrate that the contiguous Hilbert function partition $P_c(H)$ can distinguish two Hilbert functions of the same partition $P(H)$; also, the Jordan degree-type $P_{c,\deg}(H)$ can distinguish two Hilbert functions of the same contiguous partition $P_c(H)$. For $H = (1, 3, 2, 3, 3, 1)$ we have $P(H) = (6, 4, 3)$, $P_c(H) = (6, 4, 2, 1)$ and $P_{c,\deg}(H) = \big((6_0), (4_1, 1_1), (2_3)\big)$. For $H' = (1, 3, 1, 3, 3, 2)$ we have $P(H') = (6, 4, 3)$, $P_c(H') = (6, 3, 2, 1, 1)$, and $P_{c,\deg}(H') = \big((6_0), (1_1, 1_1), (3_3, 2_3)\big)$. For $H'' = (1, 3, 3, 2, 1, 3)$ (not pictured) we have $P(H'') = (6, 4, 3)$, $P_c(H'') = P_c(H') = (6, 3, 2, 1, 1)$, but $P_{c,\deg}(H'') = \big((6_0), (3_1, 2_1), (1_4, 1_4)\big)$. This last illustrates that the

Hilbert function is not determined by the contiguous partition $P_c(H)$, but is determined by the contiguous degree-type $P_{c,\deg}(H)$. We chose these examples, even thought they are not standard-graded, because their diagrams in Figure 1 are particularly transparent. Note that by Macaulay's inequalities for Hilbert functions, none of the Hilbert functions H, H', H'' above can occur for a standard graded algebra. We next give some similar comparisons whose Hilbert functions do occur for a standard graded algebra.

Standard graded Hilbert functions: we compare contiguous partitions $P_c(H)$ and contiguous degree-types $P_{c,\deg}(H)$ (2-29). Here $\{H_i, i \in [1, 8]\}$ denotes Hilbert functions.

Same partition $P(H) = H^\vee$ but different $P_c(H)$: take $H_1 = (1, 3, 6, 4, 5, 6, 2)$, $H_2 = (1, 3, 4, 5, 6, 6, 2)$ then $P_c(H_1) = (7, 6, 5, 4, 2, 1^3)$, $P_c(H_2) = (7, 6, 5, 4, 3, 2)$.

Same $P_c(H)$ but different Hilbert function (so different $P_{c,\deg}(H)$): compare $H_3 = (1, 3, 5, 7, 6)$ with $H_4 = (1, 3, 5, 6, 7)$. The Hilbert functions $H_5 = (1, 3, 6, 10, 9, 11, 12, 10)$ and $H_6 = (1, 3, 6, 10, 9, 10, 11, 12)$ have the same $P_c(H)$, but their degree types $\mathcal{S}(H)$ differ in having the subsequence $(1_3, 2_5, 1_6)$ for H_5 but $(1_3, 2_6, 1_7)$ for H_6. All of $H_1, H_2, \dots, H_6$ satisfy the Macaulay growth conditions. A simpler example compares $H_7 = (1, 3, 3, 4, 5)$ with $H_8 = (1, 3, 4, 5, 3)$: since they are unimodal, and $P(H_7) = P(H_8)$ we have also $P_c(H_7) = P_c(H_8)$, but, of course, $H_7 \neq H_8$ so $\mathcal{S}_{c,\deg}(H_7) = (5_0, 4_1^2, 2_3, 1_4)$ is not $\mathcal{S}_{c,\deg}(H_8) = (5_0, 4_1^2, 2_2, 1_3)$.

Question 2.36. For which Hilbert functions H can we find graded Artinian algebras A with $H(A) = H$ and such that for a generic $\ell \in A_1$ we have, in increasing level of refinement,

$$P_{\ell,A} = P(H), \text{ or } P_{c,\ell} = P_c(H), \text{ or } P_{c,\deg,\ell} = P_{c,\deg}(H)? \tag{2-39}$$

Note that a graded algebra $A = \mathsf{k}[x, y, z]/I$ of Hilbert function $H(A) = (1, 3, 3, 4)$ cannot be even weak Lefschetz as the minimal growth from degree 2 to degree 3 implies that $I_2 = a_1(x, y, z)$ for some $a_1 \in A_1$, so multiplication by an $\ell \in A_1$ cannot be injective from A_1 to A_2.

There has been some study of a different question, namely, which Hilbert functions H force A to have one of the Lefschetz properties [90; 117]. See also [89].

Example 2.37. We first construct the idealization of $B = \mathsf{k}[x, y, z]/\mathfrak{m}^3$ of Hilbert function $H(A) = (1, 3, 6)$ with its dual giving an algebra A of Hilbert function $H(A) = (1, 3, 6, 0) + (0, 6, 3, 1) = (1, 9, 9, 1)$ (see also Section 3A below). We may take a Macaulay dual generator $F = \sum_{i=1}^6 U_i \mu_i$ where μ_i runs through the six monomials of degree 2 in X, Y, Z, in lexicographic order, and $U_1, \dots, U_6$ are variables, as

$$F = U_1 X^{[2]} + U_2 XY + U_3 XZ + U_4 Y^{[2]} + U_5 YZ + U_6 Z^{[2]}.$$

Take $R = \mathsf{k}[x, y, z, u_1, \dots, u_6]$ acting by contraction on $S = \mathsf{k}_{DP}[X, Y, Z, U_1, \dots, U_6]$. Then $A_2^\vee = R_1 \circ F$ satisfies

$$A_2^\vee = \langle X^{[2]}, XY, XZ, Y^{[2]}, YZ, Z^{[2]}, U_1X + U_2Y + U_3Z, U_2X + U_4Y + U_5Z, U_3X + U_5Y + U_6Z\rangle \tag{2-40}$$

while $A_1^\vee = S_1 = \mathsf{k}_{DP}[X, Y, Z, U_1, \dots, U_6]$. We may take (after scaling) as a generic linear form $\ell = x + y + z + \sum_{i=1}^6 u_i$. Then the rank of $m_\ell : A_1 \to A_2$ is by duality the same as that of $m_\ell : A_2^\vee \to A_1^\vee$. But $m_\ell : A_2^\vee \to A_1^\vee$ takes a 6-dimensional space $\langle X^{[2]}, XY, XZ, Y^{[2]}, YZ, Z^{[2]}\rangle$ to the 3-dimensional space $\langle X, Y, Z\rangle$ so has a 3-dimensional kernel. Thus, using symmetry of Proposition 2.38 we have that $P_\ell = (4, 2^5, 1^6)$ and the Jordan degree type is $\mathcal{P}_\ell = (4_0, 2_2^5, 3_1, 3_2)$.

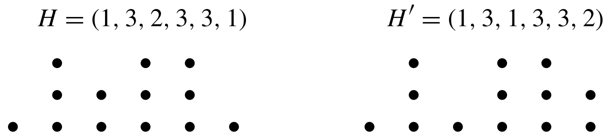


Figure 1. $P_c(H) = (6, 4, 2, 1)$ and $P_c(H') = (6, 3, 2, 1, 1)$.

Evidently, we may make similar examples using, say, a general-enough 4-dimensional subspace $\mu_1, \dots, \mu_4$ of $\mathsf{k}[x, y, z]_2$, and $F = \sum \mu_i U_i$, and finding there is a 1-dimensional kernel, this gives an algebra A of Hilbert function $(1, 7, 7, 1)$ where $\ell = x + y + z + u_1 + \cdots + u_4$ has Jordan type $(4, 2^5, 1, 1)$ and Jordan degree type $\mathcal{S}_\ell = (4_0, 2_1^5, 1_1, 1_2)$. Likewise we can determine A of Hilbert function $H = (1, 8, 8, 1)$ with ℓ a generic enough linear form having Jordan type $P_\ell = (4, 2^5, 1^4)$ and Jordan degree-type $\mathcal{S}_\ell = (4_0, 2_1^5, 1_1^2, 1_2^2)$.

Costa and Gondim show the following symmetry proposition using the numerics of central simple modules [39, Lemma 4.6]; thus, this result along with the string diagrams they introduce [39, Remark 4.9] is essentially a felicitous and visual interpretation of the work of Harima and Watanabe in [54; 55].[3]

Proposition 2.38 (symmetry of Jordan degree type for graded AG algebras). *Let A be a standard graded Artinian Gorenstein algebra of socle degree j, and let $\ell \in A_1$. Then the Jordan degree type is symmetric: The integers e_ν^n from (2-28) satisfy*

$$e_\nu^n = e_{j+1-n-\nu}^n. \tag{2-41}$$

In other words, the set of strings of $\mathcal{S} = \mathcal{S}_{\ell,A}$ of (2-25) and notation (2-26) satisfy $\mathsf{n}_\nu \in \mathcal{S} \Leftrightarrow \mathsf{n}_{j+1-n-\nu} \in \mathcal{S}$, with the same multiplicity $e_\nu^n = e_{j+1-n-\nu}^n$.

Proof. The homomorphism $m_{\ell^{f_i-1}} : V_{i,\ell} \to \ell^{f_i-1} V_{i,\ell}$ is an isomorphism of A-modules, so we have $H(V_{i,\ell})_u = H(\ell^{f_i-1} V)_{u+f_i-1}$, but from the exact pairing $A \times A \to \mathsf{k} : (a, b) \to \phi(ab)$ of Lemma 2.25, we have that $H(\ell^{f_i-1} V_{i,\ell})_{j-u} = H(V_{i,\ell})_u$. We conclude $H(V_{i,\ell})_u = H(V_{i,\ell})_{j+1-f_i-u}$; taking $n = f_i$ and using (2-38) we obtain the result. $\square$

The proposition is also a consequence of symmetric decomposition of the associated graded algebra $\mathrm{Gr}_\ell(A)$ with respect to ℓ. The reflexive $\mathrm{Gr}_\ell(A)$ module $Q_\ell(j + 1 - f_i)$ in the symmetric decomposition of $\mathrm{Gr}_\ell(A)$ has first graded component $V_{i,\ell}$ and last component $\ell^{j+1-f_i} V_{i,\ell}$ (see [61, Corollary 2.7]). This symmetry, which essentially states that there is a 1-1 map between the strings $\mathsf{n}_u(\ell)$ and the strings $\mathsf{n}_{j+1-n-u}(\ell)$ of the Jordan degree type of an AG algebra, greatly restricts the possible Jordan types; see [3; 39] for examples.

Example 2.39 [3, Example 4.6, Figure 14]. We let $H = (1, 2, 3, 2, 1)$ and consider the Jordan degree types for elements $\ell \in A_1$ for Artinian complete intersections $A = R/(f, g)$ of Hilbert function H. These are $(5_0, 3_1, 1_2)$, $(5_0, 2_1, 2_2)$, $(4_0, 4_1, 1_2)$, $(3_0, 3_1, 3_2)$, in the notation of (2-26). The partition

[3] It was in a work group at the conference Lefschetz Properties and Jordan Type at Levico, Italy, 25-29 June 2018 that Rodrigo Gondim had presented the symmetric string diagrams of [39]; we realized a few days later in a discussion with Alessandra Bernardi and Daniele Taufer at University of Trento that the Jordan degree type we introduce here is a natural context for understanding this symmetry, leading to the proposition.

$P = (3, 3, 1, 1, 1)$, which ostensibly could correspond to a symmetric Jordan degree type $(3_0, 3_2, 1_1, 1_2, 1_3)$, does not occur for a CI (Gorenstein) quotient of $R = \mathsf{k}[x, y]$. Thus, the symmetry condition of (2-41), although quite restrictive, is not sufficient to determine the Jordan types for Artinian Gorenstein algebras of given Hilbert function.

Question 2.40. How does the degree type $\mathcal{S}_{\ell,M}$ or $\mathcal{P}_{\deg,\ell,M}$ behave under

(i) deformation of $\ell \in A_1$?

(ii) deformation of M within the family of A-modules of fixed Hilbert function H? See [68].

(iii) tensor product (see Corollary 3.11)?

(iv) projection to the quotient $R/\mathrm{in}I$? Are there JDT that cannot occur for an Artin graded algebra of Hilbert function T defined by a monomial ideal?

Also, does the inequality of (2-35) of Proposition 2.32 extend to finite length modules over a local Artinian $\mathcal{A}$, taking $\ell \in \mathfrak{m}_{\mathcal{A}}$?

Question 2.41. Can we extend the notion of Jordan-degree type of (A, ℓ) for graded algebras to a "Jordan-order type" for local algebras $\mathcal{A}$, with properties analogous to those of Lemma 2.2, replacing degree by order, and omitting homogeneity? In particular, concatenating the orders of all elements in a good set of strings for (A, ℓ) should give the Hilbert function of A.

That there are issues in defining Jordan order type for a local algebra is illustrated in the following example. Recall from (1-1) and the paragraph after, that the order $\nu(a)$ of $a \in A$ is the largest power of the maximum ideal $\mathfrak{m}$ of A such that $a \in \mathfrak{m}^\nu$.

Example 2.42. Let $A = \mathsf{k}\{x, y\}/I$, $I = (x^2 - xy^2, x^4, y^4) = (x^2 - xy^2, y^4, x^3y^2)$, variables each of weight 1, with k-basis the classes of $\{1, x, y, xy, y^2, xy^2, y^3, xy^3\}$ and Hilbert function $H(A) = (1, 2, 2, 2, 1)$. Consider $\ell = x$ for which $P_x = (3, 3, 1, 1)$ and the Jordan strings $S_1 = (1, x, x^2 = xy^2)$ of orders $(0, 1, 3)$, $S_2 = (y, xy, x^2y = xy^3)$ of orders $(1, 2, 4)$, $S_3 = (x - y^2)$ and $S_4 = (xy - y^3)$ of orders (1) and (2); the choice of the strings has been made so that $x^{p_i}m_i = 0$, where m_i is the cyclic generator of S_i (hence the choice of $m_3 = (x - y^2)$, so $xm_3 = 0$). There are three basis elements (of the form $x^k m_i$, $0 \le k \le p_i - 1$) in the strings having order one; also, there is no $\mathsf{k}[x]$ linear combination of the strings with the property that the orders of the basis elements match the Hilbert function (compare with Lemma 2.2(i, iv) for the graded case).

One "solution" might be to adjust the sense of order in the presence of previous strings: for example $S_3 = \langle x - y^2\rangle$ would be considered to have *adjusted order* two, as all of R_1, in particular x, is already in $\langle S_1, S_2\rangle$.

2G. Deformations and generic Jordan type.

We assume that k is an infinite field when discussing either generic Jordan type or deformation. A *deformation* of a local Artinian algebra $\mathcal{A}$ over k is a flat family $\mathcal{A}_t$, $t \in T$ of Artinian algebras with special fiber $\mathcal{A}_{t_0} = \mathcal{A}$; then $\mathcal{A}_{t_0}$ is a *specialization* of the family $\mathcal{A}_t$. We note that an algebraic family $\mathcal{A}_t$, $t \in T$ of Artinian algebras over a parameter space is flat if the fibers $\mathcal{A}_t$, $t \in T$ have constant length.[4] If also, for $t \neq t_0$ the algebras $\mathcal{A}_t$ have constant isomorphism

[4] A flat deformation means that relations among generators at the special point extend to relations among generators at the general point (see §I.3 "Meaning of flatness in terms of relations" in [7]). The flatness of a reduced family of Artinian algebras over an algebraically closed field that has constant fiber length is well known, and noted by Briançon in [26]. See [93, Proposition 8, p. 44; 59, Exercise 5.8c, p. 125].

type, we will say $\mathcal{A}_t$, $t \neq t_0$ is a jump deformation of $\mathcal{A}_{t_0}$. We use the dominance partial order on partitions of n (Definition 2.4). The following result is well known in other contexts. See the related Lemma 2.54, and the Definition 2.55 of generic Jordan type P_M for a finite A-module M, where A is a local algebra.

Lemma 2.43. *Let V be a k-vector space of dimension n. Let T be a parameter variety (as $T = \mathbb{A}_1$). Let $\ell(t) \in \mathrm{Mat}_n(V)$, $t \in T$ be a family of nilpotent linear maps, and let P be a partition of n. Then the condition on Jordan types, $P_{\ell(t)} \leq P$ is a closed condition on T.*

Proof. This is straightforward to show from Lemma 2.3 and the semicontinuity of the rank of $\ell(t)^i$ (see [33, Theorem 6.2.5], or [62, Lemma 3.1]) □

This result is not unrelated to the work of V.I. Arnol'd [6, §4.4. Theorem], who studied the versal deformation of a matrix M having a single eigenvalue (for us, the eigenvalue is 0), and gives its dimension as $p_1 + 3p_2 + 5p_3 + \cdots$. His article discusses singularities related to the different Jordan loci in the deformation space of M, a generalization of "bifurcations", a topic we do not develop here. The centralizer of M is given in [44, §VIII.2], and is basic to the study of the nilpotent commutator of M — see the discussion after Example 3.37, and references there.

Corollary 2.44 (semicontinuity of Jordan type).

(i) *Let M_t for $t \in T$ be a family of constant length modules over a parameter space T. Then for a neighborhood $U_0 \subset T$ of $t = t_0$, we have that the generic Jordan types satisfy $t \in U_0 \Rightarrow P_{M_t} \geq P_{M_{t_0}}$.*

(ii) *Let $\mathcal{A}_t$, $t \in T$ be a constant length family of local or graded Artinian algebras. Then for a neighborhood $U_0 \subset T$ of $t = t_0$, we have $t \in U_0 \Rightarrow P_{\mathcal{A}_t} \geq P_{\mathcal{A}_{t_0}}$.*

(iii) *Let $\ell_t \in \mathcal{M}_n(\mathsf{k})$ for $t \in T$ be a family of $n \times n$ nilpotent matrices, and let P_t be their Jordan type. Then there is a neighborhood $U \subset T$ of t_0 such that $P_t \geq P_{t_0}$ for all $t \in U$.*

Applying this result to the deformation from the associated graded algebra $\mathcal{A}^*$ to the local Artinian algebra $\mathcal{A}$ we have:

Corollary 2.45. *Suppose that $\mathcal{A}$ is a local Artinian algebra with maximum ideal $\mathfrak{m}$ and $\ell \in \mathfrak{m}$. Then $P_\ell(\mathcal{A}) \geq P_\ell(\mathcal{A}^*)$.*

Proof. Consider the natural flat deformation[5] from $\mathcal{A}^*$ to $\mathcal{A}$. For $t \neq 0$, $\mathcal{A}_t$ has constant isomorphism type: this is a jump deformation, so the open neighborhood U of Corollary 2.44 includes elements where $P_{\ell,\mathcal{A}_t} = P_{\ell,\mathcal{A}}$. □

Corollary 2.45 gives the following sufficient condition for checking SLJT:

Corollary 2.46. *If an element $\ell \in \mathfrak{m}$ is SL for the associated graded Artinian algebra $\mathcal{A}^*$, then ℓ is SLJT for $\mathcal{A}$.*

[5]See [41, Theorem 15.17]. This was shown by M. Gerstenhaber [45] but [43, Chapter 5] gives a history showing prior use by D. Rim in 1956 and D. Mumford in 1959. It is easy to show in the Artinian case using the constant-length-fiber criterion for flatness.

Jordan type and initial ideal. We thank the referee for calling to our attention the work of Wiebe, connecting Lefschetz properties of ideals with those of the initial ideals. We assume that $<$ is a term order (monomial order) on the monomials of R: that is, if $\mu < \mu'$ then $\nu\mu < \nu\mu'$ for each monomial ν. The *initial ideal* in I is the monomial ideal generated by the initial monomials of its elements, and $\mathrm{gin}_< I \subset R$ is the initial monomial ideal of $\alpha \circ I$ where α is a general enough element of $\mathrm{Gl}_r(\mathsf{k}) = \mathrm{Aut}(R_1)$ acting on I. Wiebe has shown the first two parts of the next lemma. Part (iii) is newly stated here, but it follows from the discussion after [115, Proposition 2.8], based on A. Conca's [34, Lemma 1.2]; we include the proof. See also [58, §6.1.2] for a further discussion in the context of k-Lefschetz properties, which we do not treat here. The reverse-lex order is $<_{revlex}$ and satisfies $x_1 > x_2 > \cdots > x_r > x_1^2 > x_1x_2 > x_2^2 > x_1x_3 > \cdots$ (see [58, Definition 6.12]).

Proposition 2.47 [115]. *Let $I \subset R$ be an $\mathfrak{m}$-primary graded ideal.*

(i) [115, Proposition 2.8]. *Then $A = R/I$ has SLP (resp. WLP) if and only if $R/\mathrm{gin}_<(I)$ has SLP (resp. WLP)*

(ii) [115, Proposition 2.9]. *Let J be the initial ideal of I with respect to a term order. If S/J has the weak (resp. strong) Lefschetz property, then the same holds for R/I.*

(iii) *Let $K = \mathrm{gin}_< I$. Then the generic Jordan type P_A of $A = R/K$ is the same as that of R/I. For $K = \mathrm{gin}_{<_{revlex}} I$ in the reverse lex order, this is the Jordan type of R/K with respect to x_r.*

Proof of (iii). Wiebe [115, Proposition 2.8ff][6] stated that by generalizing Conca's proof [34, Lemma 1.2], using $\mathrm{in}_{revlex}(gI + (r^k)) = \mathrm{in}_{revlex}(gI) + (x_r^k)$ for all $k \geq 1$ one obtains that the Hilbert function of $S/(J, x_r^k)$ is equal to the Hilbert function of $S/(I, \ell^k)$ for a general linear form $\ell \in S$ and all $k \geq 1$. By the theorem of A. Galligo (char $\mathsf{k} = 0$) and D. Bayer and M. Stillman (arbitrary characteristic), the ideal $J = \mathrm{gin}_< I$ is Borel-fixed ([115, Theorem 2.4], proved in [41, Theorem 15.20]). It follows that for the revlex order, the generic Jordan type of R/J is $P_{x_r}(R/J)$. □

Wiebe's discussion brings in the upper semicontinuity of $\dim\big(R/(I, \ell^k)\big)_i$: attention to this semicontinuity in a more general setting shows Lemma 2.43. Wiebe used his results to give criteria for componentwise linear ideals to have the WLP or SLP in terms of the Betti numbers of a resolution; these criteria were extended to all $\mathfrak{m}$-full ideals by Harima and Watanabe [56], and further studied by J. Ahn, Young Hyun Cho, and J. P. Park [1].

However, the Jordan type may be different for $A = R/I$ and for $R/\mathrm{in}(I)$ when the latter is not strong Lefschetz.

Example 2.48. Take $R = \mathsf{k}[x, y, z]$, consider the graded lex monomial order $1 < x < y < z < x^2 < xy < xz < y^2 < yz < z^2 < \cdots$ and consider the complete intersection $A = R/I$, $I = (x^2, xy + z^2, xz + y^2)$ of Hilbert function $H(A) = (1, 3, 3, 1)$. The generic Jordan type of A is $(4, 2) = H(A)^\vee$, so A is strong Lefschetz. But $J = \mathrm{in} I = (x^2, xy, xz, y^3, z^4)$, $B = R/J$ has Jordan type $(4, 1, 1)$ since for any linear form ℓ, $m_\ell : A_1 \to A_2$ has kernel x. Thus, B is not weak Lefschetz.

[6] There are some misprints in the statements and proofs of these results in [115], that are correct in the arXiv version we have referenced.

Examples of deformation, generic Jordan type. Recall that a local Artinian algebra $\mathcal{A}$ is *curvilinear* if $H(\mathcal{A}) = (1, 1, \dots, 1_j, 0)$, in which case $\mathcal{A}$ is isomorphic to $\mathsf{k}\{x\}/(x^{j+1})$. In the following example we illustrate that a local algebra $\mathcal{A}$ determined by $I = (x, y)^2$ in $R = \mathsf{k}[x, y]$ can be deformed to a curvilinear algebra. This is a special case of Briançon's celebrated result that the fiber of the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2)$ over $(0, 0)$ is irreducible, and is the closure of the curvilinear locus — which for embedding dimension two is the locus where the ideal defining I has an element of order one [26].[7]

Example 2.49. Let $\mathcal{A}(t) = \mathsf{k}\{x, y\}/I_t$ where for $t \in \mathsf{k}$, $t \neq 0$, we let $I_t = (tx - y^2, y^3)$ and where $\mathcal{A}(0) = \lim_{t\to 0} \mathcal{A}(t) = \mathsf{k}[x, y]/(x^2, xy, y^2)$ (since for $t \neq 0$ the ideal $I_t \supset \{tx - y^2, xy, x^2, y^3\}$). For $t \neq 0$ the local algebra $\mathcal{A}(t)$ is a complete intersection with Hilbert function $(1, 1, 1)$. This family specializes to $\mathcal{A}(0)$ which is non-Gorenstein, with Hilbert function $(1, 2)$. Here for $t \neq 0$, taking $\ell = x + y$, we have that the generic linear Jordan type $P_{\ell,\mathcal{A}(t)} = (3) > (2, 1) = P_{\ell,\mathcal{A}(0)}$.

In the next examples we use the Macaulay dual generator notation from Definition 2.23. The examples use standard grading.

Example 2.50. Let $B_t = \mathsf{k}[x, y]/\operatorname{Ann} F_t$, where $F_t = tX^{[5]} + X^{[2]}Y$. Then for $t \in \mathsf{k}$, $t \neq 0$, the algebra B_t is a curvilinear complete intersection, as in the previous example, with Hilbert function $(1, 1, 1, 1, 1, 1)$. The family specializes to $B_0 = \mathsf{k}[x, y]/(y^2, x^3)$, also a complete intersection (CI), with Hilbert function $(1, 2, 2, 1)$. Here $\ell = x + y$ determines the generic (also generic linear) Jordan type $P_{\ell.B_t} = (6) > (4, 2) = P_{\ell,B_0}$.

Example 2.51. Let $B_t = \mathsf{k}\{x, y, z\}/\operatorname{Ann} F_t$, where $F_t = t^2X^{[3]}Y^{[2]} + tX^{[2]}YZ + XZ^{[2]}$. Then for $t \neq 0$ the algebra $B_t = \mathsf{k}\{x, y, z\}/(tz - xy, y^3, x^4)$ is an Artinian complete intersection with Hilbert function $(1, 2, 3, 3, 2, 1)$. The family specializes to the complete intersection $B_0 = \mathsf{k}[x, y, z]/(y^2, z^2, x^3)$, which has Hilbert function $(1, 3, 4, 3, 1)$. Here for $t \neq 0$ $P_{B_t} = (6, 4, 2) > (5, 3, 3, 1) = P_{B_0}$.

Definition 2.52 (the poset $\mathfrak{P}_M$). Let M be a finite-length $\mathcal{A}$ module for an Artinian algebra $\mathcal{A}$. We denote by $\mathfrak{P}_M$ the poset $\{P_\ell \mid \ell \in \mathfrak{m}_\mathcal{A}\}$, with the dominance partial order. We denote by $\mathfrak{P}_{i,M}$ the subposet $\mathfrak{P}_{i,M} = \{P_\ell \mid \ell \in (\mathfrak{m}_\mathcal{A})^i\}$.

There is a related poset $\mathfrak{Z}_M$ of loci within $\mathfrak{m}_A$ or $\mathfrak{m}_\mathcal{A}$ determined by the set of partitions P_ℓ: this is a subposet of $\mathfrak{P}_M$ but may be a proper subset; see Example 3.37.

Example 2.53. For $\mathcal{A}_t = \mathsf{k}\{x, y\}/(tx - y^2, y^3)$, $t \neq 0$, from Example 2.49, we have that $\mathfrak{P}_{\mathcal{A}_t} = \{P_y = (3),\ P_x = (2, 1),\ P_0 = (1, 1, 1)\}$. For $\mathcal{A}_0 = \mathsf{k}[x, y]/(x^2, xy, y^2)$, we have that $\mathfrak{P}_{\mathcal{A}_0} = \{(2, 1), (1, 1, 1)\}$.

Recall that the elements of a local Artinian algebra $\mathcal{A}$ are parametrized by the affine space $\mathbb{A}^n$, $n = \dim_\mathsf{k}(\mathcal{A})$ and with our assumption $\mathfrak{m}_\mathcal{A}$ is parametrized by the affine space $\mathbb{A}^{n-1}$. So $\mathfrak{m}_A$ is an irreducible variety. Since the rank of each power of m_ℓ acting on M is semicontinuous, and since by Lemma 2.3 these ranks determine the Jordan type of m_ℓ we have:

Lemma 2.54 (generic Jordan type of M). *Given an A or $\mathcal{A}$ module M, there is an open dense subset $U_M \subset \mathfrak{m} = \mathbb{A}^{n-1}$ for which $\ell \in U_M$ implies that the partition P_ℓ satisfies $P_\ell \geq P_{\ell'}$ for any other element $\ell' \in \mathfrak{m}$.*

[7] Briançon proved his result over $\mathsf{k} = \mathbb{C}$; his proof extends to algebraically closed fields satisfying char $\mathsf{k} > n$: this was improved to char $\mathsf{k} > n/2$ by R. Basili [12] and to all characteristics by A. Premet [108]; see also V. Baranovsky's [11].

Likewise, if $\mathcal{A}$ admits a weight function w, *then for each weight* i, *there is a dense open set* $U_{i,M} \subset \mathcal{A}_i(\mathsf{w})$ *for which* $\ell \in U_{i,M}$ *implies that* $P_\ell \geq P_{\ell'}$ *for any other* $\ell' \in \mathcal{A}_i(\mathsf{w})$. *So* $P_{\ell,M}$ *takes on a generic value* P_M *for* $\ell \in U_M$.

Definition 2.55. For M a finite module over a local Artinian algebra $\mathcal{A}$ over k, we define the *generic Jordan type* P_M by $P_M = P_\ell$ where ℓ is a generic element of the maximal ideal of $\mathcal{A}$. In the graded case, we may also define the *generic degree* i *Jordan type* for M is $P_{i,M} = P_\ell$ for ℓ a generic element of A_i (it is not defined when $A_1 = 0$); for $i = 1$, we call $P_{1,M}$ the *generic linear Jordan type*.

Evidently we have

$$P_M \geq P_{1,M} \geq \cdots \geq P_{j,M}.$$

As we next see in Example 2.57, when A is a nonstandard graded algebra the generic Jordan type P_A may not equal $P_{1,A}$ even when $A_1 \neq 0$.

Question 2.56. Under what conditions on a graded module M over a graded Artinian algebra A does its generic Jordan type satisfy $P_M = P_{1,M}$, the generic linear Jordan type? In particular, let A be a standard-graded Artinian algebra A with unimodal Hilbert function. Is the generic linear Jordan type of A always the same as the generic Jordan type of A? Proposition 2.14 shows this when the generic Jordan type of A is strong Lefschetz.

Let A be a nonstandard-graded Artinian algebra. The Jordan type P_ℓ of a nonhomogeneous element $\ell \in \mathfrak{m}_A$ may be the same as that would be expected for a strong Lefschetz element, even though A may have no linear strong Lefschetz elements, so A is not SL. This we first noticed on the following example of relative covariants proposed by the third author (see [83, Example 3.7]).

Example 2.57. We let $R = \mathsf{k}[y,z]$, with weights $\mathsf{w}(y,z) = (1,2)$, and let $A = R/I$, $I = (yz, z^3, y^7)$, having k-basis $A = \langle 1, y, y^2, z, y^3, y^4, z^2, y^5, y^6\rangle$ and having Hilbert function $H(A) = (1,1,2,1,2,1,1)$, with Macaulay dual $R \circ \langle Z^2, Y^6\rangle$. The only linear element of A, up to nonzero constant multiple is y, and the partition given by the multiplication m_y is $P_y = (7,1,1)$, so A is not strong-Lefschetz. However the nonhomogeneous element $\ell = y+z$, has strings $\{1, y+z, y^2+z^2, y^3, y^4, y^5, y^6\}$ and $\{z, z^2\}$ so $P_\ell = (7,2)$, which is the maximum possible given $H(A)$, so ℓ has strong Lefschetz Jordan type (Definition 2.12). The Macaulay dual generator for A is $F = Z^{[2]} + Y^{[6]}$

A related local algebra is $\mathcal{A} = \mathcal{R}/(yz, z^3 + y^6)$, with weights $\mathsf{w}'(y,z) = (1,1)$, of Hilbert function $H(\mathcal{A}) = (1,2,2,1,1,1,1)$; here the element $\ell' = (y+z) \in \mathfrak{m}_{\mathcal{A}}$ has Jordan type $(7,2)$, so $\mathcal{A}$ is strong Lefschetz. The associated graded algebra $\mathcal{A}^*$ with respect to $\mathfrak{m}_{\mathcal{A}}$ is $\mathsf{k}[y,z]/(yz, z^3, y^7)$, with standard grading; and $P_{\ell',\mathcal{A}^*} = P_{\ell',A} = (7,2)$. See also Theorem 2.5 and Example 2.21.

3. Constructions, examples, and commuting Jordan types

3A. Idealization and Macaulay dual generator. The principle of idealization, introduced by M. Nagata to study modules, has been used to "glue" an Artinian algebra to its dual, and so to construct graded Artinian Gorenstein (AG) algebras either having non-unimodal Hilbert functions; or with unimodal Hilbert functions but not having a strong or weak Lefschetz property [19; 20; 21; 23; 91; 113]. Examples of M. Boij [20] show that the Hilbert functions of AG algebras may have an arbitrarily high number of valleys, with local maxima at assigned values, at the cost of increasing the embedding dimension. The Jordan types and Jordan degree types — which are symmetric by Proposition 2.38 — of these examples have in general not been studied, and could be of interest. H. Ikeda (H. Sekiguchi) [111], H. Ikeda and Watanabe [70], and also Boij [21] gave examples of Artinian Gorenstein (AG) algebra having unimodal Hilbert functions, but not satisfying even weak Lefschetz. Similar examples involving a partial idealization, also not strong Lefschetz or not weak Lefschetz have been constructed more recently by R. Gondim and G. Zappalà [48; 49] and by A. Cerminara, Gondim, G. Ilardi, F. Maddaloni [32]. An example where m_L has Jordan type strictly between weak and strong Lefschetz was already given in [58, Section 5.4], referencing [70].

We will give here several idealization or partial idealization examples where we calculate the generic Jordan type. These idealizations that are Artinian Gorenstein can arise from a particular structure for a homogeneous Macaulay dual generator for the AG algebra (Section 2E). There is a far-reaching generalization by Kleiman and Kleppe to Macaulay duality over an arbitrary Noetherian base ring [73], and it would be natural to ask how the notions of Jordan type and idealization studied below might generalize from a base field, as here, to a more general base.

Nagata [94] introduced the following definition of idealization, see also [58, Definition 2.75]; for some further developments and history see D. Anderson and M. Winters [5]. For characterization by dual generator see [19; 58, Theorem 2.77] and for another approach to idealization see [80, Lemma 3.3].

Definition 3.1. Let M be an A-module. The idealization of M is the algebra $A \oplus M$ whose multiplication is given by $(a, m) \cdot (b, n) = (ab, an + bm)$.

The idealization makes $A \oplus M$ into a ring, in which the R-submodules of M correspond to the subideals of M. A particular example is formed when $M = \mathrm{Hom}(R, \mathsf{k})$ the dual of an Artinian level ring A (i.e., the socle of A is in a single degree): then the idealization is an Artinian Gorenstein algebra. Stanley and subsequently others used this construction to give examples of AG algebras having non-unimodal Hilbert function (Example 3.2).

Examples of Idealization and Jordan type. Our first is the Jordan type for the example of Stanley in codimension 13. See also [61, Example 2.28] where this was also calculated. We will find that the actual generic linear Jordan type is very far from the bound $P_c(H)$ given in Theorem 2.20. We will use the notation m^k to represent the partition $(m, \dots, m)$ with k parts.

Example 3.2 [113, Example 4.3]. We let $R = \mathsf{k}[x, y, z]$ and $S = \mathsf{k}[x, y, z, u_1, \dots u_{10}]$, $\mathfrak{D} = \mathsf{k}_{DP}[X, Y, Z]$ and $\mathfrak{F} = \mathsf{k}_{DP}[X, Y, Z, U_1, \dots, U_{10}]$. Stanley's example results from idealization of $R/\mathfrak{m}_R{}^4$ and its dual. We let $I \subset S$, $I = \mathrm{Ann}\, F$, $F = \sum U_i \Xi_i \in \mathfrak{F}_4$ where $\Xi_1, \dots \Xi_{10}$ is a basis for $\mathfrak{D}_3 \subset \mathfrak{D}$. Then $A = S/I$ has dual module $S \circ F$ satisfying

$$\begin{aligned} S_1 \circ F &= \langle R_1 \circ F, \Xi_1, \dots, \Xi_{10} \rangle, \\ S_2 \circ F &= \langle \mathfrak{D}_2, R_2 \circ F \rangle, \\ S_3 \circ F &= \langle X, Y, Z, U_1, \dots, U_{10} \rangle. \end{aligned} \tag{3-1}$$

Consequently, $H(A) = (1, 13, 12, 13, 1)$ is of length 40. Taking a general element $\ell \in S_1$, i.e., up to action of $(\mathsf{k}^*)^{\times 13}$, take $\ell = x + y + z + u_1 + \cdots + u_{10}$, it is straightforward to calculate

$$H\big(S/\operatorname{Ann}(\ell \circ F)\big) = (1, 9, 9, 1), \qquad \big(S/\operatorname{Ann}(\ell^2 \circ F)\big) = (1, 6, 1),$$
$$H\big(S/\operatorname{Ann}(\ell^3 \circ F)\big) = (1, 1), \qquad \big(S/\operatorname{Ann}(\ell^4 \circ F)\big) = (1).$$

By Lemma 2.27, the conjugate $(P_\ell)^\vee$ is the first differences Δ of the lengths of the modules $S/\operatorname{Ann}(\ell^k \circ F)$ for $0 \le k \le 4$, so here

$$(P_\ell)^\vee = \Delta(40, 20, 8, 2, 1) = (20, 12, 6, 1, 1). \tag{3-2}$$

Thus, $P_\ell = (5, 3^5, 2^6, 1^8)$ with 20 parts in contrast to $P_c(H) = (5, 3^{11}, 1^2)$. This is related to the following:

$$m_{\ell^2} : A_1 \to A_3 \text{ has rank 6 (from the five parts 3, and one part 5), but}$$
$$m_\ell : A_1 \to A_2 \text{ has rank 9, and kernel rank 4.}$$

By symmetry $m_\ell : A_2 \to A_3$ also has rank 9 and cokernel rank 4.

Note that the contraction $R_1 \circ \mathfrak{D}_3 = \mathfrak{D}_2$ takes a 10-dimensional space to a 6-dimensional space; thus any multiplication map $m_\ell : \mathfrak{D}_3 \to \mathfrak{D}_2$ has kernel rank at least 4. The Jordan degree type of this ℓ is

$$\mathcal{P}_\ell = \big(P_{\ell,0} = 5, P_{\ell,1} = (3^5, 2^3, 1^4), P_{\ell,2} = (2^3), P_{\ell,3} = (1^4)\big). \tag{3-3}$$

According to Theorem 2.20 the maximum Jordan type of a multiplication map m_ℓ for $\ell \in \mathfrak{m}_A$ (nonhomogeneous) consistent with the Hilbert function $H = H(A)$ would be $P_c(H) = (5, 3^{11}, 1^2)$ with 14 parts for a linear form. For an element $\ell \in \mathfrak{m}$ the upper bound would be $P(H) = (5, 3^{11}, 2)$ with 13 parts expected — if a quadratic term in ℓ takes the kernel of the linear part m_{ℓ_1} on A_1 to an element of A_3 not in $\ell_1 \cdot A_2$. Here the actual generic linear Jordan type $P_\ell = (5, 3^5, 2^6, 1^8)$ with 20 parts for $\ell \in A_1$ (see above) or even for $\ell \in \mathfrak{m}_A$ (verified for several random $\ell \in \mathfrak{m}_A$, by calculation in MACAULAY2) is very far from these bounds; therefore, A does not have the Lefschetz property relative to H of Definition 2.17.

Gondim, applying work of T. Maeno and Watanabe [79] relating higher Hessians and Lefschetz properties, exhibited Gorenstein algebras A with bihomogeneous dual generators of the form $F = \sum \mu_i \nu_i$, in $\mathfrak{F} = \mathsf{k}_{DP}[X, U]$, such that A does not satisfy weak Lefschetz, or, sometimes, has generic Jordan type strictly between WL and SL [48]. Here are two examples from Gondim.[8]

Example 3.3 (Gondim). Consider the cubic $f \in \mathfrak{F}$

$$f = X_1U_1U_2 + X_2U_2U_3 + X_3U_3U_4 + X_4U_4U_1. \tag{3-4}$$

The associated algebra $A = R/I$, of Hilbert function $H(A) = (1, 8, 8, 1)$ with $I = \operatorname{Ann} f$ does not have the WLP: the map $\ell : A_1 \to A_2$ is not injective for any $\ell \in A_1$. The algebra A is presented by 28 quadrics:

$$I = \big(\mathfrak{m}_x^2, u_1^2, u_2^2, u_3^2, u_4^2, u_1u_3, u_2u_4, x_1u_3, x_1u_4, x_2u_4, x_2u_1, x_3u_1, x_3u_2, x_4u_2, x_4u_3,$$
$$x_1u_1 - x_2u_3, x_2u_2 - x_3u_4, x_3u_3 - x_4u_1, x_4u_4 - x_1u_2\big),$$

and has Jordan type $(4, 2^6, 1, 1)$ so strictly less than $H(A)^\vee = (4, 2^7)$. A random element (nonhomogeneous) gives the same Jordan type (calculation in MACAULAY2).

Example 3.4 (Gondim). Let $F = XU^{[3]} + YUV^{[2]} + ZU^{[2]}V \in \mathfrak{F} = \mathsf{k}_{DP}[U, V, X, Y, Z]$. Consider $R = \mathsf{k}[u, v, x, y, z]$ and the algebra $A = R/I$, $I = \operatorname{Ann} F$, where

$$I = \langle x^2, y^2, z^2, u^4, v^3, xy, xz, yz, xv, zv^2, yu^2, u^2v^2, u^3v, xu - zv, zu - yv\rangle.$$

[8] The first example is from Gondim's talk at the workshop "Lefschetz Properties and Artinian algebras" at BIRS on March 15, 2016, at "https://www.birs.ca/workshops/2016/16w5114/files/Gondim.pdf". The second is a private communication from Gondim, following a discussion there with the first author.

Since A is a bigraded idealization it is easy to see that $H(A) = (1, 5, 6, 5, 1)$. Since the partial derivatives $x \circ F = U^{[3]}$, $y \circ F = UV^{[2]}$ and $z \circ F = U^{[2]}V$ are algebraically dependent, by the Gordan–Noether criterion [50; 48; 79] the Hessian $\mathrm{Hess}_F = 0$. By the Maeno–Watanabe criterion [79; 58, Theorem 3.76] this implies that A fails the strong Lefschetz property. On the other hand it is easy to see that $u + v$ is a WL element for A.

Since A is not strong Lefschetz, the Jordan decomposition P_ℓ for $\ell = u+v+x+y+z$ (a generic-enough linear form) is by Theorem 2.5 less in the dominance order than the conjugate $H(A)^\vee = (5, 3^4, 1)$; since A has WLP, P_ℓ has the same number of parts as $H(A)^\vee$, namely the Sperner number $H(A)_{\max} = 6$. Since $\ell^4 \neq 0$ in A, the string $S_1 = (1, \ell, \ell^2, \ell^3, \ell^4)$ so P_ℓ has a part 5; since $P_\ell < (5, 3^4, 1)$ and has 6 parts the only possibility is $P_\ell = (5, 3, 3, 3, 2, 2)$, with Jordan degree type $\mathcal{P}_\ell = (5_0, 3_1^3, 2_1, 2_2)$. By Proposition 2.14 since A is standard graded, has unimodal Hilbert function, and is not strong Lefschetz, A cannot have an element — even nonhomogeneous — that has strong Lefschetz Jordan type.

Gondim gives many further examples, using special bihomogeneous forms. Gondim and Zappalà have determined further graded Gorenstein algebras that are non-unimodal, sometimes completely non-unimodal (with decreasing Hilbert function from h_1 to $h_{j/2}$, then increasing to degree $j - 1$): they accomplish this by using properties of complexes to choose a suitable bihomogeneous dual generator $f \in \mathfrak{F}$ [49]. In a sequel work Cerminara, Gondim, Ilardi, Maddaloni study "higher order" (d_1, d_2) AG Nagata idealizations determined by Macaulay dual forms $\sum \mu_i g_j$ where μ_i runs through the d_1 powers of one set $X_1, \ldots, X_r$ of variables, while $g_i \in \mathsf{k}[U_1, \ldots, U_s]_{d_2}$ have degree d_2: they show in specific cases that the idealization is not SL, but when $d_1 \geq d_2$ they show it is WL [32, Proposition 2.7]; their main results are related to geometric properties and "simplicial" Nagata polynomials [32, Theorem 3.5]. A. Capasso, P. De Poi, and Ilardi generalize this work in [29]. J. McCullough and A. Seceleanu use idealization and a subtle choice of base level algebra to construct a new infinite sequence of quadratic Gorenstein rings with, in general, non-unimodal Hilbert functions, that are non-Koszul, with non-subadditive minimal resolutions [82, Theorem 4.3]: they have not been studied for their Jordan types. A. Dimca, Gondim and Ilardi study the connections between higher order and mixed Hessians and the Lefschetz properties of Milnor algebras in [40].

3B. Tensor products and Jordan type. It is well known that a graded Artinian algebra A with symmetric Hilbert function is SL with Lefschetz element $\ell \in A_1$ if and only if A carries an $\mathfrak{sl}_2$-representation where the raising operator E in the standard basis of $\mathfrak{sl}_2$ is multiplication by ℓ, and element of A_1, and the weight space decomposition agrees with the grading of A, see [58, Theorem 3.32]. Equivalently, such a pair (A, ℓ) is strong Lefschetz if $P_\ell = H^\vee$ (Proposition 2.10). The well-known Clebsch–Gordan formula decomposes a tensor product of $\mathfrak{sl}_2$ representations into irreducibles; it is equivalent to the strong Lefschetz property of $\mathsf{k}[x, y]/(x^m, y^n)$.

Lemma 3.5 (Clebsch–Gordan). *Assume $m \geq n$ are positive integers, and* char $\mathsf{k} = 0$ *or* char $\mathsf{k} \geq m+n-1$. *Then the Jordan type of $\ell = x + y$ in $\mathsf{k}[x, y]/(x^m, y^n)$ satisfies*

$$P_\ell = (m+n-1, m+n-3, m+n-5, \ldots m-n+1). \tag{3-5}$$

To prove this it is sufficient to know the strong Lefschetz property of standard graded quotients of $\mathsf{k}[x, y]$ in characteristic zero or characteristic larger than the socle degree (Lemma 2.15). See also [58, Theorem 3.29 and Lemma 3.70]. As a consequence we have:

Proposition 3.6 [58, Proposition 3.66].[9] *Let $z \in A$, $w \in B$ be two nonunit elements of Artinian local algebras A, B. Set $P_z = (d_1, d_2, \dots, d_t)$ and $P_w = (f_1, f_2, \dots f_s)$. Denote by $\ell = z \otimes 1 + 1 \otimes w \in A \otimes_{\mathsf{k}} B$. Assume* $\operatorname{char} \mathsf{k} = 0$, *or* $\operatorname{char} \mathsf{k} \geq \max\{d_i + f_j - 1\}$. *Then*

$$P_\ell = \bigoplus_{i,j} \bigoplus_{k=1}^{\min\{d_i, f_j\}} (d_i + f_j + 1 - 2k). \tag{3-6}$$

Also, $\dim_{\mathsf{k}} \ker(\times \ell) = \sum_{i,j} \min\{d_i, f_j\}$.

Recall that we denote by j_A the socle degree of A. The following corollary of Proposition 3.6 is not hard to show, and we leave the proof to the reader. With the additional assumption that $H(A)$ and $H(B)$ are unimodal (so A, B are SL in the narrow sense of Remark 2.9) this corollary is shown in [58, Theorem 3.34].

Corollary 3.7. *Assume that A, B are graded Artinian algebras with symmetric Hilbert functions and that* $\operatorname{char} \mathsf{k} = 0$ *or* $\operatorname{char} \mathsf{k} > j_A + j_B$. *Then the element $\ell = z \otimes 1 + 1 \otimes w \in A \otimes_{\mathsf{k}} B$ is SL if and only if z and w are both SL, respectively, in A and in B.*

For a different proof of Corollary 3.7, resting on the connection between the strong Lefschetz property of C and the weak Lefschetz properties of $C \otimes_{\mathsf{k}} \mathsf{k}[t]/(t^i)$ see Harima and Watanabe's [54, Theorem 3.10]. It is open whether $A \otimes_{\mathsf{k}} B$ is SL implies both A and B are SL, without a prior assumption on the Hilbert functions of A, B.

We may use Proposition 3.6 to determine the Jordan types of other, special elements $\ell \in \mathfrak{m}_A$ for certain Artinian algebras A.

Example 3.8 [2, Corollary 0.4]. Consider $\ell = x^2 + y^2 \in A = \mathsf{k}[x, y]/(x^3, y^3)$ and suppose $\operatorname{char} \mathsf{k} \neq 2, 3$. Then $P_\ell = (3, 2, 2, 1, 1)$. Indeed here $P_{x^2} = (2, 1)$ on $\mathsf{k}[x]/(x^3)$; likewise $P_{y^2} = (2, 1)$ on $\mathsf{k}[y]/(y^3)$, and hence by Proposition 3.6

$$P_\ell = (2+2+1-2, 2+2+1-4) \oplus (2+1+1-2) \oplus (1+2+1-2) \oplus (1+1+1-2) = (3, 2, 2, 1, 1).$$

We found that for $\operatorname{char} \mathsf{k} \neq 2, 3$ we could achieve all Jordan types of $\ell \in \mathfrak{m}_A$ from elements of the form $\ell = x^a + y^b$ or $\ell = x^a$ using Proposition 3.6, except for $P_{x^2 y} = (2, 2, 1^5)$ [2, Corollary 0.4].

The following is a special case of a family of examples due to J. Migliore, U. Nagel, and H. Schenck [92]: they show that without the assumption that the component Hilbert functions are symmetric, the tensor product in general will not preserve any Lefschetz properties.

Example 3.9. With the standard gradings, set

$$A = B = \frac{\mathsf{k}[x, y]}{(x^2, y^2, xy)} \quad \text{and} \quad C = A \otimes_{\mathsf{k}} B \cong \frac{\mathsf{k}[x, y, z, w]}{(x^2, y^2, z^2, w^2, xy, zw)}.$$

[9] Although [58] restricts to $\operatorname{char} \mathsf{k} = 0$, there is no change in showing it for $\operatorname{char} \mathsf{k} = p$ large enough.

A k-basis for the tensor product C is $\{1, x, y, z, w, xz, yz, xw, yw\}$, of Hilbert function $H(C)=(1,4,4)$. For a general linear form $\ell = ax+by+cz+dw \in C_1$ the matrix for $\times\ell : C_1 \to C_2$ with respect to that basis is given by

$$M_\ell = \begin{pmatrix} c & 0 & a & 0 \\ d & 0 & 0 & a \\ 0 & c & b & 0 \\ 0 & d & 0 & b \end{pmatrix}$$

which has $\det(M)=0$ for every $a, b, c, d \in \mathsf{k}$. Therefore C is not even WL, let alone SL. Also, since C is standard graded, Proposition 2.14 implies that C cannot have SLJT. On the other hand, A and B certainly have all of these properties.

Jordan degree type and Clebsch–Gordan. Following the Definition 2.28 (2-26) of Jordan degree type, we denote by m_s a string of length m beginning in degree s, and by n_t a string of length n beginning in degree t. We next specify the Jordan degree type of their tensor product $\mathsf{m}_s \otimes_{\mathsf{k}} \mathsf{n}_t$: our result refines the Clebsch–Gordan Lemma 3.5.

Proposition 3.10. *Under the same assumptions on characteristic as in Corollary 3.7, we have for Jordan degree type,*

$$\begin{aligned} \mathsf{m}_s \otimes_{\mathsf{k}} \mathsf{n}_t &= \big((\mathsf{n}+\mathsf{m}-1)_{s+t}, (\mathsf{n}+\mathsf{m}-3)_{s+t+1}, \dots, (\mathsf{n}-\mathsf{m}+1)_{s+t+m-1}\big) \\ &= \bigoplus_{k=1}^{m} (\mathsf{n}+\mathsf{m}+1-2k)_{s+t+k-1} \end{aligned} \tag{3-7}$$

Proof. This follows from the Definition 2.28 of contiguous Jordan degree type of a Hilbert function, and noting that when $s=t=0$ the Jordan degree type of the algebra $A=\mathsf{k}[x,y]/(x^m,y^n)$ is just the contiguous Jordan degree type $P_c(H)$ of the Hilbert function $H=H(A)$, which is the right side of (3-7). For $s,t\neq 0$ one just shifts the degrees by $s+t$. □

We leave to the reader the statement of the obvious analogue of Proposition 3.6 where we replace the d_i and f_j by $(\mathsf{d_i})_{\alpha_i}$ and $(\mathsf{f_j})_{\beta_j}$, respectively. Using this, we may define the tensor product of contiguous partitions (which have the initial degree information for strings) $\mathcal{P}_{c,\deg}\big(H(A)\big)\otimes\mathcal{P}_{c,\deg}\big(H(B)\big)$. The following corollary is shown using the layering of $\mathcal{P}_{c,\deg}(H)$ when H is unimodal: then the Jordan degree type $\mathcal{P}_{c,\deg}(H)$ is determined by the conjugate partition $H^\vee$.

Corollary 3.11. *Assume that A, B are standard graded with unimodal Hilbert functions $H(A)$, $H(B)$.*

(i) *Then we have for the contiguous partitions*

$$\mathcal{P}_{c,\deg}(H(A\otimes_{\mathsf{k}} B)) = \mathcal{P}_{c,\deg}(H(A))\otimes\mathcal{P}_{c,\deg}(H(B)). \tag{3-8}$$

(ii) *Also, if $\alpha\in A_1$ has Jordan degree-type $\mathcal{P}_{c,\deg}(H(A))$ and $\beta\in B_1$ has that of the contiguous partition $\mathcal{P}_{c,\deg}(H(B))$, then $\ell=\alpha\otimes 1_B+1_A\otimes\beta$ has the Jordan degree type $\mathcal{P}_{c,\deg}(H(A\otimes_{\mathsf{k}} B))$.*

The following example shows that the unimodal condition is necessary for (3-8).

Example 3.12. Letting $H(A)=(1,13,12,13,1)$ and $H(B)=(1,1)$, we have $P_c\big(H(A)\big)=(5_0,3_1^{11},1_1,1_3)$, and $P_c\big(H(B)\big)=(2_0)$ with product $P_c\big(H(A)\big)\otimes P_c\big(H(B)\big)=(6_0,4_1^{12},2_2^{11},2_1,2_3)$. But $H(A)\otimes H(B)=(1,14,25,25,14,1)$, and $P_c\big(H(A\otimes_{\mathsf{k}} B)\big)=(6_0,4_1^{13},2_2^{11})$.

The next example shows that even with symmetric Hilbert functions, we should not expect tensor products to preserve the relative Lefschetz property of Definition 2.17 (involving Jordan degree-type and the contiguous Hilbert function) when the grading of A or B is not standard.

Example 3.13. Let $\alpha \in \mathsf{k}$ and $A = \mathsf{k}[a, b]/(a^3 - \alpha ab, b^3)$ with weight function $\mathsf{w}(a, b) = (1, 2)$, and $B = \mathsf{k}[t]/(t^2)$ with the standard grading. It is easy to see that both A and B have the relative Lefschetz property for their Hilbert functions, which are, respectively, $H(A) = (1, 1, 2, 1, 2, 1, 1)$ and $H(B) = (1, 1)$. On the other hand, the Hilbert function of their tensor product $C = A \otimes_{\mathsf{k}} B$ is $H(C) = (1, 2, 3, 3, 3, 3, 2, 1)$ which is unimodal. If char $\mathsf{k} = 0$ or char $\mathsf{k} > 7$ then C cannot have the relative Lefschetz property: if it did, then by Proposition 2.14 C would also be SL which would imply by Corollary 3.7 (for which we need the assumptions on characteristic of k) that both A and B are SL, which is impossible since $H(A)$ is not unimodal.

Example 3.14 (SLJT in a tensor product). Let $S = \mathsf{k}[e]$ with weight $\mathsf{w}(e) = 2$, $S^\vee = \mathfrak{E} = \mathsf{k}[E]$ and $A = S/I_F$, $F = E^2$, so $A = S/(e^3)$ and $H(A) = (1, 0, 1, 0, 1)$. Now let $R = \mathsf{k}[x, y]$ with standard grading and take $B = \mathsf{k}[x, y]/(x^2, y^2)$; here $(x^2, y^2) = \mathrm{Ann}(XY)$, $XY \in R^\vee = \mathsf{k}_{DP}[X, Y]$, and $H(B) = (1, 2, 1)$. Now consider $A \otimes_{\mathsf{k}} B = \mathsf{k}[x, y, e]/(x^2, y^2, e^3)$ of Hilbert function $H(A) \otimes H(B) = (1, 2^5, 1)$. Note that $\ell = x + y + e$ has Jordan type $P_\ell = (7, 5)$; ℓ is not homogeneous but this shows that $A \otimes_{\mathsf{k}} B$ has SLJT. Since $A \otimes_{\mathsf{k}} B$ is not standard-graded, even though $H(A \otimes_{\mathsf{k}} B)$ is unimodal and symmetric, having an SLJT element does not imply that $A \otimes_{\mathsf{k}} B$ is SL. It is not SL as the only linear elements $x, y, x + y$ (up to scalar) of $A \otimes_{\mathsf{k}} B$ are not SL: for example $P_{x+y} = (3^3, 1^3)$.

If we regrade so that e has degree one, then we have a standard graded CI of generator degrees $(2, 2, 3)$, Hilbert function $H_{(2,2,3)} = (1, 3, 4, 3, 1)$ and Jordan type $P_\ell = (5, 3, 3, 1)$, and it is SL.

This example suggests:

Conjecture 3.15. If A has SLJT, and B is standard graded SL, and if $H(A \otimes_{\mathsf{k}} B) = H(A) \otimes H(B)$ (the graded product) then $A \otimes_{\mathsf{k}} B$ has SLJT.

Clebsch–Gordan in the modular case.

Remark 3.16. There is substantial work determining Clebsch–Gordan formulas in the modular case char $p \leq j$. S.P. Glasby, C.E. Praeger, and B. Xia in [46] summarize previous algorithmic results of Kei-ichiro Iima and Ryo Iwamatsu [69] using Schur functions, and of J.-C. Renaud [110]; they obtain formulas that in principle allow one to compute the generic Jordan type of $R(m, n)$ in arbitrary characteristic p — they term this the Jordan type $\lambda(m, n, p)$ of $R(m, n, p)$, which always has m parts — so $R(m, n, p)$ is always weak Lefschetz (a result they ascribe to T. Ralley [109]). In [47, Theorem 2] they show that $R(m, n, p)$ has SLP (in their language, "is standard") if $n \not\equiv \pm 1, \pm 2, \cdots \pm m \mod p$. They define a deviation vector $\epsilon(m, n, p) = \lambda(m, n, p) - (n, n, \ldots, n)$, and show

Lemma 3.17 [47, Theorems 4,6,7]. *Let $m \leq \min\{p^k, n, n'\}$.*

(*periodicity*) *If $n \equiv n' \mod p^k$ then $\epsilon(r, n, p) = \epsilon(r, n', p)$.*

(*duality*) *If $n' = -n \mod p^k$ then $\epsilon(m, n', p) = (-\epsilon_r, \ldots, -\epsilon_1)$, the "negative reverse" of $\epsilon(m, n, p)$.*

(*bound*) *There are at most 2^{m-1} different deviation vectors $\epsilon(m, n, p)$ for all $n \geq m$ and characteristics p.*

(*computation*) *For fixed m, a finite computation suffices to compute the values of $\epsilon(m, n, p)$ for all n with $n \geq m$, and all primes p.*

The authors warn that, in contrast, determining $\lambda(m, n, p)$ is not a finite computation as it involves considering n mod p for infinitely many n.

Such tables of $\lambda(m, n, p)$ for $m = 3$ and $m = 4$ have been calculated by Jung-Pil Park and Yong-Su Shin in [103]. Recent articles by L. Nicklasson [97] and S. Lundqvist and Nicklasson [76] further clarify the strong Lefschetz property for monomial complete intersections

We can similarly define for sequences $M = (m_1, \dots, m_r)$, $m_1 \le m_2 \le \dots \le m_r$ deviation vectors $\epsilon(M, p) = \epsilon(m_1, \dots, m_r; p)$ for the Jordan types of CI algebras $R(M) = \mathsf{k}[x_1, \dots, x_r]/(x_1^{m_1}, \dots, x_r^{m_r})$ when char $\mathsf{k} = p$.

Question 3.18. What does Lemma 3.17 tell us about determining modular Jordan types $\lambda(m_1, m_2, m_3, p)$ (three variables) or in more variables? This is a problem that has been studied and appears quite complex. For example in three variables work on it has involved tilings by lozenges [37; 38], see also [25; 35; 36; 75]. Further studies of the weak Lefschetz properties of monomial ideals and the relation to algebraic-geometric Togliatti systems have been made by E. Mezzetti, G. Ottaviani, R. M. Miró-Roig, and others [86; 87; 88]

G. Benkart and J. Osborn studied representations of Lie algebras in characteristic p [16]. Subsequently, Premet [107, Lemmas 3.4, 3.5], and J. Carlson, E. Friedlander and J. Pevtsova in [30, §10, Appendix "Decomposition of tensor products of $k[t]/(t^p)$ modules"] gave formulas for such products that apply to $R(m, n, p)$ when $m, n \le p$. Carlson et al. regard $T = \mathsf{k}[t]/(t^p)$ as a self-dual Hopf algebra, with coproduct $t \to 1 \otimes t + t \otimes 1$ and determine the tensor product of irreducible modules over T.

There has been substantial work on rank varieties and connections between commutative algebra and the representation theory of p-groups, beyond our scope; see, for example [9; 18].

3C. Free extensions. We have remarked that the Jordan type of the multiplication map does not depend on the grading of the algebra $\mathcal{A}$ — a remark also of Kleiman and Kleppe in [73]. Nevertheless, the Hilbert function of $\mathcal{A}$ will depend on its grading, and, in particular, whether it is regarded as a graded algebra A or an $\mathfrak{m}_{\mathcal{A}}$-adic filtered local algebra $\mathcal{A}$; thus the strong Lefschetz property is grading-dependent.

We will give some examples where $\mathcal{A}$ is the base of a free extension C with fiber B. We first recall the definition of free extensions, introduced by Harima and Watanabe in [52]. Then we give Examples 3.25 and 3.26 of algebras C where C is A-free, and we compare A with the related local algebra $\mathcal{A}$. In Theorem 3.23 we give a method of producing/verifying free extensions related to complete intersections.

Definition 3.19 [52; 66]. Given graded Artinian algebras A, B, and C, we say that C is a free extension of A with fiber B if there exist algebra maps $\iota : A \to C$ and $\pi : C \to B$

(i) ι makes C into a free A-module,

(ii) π is surjective and $\ker(\pi) = \tau(\mathfrak{m}_A) \cdot C$.

We showed in [66, Theorem 2.1] that a free extension C is a flat deformation of a finite algebra B over a local Artinian algebra $(\mathcal{A}, \mathfrak{m}, \mathsf{k})$. Thus, a free extension C of A with fiber B is an A-algebra such that the associated map $\gamma : \operatorname{Spec}(C) \to \operatorname{Spec}(\mathcal{A})$ is finite and flat, and with a closed embedding $\pi : \operatorname{Spec}(B) \to \operatorname{Spec}(C)$ inducing a cartesian diagram:

$$\begin{tikzcd} \operatorname{Spec}(B) \arrow[r, hook, "\pi"] \arrow[d] & \operatorname{Spec}(C) \arrow[d, "\gamma"] \\ \operatorname{Spec}(\mathsf{k}) \arrow[r, hook] & \operatorname{Spec}(\mathcal{A}). \end{tikzcd}$$

Related to the notion of free extension is that of a *coexact sequence*.[10] Given graded Artinian algebras A, B, and C we say that a sequence of algebra maps $A \xrightarrow{\iota} C \xrightarrow{\pi} B$ is *coexact at C* if $\ker(\pi) = \iota(\mathfrak{m}_A)\cdot C$, and a *coexact sequence* is a sequence of algebra maps

$$\mathsf{k} \longrightarrow A \xrightarrow{\iota} C \xrightarrow{\pi} B \longrightarrow \mathsf{k} \tag{3-9}$$

that is coexact in every slot (here k is the graded Artinian algebra concentrated in degree zero with $\mathfrak{m}_{\mathsf{k}} = 0$). The following result is not difficult, but we record it here for future reference.

Lemma 3.20. *Let A, B, C be graded Artinian algebras with maps $\iota : A \to C$ and $\pi : C \to B$ and suppose that π is surjective. Then the following are equivalent.*

(i) *For every k-linear section $\mathsf{s} : B \to C$ of π, the A-module map $\Phi_{\mathsf{s}} = \iota \otimes \mathsf{s} :_A (A \otimes_{\mathsf{k}} B) \to_A C$ is an isomorphism, i.e., C is an A-module tensor product.*

(ii) *The sequence (3-9) is coexact and $\iota : A \to C$ is a free extension.*

(iii) *$\iota : A \to C$ is a free extension and* $\ker(\pi) = \big(\iota(\mathfrak{m}_A)\big)\cdot C$.

(iv) $\ker(\pi) = \big(\iota(\mathfrak{m}_A)\big)\cdot C$ *and* $\dim_{\mathsf{k}} C = \dim_{\mathsf{k}} A \cdot \dim_{\mathsf{k}} B$.

Proof. (i) $\Rightarrow$ (ii). Assume (i) holds and fix an k-linear π-section $\mathsf{s} : B \to C$ so that $\Phi_{\mathsf{s}} : A \otimes_{\mathsf{k}} B \to C$ is an A-module isomorphism. Let $\pi_0 : A \otimes_{\mathsf{k}} B \to \mathsf{k} \otimes_{\mathsf{k}} B \cong B$ be the natural projection onto the B factor of the tensor product, with $\ker(\pi_0) = \mathfrak{m}_A \otimes_{\mathsf{k}} B$. Then clearly we have $\pi_0 = \pi \circ \Phi_{\mathsf{s}}$, and hence $\ker(\pi) = \Phi_{\mathsf{s}}^{-1}(\mathfrak{m}_A \otimes_{\mathsf{k}} B) = \iota(\mathfrak{m}_A)\cdot C$.

(ii) $\Rightarrow$ (iii) follows from the definitions.

(iii) $\Rightarrow$ (iv). Assume (iii). Then we have $B \cong C/\ker(\pi) = C/\iota(\mathfrak{m}_A)\cdot C$. By Nakayama's Lemma any k-linear basis for $B \cong C/\iota(\mathfrak{m}_A)\cdot C$ lifts to an A-linear basis for C, hence we have $\dim_{\mathsf{k}} C = \dim_{\mathsf{k}} A \cdot \dim_{\mathsf{k}} B$.

(iv) $\Rightarrow$ (i). Assume (iv) holds. Then $B \cong C/\iota(\mathfrak{m}_A)\cdot C$, and hence Nakayama's lemma implies any k-linear basis for B lifts to an A-generating set for C. Put another way, for any k-linear π-section $\mathsf{s} : B \to C$, Nakayama implies that we have an A-linear epimorphism $\Phi_{\mathsf{s}} : A \otimes_{\mathsf{k}} B \to C$. But since the dimensions of these two vector spaces are equal, Φ_{s} must in fact be an isomorphism. □

Note that the tensor product algebra $C = A \otimes_{\mathsf{k}} B$ is a free extension over A with fiber B (it is also a free extension over B with fiber A). In general a free extension C is isomorphic to the tensor product $A \otimes_{\mathsf{k}} B$, but only as A-modules. A free extension is a deformation of the tensor product algebra [66, Theorem 2.1]. Thus we have:

Proposition 3.21 [66, Theorem 2.4]. *Given graded Artinian algebras A, B, and C such that C is a free extension over A with fiber B, the generic linear Jordan type of C is at least as large as the generic linear Jordan type of the tensor product algebra $A \otimes_{\mathsf{k}} B$ with respect to the dominance order, i.e.,*

$$P_C \geq P_{A \otimes_{\mathsf{k}} B}.$$

This can be used to prove the following result of Harima and Watanabe.

[10] This notion was introduced by J.C. Moore, and occurs in the topology literature.

Proposition 3.22 [54, Theorem 6.1]. *Let C be a free extension of A with fiber B. Assume that* $\operatorname{char} \mathsf{k} = 0$ *or* $\operatorname{char} \mathsf{k} \geq j_A + j_B$, *that the Hilbert functions of both A and B are symmetric, and that both A and B are strong Lefschetz. Then C is also strong Lefschetz.*

We prove the following general result which gives a useful construction and criterion for obtaining A-free extensions. If $I = (f_1, \dots, f_s)$ is an ideal in a ring S and $\tau : S \to R$ is a ring homomorphism, we denote by $\big(\tau(I)\big)$ the ideal in R generated by $\{\tau(f_1), \dots, \tau(f_s)\}$. Note that in general, the image $\tau(I)$ is not itself an ideal of R (Remark 3.24).

Theorem 3.23. *Let $S = \mathsf{k}[e_1, \dots, e_d]$, $R = \mathsf{k}[x_1, \dots, x_r]$ be (not necessarily standard) graded polynomial rings, and let $\tau : S \to R$ be a map of k-algebras which makes R into a finite S-module. For any ideal $I \subseteq S$ of finite colength, set $A = S/I$ and $C = R/\big(\tau(I)\big)$, and let $\iota = \bar{\tau} : A \to C$ be the induced map between them. Set $B = R/(\tau(\mathfrak{m}_S))$ where $\mathfrak{m}_S = (e_1, \dots, e_d) \subset S$ is the homogeneous maximal ideal of S, and let $\pi : C \to B$ be the natural projection map.*

(i) *Then A, B, C are all graded Artinian algebras, and we have*

$$\ker(\pi) = \big(\iota(\mathfrak{m}_A)\big). \tag{3-10}$$

In particular, we have a coexact sequence of graded Artinian k-algebras

$$\mathsf{k} \longrightarrow A \xrightarrow{\iota} C \xrightarrow{\pi} B \longrightarrow \mathsf{k}.$$

(ii) *Furthermore, if $d = r$, and if A is a complete intersection, then so are B and C; also C is a free extension with base A and fiber B.*

Proof. (i) That A, B, and C are Artinian follows from our finiteness assumptions on I and the algebra extension $\tau : S \to R$. To see that (3-10) holds, note that we have the string of equalities

$$\ker(\pi) = \big(\tau(\mathfrak{m}_S)\big)/\big(\tau(I)\big) = \big(\iota(\mathfrak{m}_S/I)\big) = \big(\iota(\mathfrak{m}_A)\big).$$

(ii) If $d = r$ and A is a complete intersection, then $A = S/I = \mathsf{k}[e_1, \dots, e_d]/(f_1, \dots, f_d)$ for some S-regular sequence $f_1, \dots, f_d$. Then since $B = R/\big(\tau(\mathfrak{m}_S)\big) = \mathsf{k}[x_1, \dots, x_r]/\big(\tau(e_1), \dots, \tau(e_d)\big)$ and $C = R/\big(\tau(I)\big) = \mathsf{k}[x_1, \dots, x_r]/\big(\tau(f_1), \dots, \tau(f_d)\big)$ are both Artinian (so Krull dimension zero) and since $d = r$, the sequences $\tau(e_1), \dots, \tau(e_d)$ and $\tau(f_1), \dots, \tau(f_d)$ must be R-regular, hence B and C are complete intersections too. Since $\tau(e_1), \dots, \tau(e_d)$ is an R-regular sequence, they are algebraically independent and R is a free module over S. In particular, we have $C = R/\big(\tau(I)\big) \cong R \otimes_S S/I$ which shows that C is free as an $A = S/I$-module, hence C is a free extension over A with fiber B. □

Note. The statement (ii) seems to be related to a result of L. Avramov on flat extensions [8]. See also Proposition 23.8.4 in https://stacks.math.columbia.edu/tag/09PY.

Remark 3.24. It is tempting to think that the hypothesis that R and S have the same Krull dimension in Theorem 3.23(ii) could be replaced by the requirement that B is a complete intersection. But we have the following counterexample: Define $\tau : \mathsf{k}[e_1, e_2, e_3] \to \mathsf{k}[x, y]$, $\tau(e_1) = x^2$, $\tau(e_2) = y^2$, $\tau(e_3) = x^2 + y^2$ and $A = \mathsf{k}[e_1, e_2, e_3]/(e_1^2, e_2^2, e_3^2 - e_1e_2)$. Then $B = \mathsf{k}[x, y]/(x^2,\ y^2,\ x^2 + y^2) = \mathsf{k}[x, y]/(x^2, y^2)$ is a complete intersection, but $C = \mathsf{k}[x, y]/\big(x^4,\ y^4,\ (x^2 + y^2)^2 - x^2y^2\big) = \mathsf{k}[x, y]/(x^4, y^4, x^2y^2)$ is not. Moreover C has A torsion, e.g., $0 = \tau(e_3 - e_2 - e_1)$, hence it cannot be an A-free extension. Note that, as is usual,

$\tau(I)$ is not an ideal of R: here, for example $x(x^2)$ is not in $\tau(I)$; hence our notation $\big(\tau(I)\big)$ throughout for the ideal in R generated by $\tau(I)$.

A level Artinian algebra $A = R/\operatorname{Ann} F$ is one having its socle $0 : \mathfrak{m}$ in a single degree. The level algebra is a *connected sum* over a field k if its dual generator is a sum $F = F_1 + F_2$ where F_1, F_2 are in two distinct set of variables (see [28] and references cited there). For some discussion of Jordan type in more general connected sums over a Gorenstein algebra T in place of k see [67] and a related paper of E. Babson and E. Nevo [10]. In the next examples A is a connected sum over k.

Example 3.25 (C is an A-free CI). Take $S = \mathsf{k}[e_1, e]$ with weights $\mathsf{w}(e_1, e) = (1, 4)$, take $R = \mathsf{k}[x, y]$ with $\mathsf{w}(x, y) = (1, 1)$, define $\tau : S \to R$ by $\tau(e_1) = x + y$, $\tau(e) = x^2y^2$. We consider $A = S/I_F$ where $F = E^{[3]} + E_1^{[12]}$, a connected sum, we let $B = \mathsf{k}[x, y]/(x + y, x^2y^2)$, and take $C = R/\big(\tau(I_F)\big)$. Here the ideal I_F satisfies

$$I_F = (e_1e, e^3 - e_1^{12}) \quad \text{and} \quad S \circ F = A^\vee = \langle F, \{E_1^{[i]}, 0 \le i \le 11\}, F, E^{[2]}\rangle, \tag{3-11}$$

and the ideal $\big(\tau(I_F)\big) = \big((x + y)x^2y^2,\ (x^2y^2)^3 - (x + y)^{12}\big) \subset R$. We have[11]

$$H(A) = (1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1), \qquad H(B) = (1, 1, 1, 1),$$
$$H(C) = H(A) \otimes H(B) = (1, 2, 3, 4, 5, 5, 5, 5, 5, 5, 5, 5, 4, 3, 2, 1).$$

Here $H(C)$ is the Hilbert function of the complete intersection $C = \mathsf{k}[x, y]/\big(\tau(I_F)\big)$ of generator degrees $(5, 12)$: being of codimension two C is SL by Lemma 2.15; also $B \cong \mathsf{k}[x]/(x^4)$ is SL. That $H(C) = H(A) \otimes H(B)$ implies by Lemma 3.20 (iv) that C is a free extension of A with fiber B. We note that A is not strong-Lefschetz, as there is no linear SLJT element: the only linear element is e_1 (up to constant multiple), and the partition $P_{e_1,A} = (13, 1, 1)$, not $H(A)^\vee = (13, 2)$. But $\ell = e_1 + e$, a nonhomogeneous element of A, satisfies $P_{\ell,A} = (13, 2)$ so ℓ has SLJT. The corresponding element ℓ in the local ring $\mathcal{A} = \mathsf{k}\{e_1, e\}/(e_1e, e^3 - e^{12})$ (a quotient of standard-graded $\mathsf{k}\{e_1, e\}$) also has SLJT (the Jordan type does not change with grading); thus, the local ring $\mathcal{A}$ is strong Lefschetz, of Hilbert function $H(\mathcal{A}) = (1, 2, 2, 1^{10})$. This example generalizes to F being any quasihomogeneous polynomial in E_1, E.

In the following example, the Macaulay dual generators of A, B and C are simply related.

Example 3.26 (C is A-free). Let $R = \mathsf{k}[x, y, z]$ with the standard grading, $S = \mathsf{k}[e_1, e_2, e_3]$ with grading $\mathsf{w}(e_1, e_2, e_3) = (2, 2, 2)$, and $\tau : S \to R$ be the morphism defined by $\tau(e_1) = x^2$, $\tau(e_2) = y^2$, $\tau(e_3) = z^2$. Let $F = E_1^{[3]} + E_2^{[3]} + E_3^{[3]} \in \mathfrak{F} = \operatorname{Hom}_{\mathsf{k}}(S, \mathsf{k})$; the algebra $A = S/I_F$ is a connected sum

$$A = S/I_F, \quad I_F = \operatorname{Ann} F = (e_1e_2,\ e_1e_3,\ e_2e_3,\ e_1^3 - e_2^3,\ e_1^3 - e_3^3),$$

an Artinian Gorenstein non-CI ring of Hilbert function $H(A) = (1, 0, 3, 0, 3, 0, 1)$; it is not SL as $A_1 = 0$, but it is straightforward to see that for $e = e_1 + e_2 + e_3$ we have $P_{e,A} = (4, 2, 2)$ so e has SLJT. Considering $\mathcal{A}$ as the local ring A regraded to standard grading, we have that $\mathcal{A}$ is SL of Hilbert function $(1, 3, 3, 1)$.

[11] Given sequences of nonnegative integers $H(A) = (a_0, \dots, a_r)$ and $H(B) = (b_0, \dots, b_s)$ we use the shorthand notation $H(A) \otimes H(B) = (c_0, \dots, c_{r+s})$ for the convolution sequence $c_j = \sum_{i=0}^{j} a_i b_{j-i}$.

Recall $\tau : S \to R$: $\tau(e_1) = x^2$, $\tau(e_2) = y^2$, $\tau(e_3) = z^2$, and set $B = R/(x^2, y^2, z^2)$, a complete intersection of Hilbert function $H(B) = (1, 3, 3, 1)$ and Macaulay dual generator $XYZ \in \mathfrak{D} = \mathrm{Hom}_{\mathsf{k}}(R, \mathsf{k})$. Then $P_{x+y+z,B} = (4, 2, 2)$ so $x + y + z$ is SL.

$$C = R/\big(\tau(I_F)\big), \qquad \text{where } \big(\tau(I_F)\big) = (x^2y^2,\ x^2z^2,\ y^2z^2,\ x^6 - y^6,\ x^6 - z^6),$$

of Hilbert function $H(C) = (1, 3, 6, 10, 12, 12, 10, 6, 3, 1) = H(A) \otimes H(B)$, so C is A-free and is a free extension of A with fiber B.[12] We have that C is Gorenstein, as $I_C^\perp = R \circ G_C$ with dual generator $G_C = \big(XYZ \cdot (X^{[6]} + Y^{[6]} + Z^{[6]})\big)$; note that G_C is the product of the dual generator XYZ for B and $\tau'(F)$ where F is the dual generator for A, and the homomorphism $\tau' : \mathfrak{F} \to \mathfrak{D}$ corresponds to $\tau : S \to R$.

Although C is strong Lefschetz (calculated using MACAULAY2), we note that the tensor product $A \otimes_{\mathsf{k}} B$, unlike C, is not standard graded: the only degree-one elements of $A \otimes_{\mathsf{k}} B$ are of the form $\ell = 1 \otimes_{\mathsf{k}} \ell'$, $\ell' \in \langle x, y, z\rangle$ and since $\ell^3 = 0$ we have that $P_{\ell, A\otimes_{\mathsf{k}} B} \leq (4, 2, 2)^8 = (4^8, 2^8, 2^8)$ rather than the conjugate $H(A \otimes_{\mathsf{k}} B)^\vee = H(C)^\vee$. So $A \otimes_{\mathsf{k}} B$ is not SL, but its deformation C is SL. [13]

A similar example is obtained, replacing F by $F' = E_1^{[3]} + E_2^{[3]} + E_3^{[3]} + E_1E_2E_3$, defining the algebra $A' = S/I'$ where $I' = I_{F'} = \big(e_1e_2 - e_3^2,\ e_1e_3 - e_2^2,\ e_2e_3 - e_1^3,\ e_1^6 - e_2^6,\ e_1^6 - e_2^6\big)$, and defining τ as before. Then $C' = R/\big(\tau(I')\big)$ is Gorenstein, again an A'-free extension with fiber B, and with dual generator $G_{C'} = XYZ \cdot \big(X^{[6]} + Y^{[6]} + Z^{[6]} + X^{[2]}Y^{[2]}Z^{[2]}\big)$, again the product of the dual generator for B and $\tau'(F'')$.

For further discussion of free extensions or the strong Lefschetz property for complete intersection extensions see [52; 54; 58, §4.2–4.4; 106]. For examples of free extensions related to invariant theory, with some similar behavior to the examples above see [66; 83].

3D. Commuting Jordan types. Work of the last ten years has shown that there are strong restrictions on the pairs $P_{\ell,A}$, $P_{\ell',A}$ that can coexist for an Artinian algebra A [62; 71; 72; 74; 98; 99; 102]. We state several such results. For a more complete discussion, including open questions, see [13; 65; 96; 99].

We say that a partition $P = (p_1, p_2, \ldots, p_s)$ where $p_1 \geq p_2 \geq \cdots \geq p_s$ of n is *stable* if its parts differ pairwise by at least two:

$$P \text{ is stable if } p_i - p_{i+1} \geq 2 \quad \text{for } 1 \leq i \leq s - 1. \tag{3-12}$$

Let B be a nilpotent $n \times n$ matrix over an infinite field k having Jordan type $P = P_B$. Denote by $\mathcal{C}_B$ the commutator of B in $\mathrm{Mat}_n(\mathsf{k})$, and by $\mathcal{N}_B$ the nilpotent elements of C_B. It is well known that $\mathcal{N}_B$ is an irreducible variety, hence there is a generic Jordan type $Q(B)$ of matrices in $\mathcal{N}_B$.

Theorem 3.27 (Oblak and Košir [74]).[14] *Assume* char $k = 0$ *or* char $k = p > n$, *let B be an $n \times n$ nilpotent matrix of Jordan type P. Then $Q(B)$ is stable and depends only on the Jordan type P.*

---

[12] Here $H(C)$ is the Hilbert function of a CI of generator degrees $(4, 4, 4)$ but I_F has five generators: is there some importance to this: can we make a similar example where I_F is a CI of this Hilbert function?

[13] In [66, Theorem 2.1] the deformation is explicitly given as a one-parameter family $\mathfrak{C} = \{C(t), t \in \mathbb{A}^1 = \mathsf{k}\}$. The embedding dimension in a constant length family $\mathfrak{C}$ is semicontinuous, has the value 4 for $A \otimes_{\mathsf{k}} B$ above (namely $\{x, y, z, e\}$) but three ($\{x, y, z\}$) for C; so this must be a jump deformation: for $t \in U$, an open dense of $\mathbb{A}^1$ not containing $t = 0$, the algebra $C(t)$ is standard-graded and also is SL (Corollary 2.44).

[14] This is shown in [74] over an algebraically closed field of char $k = 0$, but their proof carries through for any infinite field of char $k = 0$ or char $k = p > n$. See [65, Remark 2.7].

Their proof relied on showing that for a general enough matrix $A \in \mathcal{N}_B$, the local ring $\mathcal{A} = \mathsf{k}\{A, B\}$ is Gorenstein (note that, in general, it is nonhomogeneous). A result of Macaulay shows that a Gorenstein (so complete intersection) quotient of $\mathsf{k}\{x, y\}$ has a Hilbert function $H(\mathcal{A})$ whose conjugate $H(\mathcal{A})^\vee$ is stable; and by Lemma 2.15 the generic Jordan type $P_{\mathcal{A}} = H(\mathcal{A})^\vee$. See also [14, Theorem 2.27] for a discussion of these steps and [15, Section 2.4] for a discussion of the Oblak-Košir result that $\mathsf{k}[A, B]$ is Gorenstein for A general enough in $\mathcal{N}_B$.

Corollary 3.28. *There can be at most one stable partition among the partitions P_ℓ, $\ell \in \mathfrak{m}_{\mathcal{A}}$ for a local Artinian algebra $\mathcal{A}$.*

For example no two of {8, (7, 1), (6, 2), (5, 3)} can occur for partitions P_ℓ for the same (commutative) local algebra $\mathcal{A}$.

Oblak has made a conjecture giving a recursive way to determine $Q(P)$ from P. The largest and smallest part of $Q(P)$ were determined in [98; 72], and "half" the conjecture was shown in [60]. Showing the other half is equivalent to proving a combinatorial result about a certain poset associated to $\mathcal{N}_B$ [72; 62], that would be independent of the characteristic of k. Basili established the Oblak conjecture very recently in [13].

Given an Artinian graded algebra A, or a local algebra $\mathcal{A}$ there is a generic Jordan type P_A or $P_{\mathcal{A}}$ (Definition 2.55) by Lemma 2.54, simply because $\mathfrak{m}_A$ is an affine space, so is irreducible. However, P_A or $P_{\mathcal{A}}$ need not be stable in the sense of (3-12).

Example 3.29. For $A = \mathsf{k}[x, y]/(x^2, xy, y^2)$, or, more generally, for $A_{r,k} = \mathsf{k}[x_1, \dots, x_r]/\mathfrak{m}^k$ for $r, k \geq 2$ the algebra A has generic Jordan type $P = H(A)^\vee$, which is nonstable. For example $H(A_{2,k})$ for $k > 1$ satisfies $H = (1, 2 \dots, k)$, whose conjugate $H^\vee$ is $(1, 2, \dots, k)$.

These are examples of algebras having *constant Jordan type* (CJT): the Jordan type $P_{\ell,A}$ is the same for each linear element $\ell \in A$. Modules of CJT have been extensively studied, and connected to vector bundles over projective space [30].

When the partition P_ℓ occurs for a pair (ℓ, M) where M is a finite A-module and A is Artinian, and $\ell \in A$ is nilpotent, then P_{ℓ^k} for a power ℓ^k can be simply described in terms of P_ℓ, and of course must also occur for M or for A (and likewise for $\mathcal{A}$ local and $\ell \in \mathfrak{m}_{\mathcal{A}}$). We briefly describe this.

Definition 3.30 (almost rectangular partition, [74]). A partition P of n is *almost rectangular* if its parts differ pairwise by at most 1. We denote by $[n]^k$ the unique almost rectangular partition of n having k parts. If $n = qk + r$, $0 \leq r < k$ then

$$[n]^k = \big((q+1)^r, q^{k-r}\big). \tag{3-13}$$

Given a partition $P = (p_1, p_2, \dots, p_s)$, with $p_1 \geq p_2 \geq \cdots \geq p_s$, we denote by $[P]^k$ the partition $\big([p_1]^k, [p_2]^k, \dots, [p_s]^k\big)$ having ks parts.

For example $[7]^2 = (4, 3)$, $[7]^3 = (3, 2, 2)$, $[7]^4 = (2, 2, 2, 1)$, $[7]^5 = (2, 2, 1, 1, 1)$, and if $P = (7, 5)$ then $[P]^2 = (4, 3, 3, 2)$.

Lemma 3.31. *Suppose a nilpotent $n \times n$ matrix M is **regular**: has Jordan type $[n]$. Then M^k has Jordan type $[n]^k$. Suppose that the Jordan type of M is P_M, then $P_{M^k} = [P_M]^k$.*

The term "almost rectangular partition" was introduced by Košir and Oblak; the notion is key in studying $Q(P)$, and occurs earlier in Basili's [12], who showed that the number of parts of $Q(P)$ is r_P, the smallest number of almost-rectangular subpartitions needed to cover P.

The question of determining which Jordan types may commute (may coexist in the same Artinian local algebra A) is quite open in general. See J. Britnell and M. Wildon's [27, §4] and Oblak's [99] for some discussion. The former shows that the problem of when two nilpotent matrices commute is in general characteristic dependent: when k is infinite the orbits (d, d) and $(d+1, d-1)$ are commuting, but when k is a finite field, whether those orbits commute depend on the residue classes of d mod powers of p [27, Proposition 4.12, Remark 4.15]. A result of G. McNinch [84, Lemma 22] shows that the Jordan type of a generic element in the pencil of matrices $M+tN$ can depend on the characteristic. See also [62, Remark 3.16].

There appears to be substantial structure to the set of partitions P having $Q(P) = Q$ where Q is a given stable partition — see [65] which shows this structure for stable partitions Q with two parts, and poses a "box conjecture" for general stable Q.

3E. Problems. We end with some further problems concerning Jordan type.

Compatibility of the partition P_ℓ and its refinements with the Hilbert function H.

Question 3.32. For which Hilbert functions H can we find graded Artinian algebras A with $H(A) = H$, such that for a generic $\ell \in A_1$ we have, in increasing level of refinement,

$$P_{\ell,A} = P(H), \quad \text{or} \quad P_{c,\ell} = P_c(H), \quad \text{or} \quad P_{c,\deg,\ell} = P_{c,\deg}(H)? \tag{3-14}$$

Note that a graded algebra $A = \mathsf{k}[x, y, z]/I$ of Hilbert function $H(A) = (1, 3, 3, 4)$ cannot be even weak Lefschetz as the minimal growth from degree 2 to degree 3 implies that $I_2 = a_1(x, y, z)$ for some $a_1 \in A_1$, so multiplication by an $\ell \in A_1$ cannot be injective from A_1 to A_2.

Remark 3.33. In codimension two, the Hilbert function of a graded ideal (actually, also of nongraded ideal) is determined by the partition $P(H)$; here, the family $\mathrm{Gr}(H)$ parametrizing all graded quotients of $R = \mathsf{k}[x, y]$ having Hilbert function H is (smooth and) irreducible. However, in codimension three, even for graded algebras the family $\mathrm{Gr}(H)$ of quotients of $R = \mathsf{k}[x, y, z]$ having Hilbert function H may be reducible. One example is given in [22, Theorem 2.3], where $H = (1, 3, 4, 4)$. Here the behavior of one component of $\mathrm{Gr}(H)$ with respect to Jordan type may be different than that of another. Considering the family Gor(H) of nongraded Gorenstein height three algebra quotients of $R = \mathsf{k}\{x, y, z\}$ (the regular local ring) of Hilbert function H, in [64], the first and second authors use the Jordan type and the semicontinuous property of the symmetric decomposition, to show that Gor(H) has several irreducible components, for suitable H, in particular for $H = (1, 3, 3, 2, 2, 1)$.

There has been some study of a different question, which Hilbert functions force one of the Lefschetz properties [90; 117]. See also [89].

Additivity of Jordan type.

Question 3.34. Let $0 \to L \to M \to N \to 0$ be an exact sequence of finite A-modules (or $\mathcal{A}$-modules). How can we compare the generic Jordan types P_L, P_M, and P_N? Under what conditions could we have additivity $P_M = P_L + P_N$, in a suitable sense for the Jordan types, or have $\mathcal{P}_M = \mathcal{P}_L + \mathcal{P}_N$ for the Jordan degree types (Definition 2.28)?

We can ask the analogous question for nongeneric Jordan types (see, for example, [3]).

Example 3.35 (additivity). Let $R = \mathsf{k}[x, y]$, and consider the ideals $I = (x^4, xy, y^3)$ and $J = (x^6, xy, y^6)$. Let $L = I/J$, $M = R/J$, and $N = R/I$. Then $\ell = x + y$ computes the generic Jordan type in each of the modules L, M, and N. We have, for the Jordan degree types,

$$L = I/J = \langle y^3, y^4, x^4, y^5, x^5\rangle, \qquad \text{and} \qquad \mathcal{P}_{\ell,L} = (3_3, 2_4);$$
$$N = R/I = \langle 1, x, y, x^2, y^2, x^3\rangle, \qquad \text{and} \qquad \mathcal{P}_{\ell,N} = (4_0, 2_1);$$
$$M = R/J = \langle 1, x, y, x^2, y^2, x^3, y^3, x^4, y^4, x^5, y^5\rangle, \qquad \text{and} \qquad \mathcal{P}_{\ell,M} = (6_0, 5_1).$$

Thus, $\mathcal{P}_M = \big(6_0 = 4_0 +_c 2_4, 5_1 = 2_1 +_c 3_3\big)$ showing that the Jordan degree type $\mathcal{P}_{\ell,M}$ is equal to $\overline{\mathcal{P}_{\ell,N} + P_{\ell,L}}$ in a natural sense. On the other hand the Jordan type $P_{\ell,M} = (6, 5)$ is not equal to the sum $P_{\ell,L} + P_{\ell,N} = (4, 3, 2, 2)$, nor is it equal to the dominance sum $P_{\ell,L} +_b P_{\ell,N} = (4+3, 2+2) = (7, 4)$. Each Jordan type $P_{\ell,N}$, $P_{\ell,L}$, and $P_{\ell,M}$ is SL.

Question 3.36. What partitions $P_{\ell,M}$ and degree-partitions $\mathcal{P}_{\ell,M}$ can we obtain for M, fixing those invariants for L and N?

Loci in $\mathbb{P}(\mathfrak{m}_{\mathcal{A}})$ *defined by Jordan type.* Recall from Definition 2.52 that the set of Jordan types of elements of A acting on M is a poset $\mathfrak{P}_M$ under the "dominance" partial order; this poset $\mathfrak{P}_M$ is an invariant of the module M. Given a partition P of $m = \dim_{\mathsf{k}} M$, the locus $\mathfrak{Z}_{P,M} \subset \mathbb{P}(\mathfrak{m}_A)$, the projective space of the maximal ideal, parametrizes those elements $\ell \in \mathfrak{m}_A$ such that the action of m_ℓ on M has Jordan type $P_\ell = P$. The closures $\overline{\mathfrak{Z}_{P,M}}$ form a poset under inclusion. Of course, the actual loci $\mathfrak{Z}_{P,M}$ in either the A graded or $\mathcal{A}$ local case give more information than just the poset.

Example 3.37. Let $M = \mathsf{k}[x, y]/(x^2, y^3)$: then $P_x = (2, 2, 2)$, $P_y = (3, 3)$ and $P_\ell = (4, 2)$ for $\ell = x + by$, $b \neq 0$. Here $\mathfrak{P}_M = \{(4, 2) \geq (3, 3) \geq (2, 2, 2)\}$. However $\mathfrak{Z}_{P,M} = \{\overline{\mathfrak{Z}_{4,2}}\} \supset \{\overline{\mathfrak{Z}_{3,3}} \cup \overline{\mathfrak{Z}_{2,2,2}}\}$ as $\mathfrak{Z}_{3,3}$ is a single point. So the two posets P_M and the poset of closures of loci $\mathfrak{Z}_{P,M}$ may be different, the latter being necessarily a subposet of the former, by the semicontinuity of Jordan type.

There has been some study of these Jordan type loci by commutative algebraists, for example Boij, Migliore, Miró-Roig, and Nagel, on the nonweak Lefschetz locus [24], and the notes [2]. On the other hand, there has been recent study of the Jordan type loci $\mathfrak{Z}_P$ in the nilpotent commutator $\mathcal{N}_B$ of an $n \times n$ matrix B [98; 99]; when B is a Jordan matrix of stable Jordan type Q, then it is conjectured that the set $\mathcal{B}(Q)$ of loci in the nilpotent commutator $\mathcal{N}_B$ can be arranged in a rectangular r-box, whose dimensions are determined by the r parts of Q (see [65, Conjecture 4.11]), and that the equations for these loci are complete intersections [65, Remark 4.13]. The first conjecture is shown for stable Q having $r = 2$ parts [65, Theorem 1.1].

Question 3.38. Let $x \in \mathfrak{m}_{\mathcal{A}}$ have Jordan type Q. Denote the matrix of m_x by B and recall that $\mathcal{N}_B$ is the nilpotent commutator of B (Section 3D). Is there a morphism $\tau_{x,\mathcal{A}} : \mathfrak{m}_{\mathcal{A}} \to \mathcal{N}_B$, such that the Jordan type of $y \in \mathfrak{m}_{\mathcal{A}}$ satisfies $P_y = P_{\tau_{x,\mathcal{A}}(y)}$?

Recall that for $x \in \mathcal{A}$ we denote by $\nu(x)$ its order: the maximum i such that $x \in {\mathfrak{m}_{\mathcal{A}}}^i$.

Question 3.39. Given an Artinian algebra $\mathcal{A}$ with socle degree $j_{\mathcal{A}}$, by a simple linear algebra argument, we can always find a k-basis B for $\mathcal{A}$ such that for every $i \in \{0, \dots, j_{\mathcal{A}}\}$

$$\#\{x \in B \mid \nu(x) = i\} = H(\mathcal{A})_i. \tag{3-15}$$

Given $\ell \in \mathfrak{m}_{\mathcal{A}}$, is it possible to find a Jordan basis for m_ℓ also satisfying (3-15)?

This is clearly possible when A is graded, and ℓ is homogeneous.

Tensor products of local Artinian algebras and Hilbert function.

Question 3.40. Assume that for local Artinian k algebras $\mathcal{A}$ and $\mathcal{B}$ their tensor product $\mathcal{A} \otimes_{\mathsf{k}} \mathcal{B}$ is also a local Artinian k algebra. By a criterion of M. Sweedler [114, Theorem, b.iii.] this is equivalent to $(\mathcal{A}/\mathfrak{m}_{\mathcal{A}}) \otimes_{\mathsf{k}} (\mathcal{B}/\mathfrak{m}_{\mathcal{B}})$ is local. How is the Hilbert function for $\mathcal{A} \otimes_{\mathsf{k}} \mathcal{B}$ related to the Hilbert functions of $\mathcal{A}$ and of $\mathcal{B}$? Here we assume the residue fields are equal $(\mathcal{A}/\mathfrak{m}_{\mathcal{A}}) = (\mathcal{B}/\mathfrak{m}_{\mathcal{B}}) \cong \mathsf{k}$: then what is the relation between the associated graded algebra $(\mathcal{A} \otimes_{\mathsf{k}} \mathcal{B})^*$ and $\mathcal{A}^* \otimes_{\mathsf{k}} \mathcal{B}^*$, the tensor product of the associated graded algebras of $\mathcal{A}$, $\mathcal{B}$?

Acknowledgements

We appreciate conversations with Shujian Chen and Ivan Martino. We thank Oana Veliche for conversations and help with MACAULAY2. A. Iarrobino appreciates conversations with participants of the informal work group on Jordan Type at Institute Mittag Leffler in July, 2017 [2]. A subsequent remark of Yong-Su Shin that there was no published reference available in the Lefschetz-related commutative algebra literature about Jordan types was a spark for this work. Discussion during his visit to Northeastern University in January 2018 led to Section 3B about the literature on modular tensor products. We thank Rodrigo Gondim for permission to use several examples. We appreciate comments and answers to our questions by Larry Smith, Junzo Watanabe, Alexandra Seceleanu and Steven L. Kleiman. We thank Lorenzo Robbiano, Gordana Todorov, Shijie Zhu and Jerzy Weyman for their comments. We thank the referee for comments, in particular the referee suggestions led to Proposition 2.47.

P. Macias Marques was partially supported by CIMA – Centro de Investigação em Matemática e Aplicações, Universidade de Évora, project UID/MAT/04674/2019 (Fundação para a Ciência e Tecnologia). Part of this work was done while Macias Marques was visiting KU Leuven, he wishes to thank Wim Veys and the Mathematics Department for their hospitality.

References

[1] J. Ahn, Y. H. Cho, and J. P. Park, "Generic initial ideals of Artinian ideals having Lefschetz properties or the strong Stanley property", *J. Algebra* **318**:2 (2007), 589–606.

[2] N. Altafi, H. L. Dao, A. Iarrobino, L. Khatami, A. Seceleanu, and Y.-S. Shin, "Work group on Jordan type at "Lefschetz properties in algebra, geometry and combinatorics"", 2017.

[3] N. Altafi, A. Iarrobino, and L. Khatami, "Complete intersection Jordan types in height two", *J. Algebra* **557** (2020), 224–277.

[4] N. Altafi, A. Iarrobino, L. Khatami, and J. Yaméogo, "Jordan types for graded Artinian algebras in height two", 2021. arXiv 2006.11794v2

[5] D. Anderson and M. Winders, "Idealization of a module", *J. Commut. Algebra* **1**:1 (2009), 3–56.

[6] V. I. Arnold, "Matrices depending on parameters", *Uspehi Mat. Nauk* **26**:2(158) (1971), 101–114. In Russian; translated in *Russ. Math. Surv.* **26**:2 (1971), 29–43.

[7] M. Artin, "Lectures on deformations of singularities", 1976, available at http://www.math.tifr.res.in/~publ/ln/tifr54.pdf.

[8] L. Avramov, "Flat morphisms of complete intersections", *Dokl. Akad. Nauk SSSR* **225**:1 (1975), 11–14. In Russian; translated in *Soviet Math. Dokl.* **16**:6 (1976), 1413–1417.

[9] L. Avramov and S. B. Iyengar, "Restricting homology to hypersurfaces", pp. 1–23 in *Geometric and topological aspects of the representation theory of finite groups*, Springer Proc. Math. Stat. **242**, Springer, 2018.

[10] E. Babson and E. Nevo, "Lefschetz properties and basic constructions on simplicial spheres", *J. Algebraic Combin.* **31**:1 (2010), 111–129.

[11] V. Baranovsky, "The variety of pairs of commuting nilpotent matrices is irreducible", *Transform. Groups* **6**:1 (2001), 3–8.

[12] R. Basili, "On the irreducibility of commuting varieties of nilpotent matrices", *J. Algebra* **268**:1 (2003), 58–80.

[13] R. Basili, "On the maximal nilpotent orbit that intersects the centralizer of a matrix", *Transform. Groups* **27**:1 (2022), 1–30.

[14] R. Basili and A. Iarrobino, "Pairs of commuting nilpotent matrices, and Hilbert function", *J. Algebra* **320**:3 (2008), 1235–1254.

[15] R. Basili, A. Iarrobino, and L. Khatami, "Commuting nilpotent matrices and Artinian algebras", *J. Commut. Algebra* **2**:3 (2010), 295–325.

[16] G. Benkart and J. Osborn, "Representations of rank one Lie algebras of characteristic p", pp. 1–37 in *Lie algebras and related topics* (New Brunswick, NJ, 1981), Lecture Notes in Math. **933**, Springer, 1982.

[17] D. Benson, *Representations of elementary abelian p-groups and vector bundles*, Cambridge Tracts in Mathematics **208**, Cambridge University Press, 2017.

[18] D. Benson, S. B. Iyengar, H. Krause, and J. Pevtsova, "Local duality for representations of finite group schemes", *Compos. Math.* **155**:2 (2019), 424–453.

[19] D. Bernstein and A. Iarrobino, "A nonunimodal graded Gorenstein Artin algebra in codimension five", *Comm. Algebra* **20**:8 (1992), 2323–2336.

[20] M. Boij, "Graded Gorenstein Artin algebras whose Hilbert functions have a large number of valleys", *Comm. Algebra* **23**:1 (1995), 97–103.

[21] M. Boij, "Components of the space parametrizing graded Gorenstein Artin algebras with a given Hilbert function", *Pacific J. Math.* **187**:1 (1999), 1–11.

[22] M. Boij and A. Iarrobino, "Reducible family of height three level algebras", *J. Algebra* **321**:1 (2009), 86–104.

[23] M. Boij and D. Laksov, "Nonunimodality of graded Gorenstein Artin algebras", *Proc. Amer. Math. Soc.* **120**:4 (1994), 1083–1092.

[24] M. Boij, J. Migliore, R. Miró-Roig, and U. Nagel, "The non-Lefschetz locus", *J. Algebra* **505** (2018), 288–320.

[25] H. Brenner and A. Kaid, "A note on the weak Lefschetz property of monomial complete intersections in positive characteristic", *Collect. Math.* **62**:1 (2011), 85–93.

[26] J. Briançon, "Description de $\mathrm{Hilb}^n \mathcal{C}\{x, y\}$", *Invent. Math.* **41**:1 (1977), 45–89.

[27] J. Britnell and M. Wildon, "On types and classes of commuting matrices over finite fields", *J. Lond. Math. Soc.* (2) **83**:2 (2011), 470–492.

[28] W. Buczyńska, J. a. Buczyński, J. Kleppe, and Z. Teitler, "Apolarity and direct sum decomposability of polynomials", *Michigan Math. J.* **64**:4 (2015), 675–719.

[29] A. Capasso, P. De Poi, and G. Ilardi, "CW-complex Nagata idealizations", *Adv. in Appl. Math.* **120** (2020), art. id. 102079.

[30] J. F. Carlson, E. Friedlander, and J. Pevtsova, "Modules of constant Jordan type", *J. Reine Angew. Math.* **614** (2008), 191–234.

[31] E. Cattani, "Mixed Lefschetz theorems and Hodge–Riemann bilinear relations", *Int. Math. Res. Not.* **2008**:10 (2008), art. id. rnn025.

[32] A. Cerminara, R. Gondim, G. Ilardi, and F. Maddaloni, "Lefschetz properties for higher order Nagata idealizations", *Adv. in Appl. Math.* **106** (2019), 37–56.

[33] D. H. Collingwood and W. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold, New York, 1993.

[34] A. Conca, "Reduction numbers and initial ideals", *Proc. Amer. Math. Soc.* **131**:4 (2003), 1015–1020.

[35] D. Cook, II, "The Lefschetz properties of monomial complete intersections in positive characteristic", *J. Algebra* **369** (2012), 42–58.

[36] D. Cook, II, "The strong Lefschetz property in codimension two", *J. Commut. Algebra* **6**:3 (2014), 323–345.

[37] D. Cook, II and U. Nagel, "The weak Lefschetz property, monomial ideals, and lozenges", *Illinois J. Math.* **55**:1 (2011), 377–395.

[38] D. Cook, II and U. Nagel, "Signed lozenge tilings", *Electron. J. Combin.* **24**:1 (2017), art. id. 1.9.

[39] B. Costa and R. Gondim, "The Jordan type of graded Artinian Gorenstein algebras", *Adv. in Appl. Math.* **111** (2019), art. id. 101941.

[40] A. Dimca, R. Gondim, and G. Ilardi, "Higher order Jacobians, Hessians and Milnor algebras", *Collect. Math.* **71**:3 (2020), 407–425.

[41] D. Eisenbud, *Commutative algebra: with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, 1995.

[42] E. Friedlander, J. Pevtsova, and A. Suslin, "Generic and maximal Jordan types", *Invent. Math.* **168**:3 (2007), 485–522.

[43] W. Fulton, *Intersection theory*, 2nd ed., Ergeb. Math. Grenzgeb. (3) **2**, Springer, 1998.

[44] F. R. Gantmacher, *The theory of matrices*, vol. 1, Chelsea, New York, 1959.

[45] M. Gerstenhaber, "On the deformation of rings and algebras, II", *Ann. of Math.* **84** (1966), 1–19.

[46] S. P. Glasby, C. E. Praeger, and B. Xia, "Decomposing modular tensor products: 'Jordan partitions', their parts and p-parts", *Israel J. Math.* **209**:1 (2015), 215–233.

[47] S. P. Glasby, C. E. Praeger, and B. Xia, "Decomposing modular tensor products, and periodicity of 'Jordan partitions'", *J. Algebra* **450** (2016), 570–587.

[48] R. Gondim, "On higher Hessians and the Lefschetz properties", *J. Algebra* **489** (2017), 241–263.

[49] R. Gondim and G. Zappalà, "Lefschetz properties for Artinian Gorenstein algebras presented by quadrics", *Proc. Amer. Math. Soc.* **146**:3 (2018), 993–1003.

[50] P. Gordan and M. Nöther, "Ueber die algebraischen Formen, deren Hesse'sche Determinante identisch verschwindet", *Math. Ann.* **10**:4 (1876), 547–568.

[51] L. Göttsche, "Betti numbers for the Hilbert function strata of the punctual Hilbert scheme in two variables", *Manuscripta Math.* **66**:3 (1990), 253–259.

[52] T. Harima and J. Watanabe, "The finite free extension of Artinian K-algebras with the strong Lefschetz property", *Rend. Sem. Mat. Univ. Padova* **110** (2003), 119–146. Erratum in **112** (2004), 237–238.

[53] T. Harima and J. Watanabe, "The central simple modules of Artinian Gorenstein algebras", *J. Pure Appl. Algebra* **210**:2 (2007), 447–463.

[54] T. Harima and J. Watanabe, "The strong Lefschetz property for Artinian algebras with non-standard grading", *J. Algebra* **311**:2 (2007), 511–537.

[55] T. Harima and J. Watanabe, "The commutator algebra of a nilpotent matrix and an application to the theory of commutative Artinian algebras", *J. Algebra* **319**:6 (2008), 2545–2570.

[56] T. Harima and J. Watanabe, "The weak Lefschetz property for $\mathfrak{m}$-full ideals and componentwise linear ideals", *Illinois J. Math.* **56**:3 (2012), 957–966.

[57] T. Harima, J. Migliore, U. Nagel, and J. Watanabe, "The weak and strong Lefschetz properties for Artinian K-algebras", *J. Algebra* **262**:1 (2003), 99–126.

[58] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, and J. Watanabe, *The Lefschetz properties*, Lecture Notes in Mathematics **2080**, Springer, 2013.

[59] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, 1977.

[60] A. Iarrobino, *Associated graded algebra of a Gorenstein Artin algebra*, vol. 107, Mem. Amer. Math. Soc. **514**, Amer. Math. Soc., Providence, RI, 1994.

[61] A. Iarrobino and M. Boij, "Duality for central simple modules of Artinian Gorenstein algebras", In preparation.

[62] A. Iarrobino and L. Khatami, "Bound on the Jordan type of a generic nilpotent matrix commuting with a given matrix", *J. Algebraic Combin.* **38**:4 (2013), 947–972.

[63] A. Iarrobino and P. Macias Marques, "Symmetric decomposition of the associated graded algebra of an Artinian Gorenstein algebra", *J. Pure Appl. Algebra* **225**:3 (2021), art. id. 106496.

[64] A. Iarrobino and P. M. Marques, "Reducibility of a family of local Artinian Gorenstein algebras", 2021. arXiv 2112.14664

[65] A. Iarrobino, L. Khatami, B. Van Steirteghem, and R. Zhao, "Nilpotent matrices having a given Jordan type as maximum commuting nilpotent orbit", *Linear Algebra Appl.* **546** (2018), 210–260.

[66] A. Iarrobino, P. M. Marques, and C. McDaniel, "Free extensions and Jordan type", *J. Algebra* **549** (2020), 346–364.

[67] A. Iarrobino, C. McDaniel, and A. Seceleanu, "Connected sums of graded Artinian Gorenstein algebras and Lefschetz properties", *J. Pure Appl. Algebra* **226**:1 (2022), art. id. 106787.

[68] A. Iarrobino, P. M. Marques, and J. Steinmeyer, "Sequential Jordan type of an Artinian algebra", in preparation.

[69] K.-i. Iima and R. Iwamatsu, "On the Jordan decomposition of tensored matrices of Jordan canonical forms", *Math. J. Okayama Univ.* **51** (2009), 133–148.

[70] H. Ikeda and J. Watanabe, "The Dilworth lattice of Artinian rings", *J. Commut. Algebra* **1**:2 (2009), 315–326.

[71] L. Khatami, "The poset of the nilpotent commutator of a nilpotent matrix", *Linear Algebra Appl.* **439**:12 (2013), 3763–3776.

[72] L. Khatami, "The smallest part of the generic partition of the nilpotent commutator of a nilpotent matrix", *J. Pure Appl. Algebra* **218**:8 (2014), 1496–1516.

[73] S. L. Kleiman and J. O. Kleppe, "Macaulay duality and its geometry", preprint, 2022.

[74] T. Košir and P. Oblak, "On pairs of commuting nilpotent matrices", *Transform. Groups* **14**:1 (2009), 175–182.

[75] A. R. Kustin and A. Vraciu, "The weak Lefschetz property for monomial complete intersection in positive characteristic", *Trans. Amer. Math. Soc.* **366**:9 (2014), 4571–4601.

[76] S. Lundqvist and L. Nicklasson, "On the structure of monomial complete intersections in positive characteristic", *J. Algebra* **521** (2019), 213–234.

[77] F. H. S. Macaulay, *The algebraic theory of modular systems*, Cambridge University Press, 1994.

[78] T. Maeno and Y. Numata, "Sperner property and finite-dimensional Gorenstein algebras associated to matroids", *J. Commut. Algebra* **8**:4 (2016), 549–570.

[79] T. Maeno and J. Watanabe, "Lefschetz elements of Artinian Gorenstein algebras and Hessians of homogeneous polynomials", *Illinois J. Math.* **53**:2 (2009), 591–603.

[80] M. Mastroeni, H. Schenck, and M. Stillman, "Quadratic Gorenstein rings and the Koszul property, I", *Trans. Amer. Math. Soc.* **374**:2 (2021), 1077–1093.

[81] N. Mastronardi and P. Van Dooren, "Computing the Jordan structure of an eigenvalue", *SIAM J. Matrix Anal. Appl.* **38**:3 (2017), 949–966.

[82] J. McCullough and A. Seceleanu, "Quadratic Gorenstein algebras with many surprising properties", *Arch. Math.* (*Basel*) **115**:5 (2020), 509–521.

[83] C. McDaniel, S. Chen, A. Iarrobino, and P. Macias Marques, "Free extensions and Lefschetz properties, with an application to rings of relative coinvariants", *Linear Multilinear Algebra* **69**:2 (2021), 305–330.

[84] G. McNinch, "On the centralizer of the sum of commuting nilpotent elements", *J. Pure Appl. Algebra* **206**:1-2 (2006), 123–140.

[85] D. Meyer and L. Smith, *Poincaré duality algebras, Macaulay's dual systems, and Steenrod operations*, Cambridge Tracts in Mathematics **167**, Cambridge University Press, 2005.

[86] E. Mezzetti and R. M. Miró-Roig, "The minimal number of generators of a Togliatti system", *Ann. Mat. Pura Appl.* (4) **195**:6 (2016), 2077–2098.

[87] E. Mezzetti, R. M. Miró-Roig, and G. Ottaviani, "Laplace equations and the weak Lefschetz property", *Canad. J. Math.* **65**:3 (2013), 634–654.

[88] M. Michałek and R. M. Miró-Roig, "Smooth monomial Togliatti systems of cubics", *J. Combin. Theory Ser. A* **143** (2016), 66–87.

[89] J. Migliore and U. Nagel, "Survey article: a tour of the weak and strong Lefschetz properties", *J. Commut. Algebra* **5**:3 (2013), 329–358.

[90] J. Migliore and F. Zanello, "The strength of the weak Lefschetz property", *Illinois J. Math.* **52**:4 (2008), 1417–1433.

[91] J. Migliore and F. Zanello, "Stanley's nonunimodal Gorenstein h-vector is optimal", *Proc. Amer. Math. Soc.* **145**:1 (2017), 1–9.

[92] J. Migliore, U. Nagel, and H. Schenck, "The weak Lefschetz property for quotients by quadratic monomials", *Math. Scand.* **126**:1 (2020), 41–60.

[93] D. Mumford, *Lectures on curves on an algebraic surface*, Annals of Mathematics Studies **59**, Princeton University Press, 1966.

[94] M. Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics **13**, Interscience Publishers, New York, 1962.

[95] S. Nasseh, A. Seceleanu, and J. Watanabe, "Determinants of incidence and Hessian matrices arising from the vector space lattice", *J. Commut. Algebra* **11**:1 (2019), 131–154.

[96] N. V. Ngo and K. Šivic, "On varieties of commuting nilpotent matrices", *Linear Algebra Appl.* **452** (2014), 237–262.

[97] L. Nicklasson, "The strong Lefschetz property of monomial complete intersections in two variables", *Collect. Math.* **69**:3 (2018), 359–375.

[98] P. Oblak, "The upper bound for the index of nilpotency for a matrix commuting with a given nilpotent matrix", *Linear Multilinear Algebra* **56**:6 (2008), 701–711.

[99] P. Oblak, "On the nilpotent commutator of a nilpotent matrix", *Linear Multilinear Algebra* **60**:5 (2012), 599–612.

[100] K. C. O'Meara and J. Watanabe, "Weyr structures of matrices and relevance to commutative finite-dimensional algebras", *Linear Algebra Appl.* **532** (2017), 364–386.

[101] K. C. O'Meara, J. Clark, and C. I. Vinsonhaler, *Advanced topics in linear algebra*: *Weaving matrix problems through the Weyr form*, Oxford University Press, 2011.

[102] D. I. Panyushev, "Two results on centralisers of nilpotent elements", *J. Pure Appl. Algebra* **212**:4 (2008), 774–779.

[103] J. P. Park and Y.-S. Shin, "Modular Jordan type for $\Bbbk[x, y]/(x^m, y^n)$ for $m = 3, 4$", *J. Korean Math. Soc.* **57**:2 (2020), 283–312.

[104] J. Pevtsova, "Representations and cohomology of finite group schemes", pp. 231–261 in *Advances in representation theory of algebras*, Eur. Math. Soc., Zürich, 2013.

[105] J. Pevtsova, "Support varieties, local Jordan type, and applications", in *Joint AMS-EMS-SPN Meeting* (Porto, 2015), 2015.

[106] D. Popescu, "The strong Lefschetz property and certain complete intersection extensions", *Bull. Math. Soc. Sci. Math. Roumanie* (*N.S.*) **48(96)**:4 (2005), 421–431.

[107] A. Premet, "The Green ring of a simple three-dimensional Lie p-algebra", *Izv. Vyssh. Uchebn. Zaved. Mat.* 10 (1991), 56–67. In Russian; translated in *Soviet Math.* (*Iz. VUZ*) **35**:10 (1991), 51–60.

[108] A. Premet, "Nilpotent commuting varieties of reductive Lie algebras", *Invent. Math.* **154**:3 (2003), 653–683.

[109] T. Ralley, "Decomposition of products of modular representations", *J. London Math. Soc.* **44** (1969), 480–484.

[110] J.-C. Renaud, "The decomposition of products in the modular representation ring of a cyclic group of prime power order", *J. Algebra* **58**:1 (1979), 1–11.

[111] H. Sekiguchi, "The upper bound of the Dilworth number and the Rees number of Noetherian local rings with a Hilbert function", *Adv. Math.* **124**:2 (1996), 197–206.

[112] H. Shapiro, "The Weyr characteristic", *Amer. Math. Monthly* **106**:10 (1999), 919–929.

[113] R. P. Stanley, "Hilbert functions of graded algebras", *Advances in Math.* **28**:1 (1978), 57–83.

[114] M. E. Sweedler, "When is the tensor product of algebras local?", *Proc. Amer. Math. Soc.* **48** (1975), 8–10.

[115] A. Wiebe, "The Lefschetz property for componentwise linear ideals and Gotzmann ideals", *Comm. Algebra* **32**:12 (2004), 4601–4611. Revised version available at `https://arxiv.org/abs/math/0307223`.

[116] J. Yaméogo, "Décomposition cellulaire de variétés paramétrant des idéaux homogènes de $\mathbb{C}[[x, y]]$, Incidence des cellules, I", *Compositio Math.* **90**:1 (1994), 81–98.

[117] F. Zanello and J. Zylinski, "Forcing the strong Lefschetz and the maximal rank properties", *J. Pure Appl. Algebra* **213**:6 (2009), 1026–1030.

ANTHONY IARROBINO: a.iarrobino@northeastern.edu
Department of Mathematics, Northeastern University, Boston, MA, United States

PEDRO MACIAS MARQUES: pmm@uevora.pt
Departamento de Matemática, Universidade de Évora, Évora, Portugal

CHRIS MCDANIEL: cmcdanie@endicott.edu
Department of Mathematics, Endicott College, Beverly, MA, United States

JCA — prepared by msp for the
Rocky Mountain Mathematics Consortium

APPENDIX C

Paper “Jordan type of an Artinian algebra, a survey”, by Nasrin Altafi, Anthony Iarrobino, and Pedro Macias Marques

Jordan type of an Artinian algebra, a survey*

Nasrin Altafi

Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada and KTH Royal Institute of Technology, Stockholm, Sweden

Anthony Iarrobino

Department of Mathematics, Northeastern University, Boston, MA 02115, USA.

Pedro Macias Marques

Departamento de Matemática, Escola de Ciências e Tecnologia, Centro de Investigação em Matemática e Aplicações, Instituto de Investigação e Formação Avançada, Universidade de Évora, Rua Romão Ramalho, 59, P–7000–671 Évora, Portugal

July 3, 2023

Abstract

We consider Artinian algebras A over a field k, both graded and local algebras. The Lefschetz properties of graded Artinian algebras have been long studied, but more recently the Jordan type invariant of a pair (ℓ, A) where ℓ is an element of the maximal ideal of A, has been introduced. The Jordan type gives the sizes of the Jordan blocks for multiplication by ℓ on A, and it is a finer invariant than the pair (ℓ, A) being strong or weak Lefschetz. The Jordan degree type for a graded Artinian algebra adds to the Jordan type the initial degree of "strings" in the decomposition of A as a $\mathsf{k}[\ell]$ module. We here give a brief survey of Jordan type for Artinian algebras, Jordan degree type for graded Artinian algebras, and related invariants for local Artinian algebras, with a focus on recent work and open problems.

Contents

***Keywords**: Artinian Gorenstein, local algebra, symmetric decomposition, irreducible components, deformation, Hilbert function, Jordan type, Lefschetz property, Macaulay dual generator, parametrization. **2020 Mathematics Subject Classification**: Primary: 13H10; Secondary: 13E10, 13M05, 14B07, 14C05

1 Introduction.

Notation.

Let A be a graded or local Artinian algebra quotient of $R = \mathsf{k}[x_1, \ldots, x_r]$ (polynomial ring) or of $\mathcal{R} = \mathsf{k}\{x_1, \ldots, x_r\}$ (regular local ring) with maximal ideal $\mathfrak{m}$ and highest socle degree j: that is $A_j \neq 0$, but $A_i = 0$ for $i > j$. Here, for A local we take A_i to be the i-th graded piece of the associated graded algebra $A^* = \bigoplus \mathfrak{m}^i/\mathfrak{m}^{i+1}$ of A. For A graded we let $\mathfrak{m} = \oplus_1^j A_i$. The Hilbert function of A is the sequence $H(A) = (h_0, h_1, \ldots, h_j)$ where $h_i = \dim_{\mathsf{k}} A_i$; the *Sperner number* of $H(A)$ is the maximum value of $H(A)$. The *Jordan type* $P_{\ell,A}$ of a nilpotent element $\ell \in \mathfrak{m}$ of A is the partition P giving the sizes of the Jordan blocks of the (nilpotent) multiplication map m_ℓ. The properties of (ℓ, A) being strong-Lefschetz ($P = H(A)^\vee$, the conjugate of the Hilbert function viewed as a partition) or weak-Lefschetz (the number of parts of P is the Sperner number) of a pair (ℓ, A), have been investigated as such since at least 1978 - see [St, H-W, MiNa]. Earlier, J. Briançon in 1972 showed the strong Lefschetz property $P_{\ell,A} = H(A)^\vee$ in characteristic zero for each codimension two Artinian algebra A and a generic $\ell \in R_1$ [Bri]. But Jordan type is a finer concept: there are in general many partitions that can occur for $P_{\ell,A}$ given just the Hilbert function $H = H(A)$. A basic introductory paper is the second two authors' joint paper with C. McDaniel [IMM2]; other resources include [BMMN, IMM1, IMM3, CGo]. Our attention in this note will be to the more general notion of Jordan type, as opposed to merely the Lefschetz properties.

Let H be a sequence that occurs as the Hilbert function of an Artinian quotient of R or $\mathcal{R}$. First, take R to be the polynomial ring. We denote by G(H) and GGor(H) the family of graded or graded Gorenstein, respectively, quotients of R having Hilbert function H. Now take $\mathcal{R}$ to be the regular local ring, and denote by Z(H) or ZGor(H), respectively, the family of all (not necessarily graded) quotients of $\mathcal{R}$ having Hilbert function H, or, respectively, the Gorenstein quotients of $\mathcal{R}$ having Hilbert function H. We regard these in this survey as subvarieties (not necessarily irreducible) of the Grassmanian Grass($R/\mathfrak{m}^n$), $n = |H|$; but some have also looked at the scheme structures, namely the Hilbert scheme $\mathrm{Hilb}^n(R)$ (see, for example [Hui, Je, BCR, Kl, CaJN] and [IKa, Appendix C]). We will write R for both R and

$\mathcal{R}$, when considering both at the same time. There is a natural notion of dominance of Jordan types (see Definition 1.5). Our goals in this survey are

(a). Review the definitions and properties of Jordan type and Jordan degree type.

(b). Report on progress on the several major questions below, and

(c). Suggest some further problems.

1.1 Major problems.

The development of the subject has been related to some questions:

(i). How does Jordan type behave as one deforms the the element $\ell \in \mathfrak{m}$, or the algebra $A = R/I$ among algebras of a given Hilbert function? Two cases: graded A, and local $A = \mathcal{R}/I$. In particular, does the Hilbert function determine a bound (in the sense of domination) on the possible Jordan types?

(ii). For graded A, there is a refinement of Jordan type to a Jordan degree type [IMM2]. Determine its properties and avatars (Sections 2.3 and 2.4 below).

There is a natural generalization of Jordan type to "contiguous Jordan type" for graded algebras having non-unimodal Hilbert function. There are similar questions of deformation (see [IMM2, Section 2F, Definition 2.28ii], not treated here).

(iii). Generalizations and refinements of Jordan type from graded algebras to local algebras [IMS] (see Section 3 below).

(iv). When A is local Gorenstein, what is the relation of these refinements of Jordan type to the symmetric decomposition of A (see [IM1, IMS])?

(v). Using Jordan type and other invariants to show that various families $Z(H)$ or $ZGor(H)$ have several irreducible components [IM2] (Section 3.1 below).

(vi). Given the Artinian algebra A, and a fixed partition P of $|A|$, what is the locus $\mathfrak{Z}_P \subset \mathbb{P}(A_1) \cong \mathbb{P}^{r-1}$ of linear forms ℓ for which $P_{\ell,A} = P$? The non-Lefschetz locus [BMMN]?

(vii). What is the relation between Jordan type and the Betti minimal resolution of A? [Ab, AbSc]

(viii). What pairs of Jordan type partitions $P_{\ell,A}$ and $P_{\ell',A}$ may occur together in an Artinian A? OR, what Jordan types P_M, P_N may occur for a pair (M,N) of $n \times n$ commuting matrices (see [Kh]).

Some of these questions are now partially answered, ideas behind them have inspired other questions that remain open. We discuss (i)-(v) in more detail below, and then pose some specific questions.

1.2 What is Jordan type?

We first present the definitions and some properties of Jordan type, and then in Section 2.1 discuss the relationship to the weak and strong Lefschetz properties for graded algebras. Since the definition of Jordan type does not require grading, we start by stating it in the general setting, for a module over an algebra that may not be graded.

Definition 1.1 (Jordan type)**.** (See also [IMM2, Definition 2.1] and [H-W, Section 3.5]) Let M be a finitely generated module over the Artinian algebra A, and let $\ell \in \mathfrak{m}$. The Jordan type of ℓ in M is the partition of $\dim_{\mathsf{k}} M$, denoted $P_\ell = P_{\ell,M} = (p_1, \ldots, p_s)$, where $p_1 \geq \cdots \geq p_s$, whose parts p_i are the block sizes in the Jordan canonical form matrix of the multiplication map $m_\ell : M \to M$, $x \mapsto \ell x$. The *generic Jordan type* of A, denoted P_A, is the Jordan type $P_{\ell,A}$ for a generic element ℓ of A_1 (when A is graded), or of $\mathfrak{m}_A$ (A local).

The Jordan block form for the similarity class of a matrix is sometimes called the Segre characteristic, in contrast to its conjugate, the Weyr characteristic (see Note 1.10 below).

Definition 1.2 (Jordan basis, pre-Jordan basis)**.** With the notation of the previous definition, a *pre-Jordan basis* for ℓ is a basis of M as a vector space over k of the form

$$\mathcal{B} = \{\ell^i z_k \mid 1 \leq k \leq s,\, 0 \leq i \leq p_k - 1\}, \tag{1.1}$$

where $P_{\ell,M} = (p_1, \ldots, p_s)$ is the Jordan type of ℓ. We call the subsets $S_k = \{z_k, \ell z_k, \ldots, \ell^{p_k-1} z_k\}$ *strings* of the basis $\mathcal{B}$, and each element $\ell^i z_k$ a *bead* of the string. The Jordan blocks of the multiplication m_ℓ are determined by the strings S_k, and M is the direct sum

$$M = \langle S_1 \rangle \oplus \cdots \oplus \langle S_s \rangle. \tag{1.2}$$

If the elements $z_1, \ldots, z_s \in M$ satisfy $\ell^{p_k} z_k = 0$ for each k, we call $\mathcal{B}$ a *Jordan basis* for ℓ, recovering the usual definition in linear algebra, since a matrix representing the multiplication by ℓ with respect to $\mathcal{B}$, ordering elements as $(\ell^{p_1-1}z_1, \ldots, z_1, \ell^{p_2-1}z_2, \ldots, z_2, \ldots, \ell^{p_s-1}z_s, \ldots, z_s)$, is a canonical Jordan form. In that case the $\langle S_k \rangle$ are cyclic $\mathsf{k}[\ell]$-submodules of M.

The following is well-known (see [Ar, §4.7], [We]).

Lemma 1.3. *If B has a pre-Jordan basis $\mathcal{B}$ as in* (1.1)*, then for each k, we have*

$$\ell^{p_k} z_k \in \langle \ell^a z_i \mid a \geq p_k, i < k \rangle.$$

There is a Jordan basis of M derived from the pre-Jordan basis, and having the same partition invariant $P_{\ell,M}$ giving the lengths of strings.

Algorithm 1.4. Often it is useful to consider a pre-Jordan basis (or a Jordan basis) to study the Jordan type of an element $\ell \in \mathfrak{m}$. However, to compute the Jordan type of an element in a module, we do not need to choose a basis. We can consider the sequence $(d_0, \ldots, d_{j+1})$, where $d_i = \dim_{\mathsf{k}} M/\ell^i M$, and compute the sequence of differences $\Delta_\ell = (\delta_1, \ldots, \delta_{j+1})$, where $\delta_i = d_i - d_{i-1}$. Then taking the conjugate partition of this sequence, we get the Jordan type of ℓ in M (see [IMM2, Lemma 2.3]):

$$P_{\ell,M} = \Delta_\ell^\vee.$$

This is the algorithm used in Macaulay 2.

A key notion is specialization of Jordan types, which follows the dominance partial order on partitions (Lemma 1.6).

Definition 1.5 (Dominance order)**.** Let $P = (p_1, \ldots, p_s)$, $p_1 \geq \cdots \geq p_s$, and $Q = (q_1, \ldots, q_r)$, $q_1 \geq \cdots \geq q_r$, be two partitions of $n = \sum p_i = \sum q_i$. We say that P dominates Q (written $P \geq Q$, if for each $k \in [1, \min\{s, r\}]$, we have

$$\sum_{i=1}^{k} p_i \geq \sum_{i=1}^{k} q_i.$$

For example, the partition $(5, 4, 2) \geq (5, 3, 2, 1)$, but $(5, 3, 3, 2)$ and $(4, 4, 4, 1)$ are incomparable.

Let P be a partition of n we denote by $P^\vee$ the conjugate partition of n: switch rows and columns in the Ferrers diagram of P. Let H be a sequence that occurs as the Hilbert function of an Artinian algebra, and denote by P_H the associated partition of $n = |A|$, $H^\vee$ its conjugate.

$$H = (1, 2, 3, 2, 1), \qquad P_H = \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & & \\ \bullet & & \end{array} \qquad H^\vee = (P_H)^\vee = (5, 3, 1) : \begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & & & & \end{array}$$

Figure 1: Hilbert function, its partition $(3, 2, 2, 1, 1)$, and conjugate (Example 1.8).

The following result is well known.

Lemma 1.6. *[IMM2, Theorem 2.5] Let A be a standard graded Artinian algebra, and let $\ell \in A_1$ be a linear form. Then $P_{\ell,A} \leq H(A)^\vee$ in the dominance partial order on partitions.*

There is an analogous statement for local algebras A (ibid.).

Corollary 1.7. *Let A be an Artinian quotient of A and let $\ell \in \mathfrak{m}_A$. Then $P_{\ell,A}$ has at least as many parts as the Sperner number of $H(A)$.*

Proof. That $H(A)^\vee \geq P_{\ell,A}$ and are partitions of $n = \dim_{\mathsf{k}} A$ is equivalent to $H(A) = \big(H(A)^\vee\big)^\vee \leq P_{\ell,A}^\vee$ [CM, Lemma 6.3.1]. So the largest part of $H(A)$ (viewed as a partition) is less or equal the largest part in $P_{\ell,A}^\vee$, which is just the number of parts of $P_{\ell,A}$. □

Example 1.8 (Comparison of Jordan type for algebra B and associated graded algebra $A = B^*$)**.** (a) Consider the graded CI algebra $A = \mathsf{k}[x, y]/I$, $I = (x^3, y^3) = \mathrm{Ann}(X^2Y^2)$, with $H(A) = (1, 2, 3, 2, 1)$ and $H^\vee = (5, 3, 1)$ (Figure 1). Here

$$P_{\ell,A} = (5, 3, 1) \text{ for } \ell = ax + by \text{ when } ab \neq 0, \text{ but } P_{x,A} = P_{y,A} = (3, 3, 3).$$

The strings for $\ell = x$ are $\{1, x, x^2\}$, $\{y, yx, yx^2\}$, $\{y^2, y^2x, y^2x^2\}$, and $(5, 3, 1) > (3, 3, 3)$.

(b) Consider the non-homogeneous CI algebra $B = \mathcal{R}/J$, with $\mathcal{R} = \mathsf{k}\{x, y\}$, and ideal $J = (x^3, y^3 - x^2y^2) = \operatorname{Ann}(X^2Y^2 + Y^3)$ satisfying $B^* = A$. We have for char $\mathsf{k} = 0$, again $P_{\ell,B} = (5, 3, 1)$ for $\ell = ax + by$ when $ab \neq 0$, and $P_{x,B} = (3, 3, 3)$. But now $P_{y,B} = (4, 3, 2)$, as the multiplication m_y has pre-Jordan basis strings $\{1, y, y^2, y^3 = x^2y^2\}$, $\{x, xy, xy^2\}$, and $\{x^2, x^2y\}$. Applying Algorithm 1.4, a Jordan basis for m_y has the strings $\{1, y, y^2, y^3 = x^2y^2\}$, $\{x, xy, xy^2\}$, and $\{x^2 - y, x^2y - y^2\}$, as y^4, xy^3, and $(x^2 - y)y^2$ are zero. The algebra B is a deformation of A, and $P_{y,B} = (4, 3, 2) > P_{y,A} = (3, 3, 3)$ in the dominance partial order, consistent with Corollary 3.10.

The following example illustrates some of the methods of determining Jordan type for a non-homogeneous AG algebra. See also [IM2, §2.4].

Example 1.9. (Determining Jordan type, C non-homogeneous.) Let $\mathcal{R} = \mathsf{k}\{x, y, z\}$ and $C = \mathcal{R}/\operatorname{Ann} G$, where $G = X^3Y + Y^2Z$. Then C is a non-homogeneous AG algebra, not CI, defined by $\operatorname{Ann} G = (xz,\ yz - x^3,\ z^2,\ xy^2,\ y^3)$, with $H(C) = (1, 3, 3, 2, 1)$ and $H(C)^\vee = (5, 3, 2)$.

i. **Generic Jordan type of C.** Assume char $\mathsf{k} \notin \{2, 3\}$ and consider a general element $\ell \in \mathfrak{m}_C$. We write $\ell = ax + by + cz + h$, with $h \in \mathfrak{m}_C^2$. Suppose $ab \neq 0$. Then $\ell^4 = 4a^3bx^3y \neq 0$. Also, $\ell^3 = a^3x^3 + 3a^2bx^2y + h'$ and $\ell^2x = a^2x^3 + 2abx^2y + h''$, with $h', h'' \in \mathfrak{m}_C^4$ (note that $yz = x^3$ in A, so $y^2z = x^3y \in \mathfrak{m}_C^4$). We can easily check that ℓ^3 and ℓ^2x are linearly independent, so we have already two strings in a pre-Jordan basis for ℓ, namely $\{1, \ell, \ell^2, \ell^3, \ell^4\}$ and $\{x, \ell x, \ell^2x\}$. According to Lemma 3.9 the Jordan type of ℓ in C is at most $(5, 3, 2)$, and we already have two string of lengths 5 and 3, so we will check if we can get a new string of length 2. Note that $\mathfrak{m}_C^3 = \langle \ell^3, \ell^2x, \ell^4\rangle$, so if there is a further string of length two, there must be an order-one element $\alpha \in \mathfrak{m}_C \setminus \mathfrak{m}_C^2$ such that $\ell\alpha \notin \langle \ell^2, \ell^3, \ell^4, \ell x, \ell^2x\rangle$. Using ℓ and x to cancel terms in α if necessary, we can assume that $\alpha = z + g$, with $g \in \mathfrak{m}_C^2$. Now $\ell\alpha = bx^3 + \ell g \in \mathfrak{m}_C^3$, meaning there is no new length-two string. Therefore the Jordan type of ℓ is

$$P_{\ell,C} = (5, 3, 1, 1),$$

and since the set $\{ax + by + cz + h \in \mathfrak{m}_C : ab \neq 0, h \in \mathfrak{m}_C^2\}$ is an open dense subeset of $\mathfrak{m}_C$, this is the generic Jordan type of C (Definition 1.1). We can consider $\{z\}$ and $\{y^2\}$ as new strings to complete the pre-Jordan basis.

ii. **Why we cannot attain a last length-two string**. That a last two-length string is not attainable is related to a construction from [IM1, Proposition 1.33]. The module $Q_C(1)$ can be explained by the relations between the terms Y^2Z and X^3Y in G (we refer to [IM1] for details on the $Q(a)$ modules, introduced by the second author in [I1]; see also Lemma 3.1 below). Here, $Q_C(1)$ has two homogeneous terms:

$$Q_C(1)_1 = \frac{(0 : \mathfrak{m}_C^3)}{\mathfrak{m}_C^2 + (0 : \mathfrak{m}_C^2)} \quad \text{and} \quad Q_C(1)_2 = \frac{\mathfrak{m}_C^2 \cap (0 : \mathfrak{m}_C^2)}{\mathfrak{m}_C^3 + (0 : \mathfrak{m}_C)}. \tag{1.3}$$

Note that Y^2 is not a partial of X^3Y, but all further partials of Y^2 belong to $\langle 1, Y\rangle$, and thus are also partials of X^3Y. So acting on G with z yields $z \circ G = Y^2$, and this means that the class of z is non-zero in $Q_C(1)_1$ (in fact, it generates this module). However, $\mathfrak{m}_\mathcal{R} z \circ G = \langle 1, Y\rangle$, so if $\ell' \in \mathfrak{m}_\mathcal{R}$ is a lifting ot ℓ, we have $\ell' z \circ G = bY + d = (bx^3 + dx^3y) \circ G$, for some $d \in \mathsf{k}$, which explains why $\ell z \in \mathfrak{m}_C^3$ and its class is zero in $Q_C(1)_2$, so the module $Q_C(1)$ is acyclic.

Coincidently, $Q_C(1) = \langle z, y^2 \rangle$, so its generators are the elements we chose for the last two strings of the pre-Jordan basis.[1]

iii. **Special Jordan types of C.** When $\ell = ax + by + cz + h$, $h \in \mathfrak{m}_C^2$ and $ab = 0$, we find lower Jordan types in the dominance order. For instance,

$$P_{x,C} = (4^2, 1^2), \quad P_{y+z,C} = (4, 2^3), \quad P_{y,C} = (3^2, 2^2), \quad P_{x^2,C} = (2^4, 1^2), \quad P_{z,C} = (2^3, 1^4). \quad (1.4)$$

The strings for a pre-Jordan basis for z are particularly interesting, and illustrate the issues of the non-graded case: since $yz = x^3$ a possible choice is $\{1, z\}$, $\{y, x^3\}$, $\{y^2, x^3 y\}$, $\{x\}$, $\{x^2\}$, $\{xy\}$, $\{x^2 y\}$. Note that in the strings $\{y, x^3\}$ and $\{y^2, x^3 y\}$ there is a jump in order: the orders of y and y^2 are 1 and 2, but multipliyiong by z makes these orders jump to 3 and 4, respectively.

iv. **Deformation $C(t)$.** Consider the family of Artinian Gorenstein algebras $\big(C(t)\big)_{t \in \mathsf{k}}$, where $C(t) = \mathcal{R}/\operatorname{Ann} G(t)$ is defined by the dual generator $G(t) = X^3Y + Y^2Z + tYZ^2$. Then $C(0) = C$, and for $t \neq 0$, $C(t)$ is a CI algebra, as $\operatorname{Ann} G(t) = (xz,\ ty^2 - yz + x^3,\ z^2 - tx^3)$. We have $H\big(C(t)\big) = H(C) = (1, 3, 3, 2, 1)$ for all t. We can check that for $t \neq 0$ the Jordan type of $\ell = ax + by + cz + h$, with $h \in \mathfrak{m}_{C(t)}^2$ and $ab \neq 0$, is $P_{\ell, C(t)} = (5, 3, 2) = H\big(C(t)\big)^\vee$, admitting strings $\{1, \ell, \ell^2, \ell^3, \ell^4\}$, $\{x, \ell x, \ell^2 x\}$, and $\{z, \ell z\}$. So the generic Jordan type of $C(t)$, for $t \neq 0$, strictly dominates that of $C = C(0)$ which is $(5, 3, 1, 1)$ (simply domination is required by Lemma 3.9). For x, $y + z$, y, and x^2, we find the same Jordan types in $C(t)$ as in (1.4), but $P_{z, C(t)} = (3^2, 1^4) > P_{z,C}$.

The associated graded algebra is $C(t)^* = R/(xz,\ ty^2 - yz,\ z^2,\ x^4)$, with $R = \mathsf{k}[x, y, z]$, for $t \neq 0$, and $C(0)^* = C^* = R/(xz,\ yz,\ z^2,\ xy^2,\ y^3,\ x^4)$. The generic Jordan type of $C(t)^*$ is the same as that of $C(t)$ as are the special Jordan types of x and x^2, but $P_{y+z, C(t)^*} = (3, 2^3, 1)$, $P_{y, C(t)^*} = (3, 2^3, 1)$, for any $t \in \mathsf{k}$, and $P_{z, C(t)^*} = (2^2, 1^6)$, for $t \neq 0$, $P_{z, C(0)^*} = (2, 1^8)$. All these are dominated by the respective Jordan types in $C(t)$, as expected from Corollary 3.10.

Note 1.10. The *Weyr characteristic.*, an invariant of the similarity class of a matrix introduced by Eduard Weyr in 1885, for our nilpotent maps m_ℓ on A is just the conjugate (switch rows and columns of the Ferrers diagram) of the Jordan partition $P_{\ell, A}$ ([OCV, §2.4]). See [Sh] for an excellent introduction; K. O'Meara and J. Watanabe point out that for some problems the Weyr form may be more useful than the Jordan type [OW]; see also [IMM2, Note p. 371] for further references.

1.3 Historical note.

Lefschetz properties of the cohomology rings of algebraic varieties had been long studied before the algebraists adapted it. R. Stanley showed that graded Artinian complete intersection algebras $A = R/(x_1^{a_1}, \dots, x_r^{a_r})$ satisfy a strong Lefschetz property [St]: he proved this using the hard Lefschetz theorem for the cohomology of the product $\mathbb{P} = \mathbb{P}^{a_1 - 1} \times \cdots \times \mathbb{P}^{a_r - 1}$ of projective spaces. This inspired many to explore the Lefschetz properties of Artinian algebras. Results and open problems at the time concerning Lefschetz properties of graded Artinian algebras were well set out in the 2013 foundational opus by T. Harima-J. Watanabe et al [H-W] and also surveyed by J. Migliore and U. Nagel [MiNa]. Other articles on the Lefschetz properties include

[1] Further examples and discussion of these points are found in [IM2, §2.4, Remark 2.11ff.].

the T. Harima articles [Ha1, Ha2] in 1995 and 1999, the 2011 B. Harbourne, H. Schenck, and A. Seceleanu on Gelfand-Tsetlin patterns and the weak Lefschetz property [HScSe], and the direction of singular hypersurfaces and Lefschetz properties by R. Di Gennaro, G. Ilardi, and J. Vallès [DIV], a direction continued by many as E. Mezzetti, R.M. Miró-Roig, G. Ottaviani on Laplace Equations and weak Lefschetz [MeMO] and R.M. Miró-Roig and M. Salat on Togliatti equations [MR-S].

Despite advocacy since 2012 at the Lefschetz conference organized by Junzo Watanabe at Tokai University, of the second author for using the finer Jordan type invariant for a pair (ℓ, A), it was not until [IMM2] that an introduction to the topic was written. This was at the instigation of Yong-Su Shin, who asked prior to coauthoring [PaSh], where one could find an introduction to Jordan type! There was none. The authors of [IMM2] attempted to give a comprehensive introduction, including new results, doing for Jordan type what J. Migliore and U. Nagel had done earlier in the same journal in "Tour of the strong and weak Lefschetz Properties" [MiNa]. For some topics, such as modular tensor products, they were able to exhibit several threads of work by different communities who seemed unaware of each other's work on the same subject [IMM2, §3B]. Several other articles by the same group treated Jordan type for certain free extensions, which are deformations of tensor products [IMM1, Theorem 2.1]; see also [MCIM] which gives a connection of free extensions to invariant theory.

A main advance in the study of Lefschetz properties of Artinian Gorenstein (or AG) algebras was the article of T. Maeno and J. Watanabe, showing that the ranks of multiplication by powers of a linear form ℓ on the degree components A_i of a graded Gorenstein algebra A was given by the ranks of certain higher Hessians formed from the Macaulay dual generator of A, at a point p_ℓ [MW]. This result was extended and used by many, including R. Gondim [Go], Gondim and G. Zappalà [GoZ1], and it was generalized in [GoZ2], to the mixed Hessians. These have been used to prove that some Nagata idealization examples of graded AG algebras in embedding dimension at least four, are not strong Lefschetz (as [CeGIM]. The Hessian tools have been used recently by a growing cohort to study Jordan types for pairs (A, ℓ) where A is a graded AG algebra and $\ell \in A_1$ (see, for example [Al1, AlIK, AlIKY, Y1, Y2] and Section 2.2 below).

Recent articles on Jordan type and Artinian algebras.

We here mention several recent articles and research areas, with emphasis on those that mention Jordan type. Fixing codimension two, and a Hilbert function H, we can study the "Jordan cells" $\mathbb{V}(E_P)$ of the family G_H, comprised of ideals having initial monomial ideal E_P in a direction given by a linear form ℓ, determined by the partition P, which must have "diagonal lengths" T: see [AlIKY, Theorem 2.8]. The cell $\mathbb{V}(E_P)$ is comprised of all graded Artinian algebras $A = \mathsf{k}[x, y]/I$ such that $P_{\ell,A} = P$. The dimension of these cells, and some of their geometric properties were known [Y1, Y2, IY]; the article [AlIKY] determines the generic number of generators of ideals in each cell [AlIKY, Theorems 3.11, 5.15] using a decomposition of cells into a product of simpler components. See Question 4.3.

There has been the beginning of tying the Jordan type with the Betti resolution of A, see N. Abdallah and H. Schenck [AbSc] and N. Abdallah is [Ab], and as well J. Jelisiejew, S. Masuti and M. Rossi's [JeMR], where they investigate local complete intersections of codimension three, also the book-length [KKRSSY] has some Betti vs. Jordan type calculations.

Not totally unrelated, the preprint [AAIY] studies Jordan types for codimension three graded Gorenstein algebras of Sperner number at most 5 and all linear forms. This is facilitated by the D. Buchsbaum-D. Eisenbud Pfaffian structure theorem and related work [BuEi, CoV, Di] which specifies the Betti resolutions possible given $H(A)$. The results are still complex with 26 Jordan types for $H = (1, 3, 4^k, 3, 1)$ when $k \geq 3$ and 47 for $H = (1, 3, 5^4, 3, 1)$.

In [A-N] the weak Lefschetz property and Jordan types for linear forms of a class of graded AG algebras, called Perazzo algebras, of codimension five were studied. For Perazzo algebras, the multiplication map ℓ^{j-2} from degree 1 to degree $j-1$ does not have maximal rank, where j is the socle degree. Thus, the strong Lefschetz property for this family is not satisfied. In [A-N] all Jordan types for linear forms of Perazzo algebras of codimension five with the smallest possible Hilbert function were determined.

2 Properties of Jordan type, and of Jordan degree type.

2.1 Lefschetz properties and Jordan type.

Definition 2.1 (Lefschetz properties)**.** Let A be a graded Artinian algebra of highest socle degree j and let $\ell \in A_1$. We say that the pair (A, ℓ) is a *weak Lefschetz* (WL) if for each $i \geq 0$ the multiplication map $\times \ell : A_i \to A_{i+1}$ has maximal rank. The algebra A satisfies the *weak Lefschetz property* (WLP) if it has a WL element. We say that the pair (ℓ, A) is *strong Lefschetz* (SL) if for each $i, d \geq 0$ the multiplication map $\times \ell^d : A_i \to A_{i+d}$ has maximal rank. The algebra A satisfies the *strong Lefschetz property* (SLP) if it has a SL element.

The following result part A is a portion of [IMM2, Prop. 2.10]; part B is essentially [IMM2, Lemma 2.11], shown when $H(A)$ is also symmetric in [H-W, Prop. 3.5]. We say that a Hilbert function $H(A) = (h_0, h_1, \ldots, h_j)$ is *unimodal* if there is an integer k such that $h_0 \leq \cdots \leq h_k$ and $h_k \geq h_{k+1} \geq \cdots \geq h_j$. Recall that the Sperner number $\mathrm{Sperner}(A)$ is the maximum value of $H(A)$.

Lemma 2.2. *A. Let A be a graded Artinian algebra (possibly non-standard), and $\ell \in A_1$. Then the following are equivalent*

(i.) The pair (A, ℓ) is strong Lefschetz;

(ii.) The Jordan type $P_{A,\ell} = H(A)^\vee$, the conjugate of the Hilbert function viewed as a partition.

B. Assume further that $H(A)$ is unimodal. Then the following are equivalent

(i). The pair (A, ℓ) is weak Lefschetz.

(ii). The dimension $\dim_k A/\ell A = \mathrm{Sperner}(A)$.

(iii). The number of parts of the Jordan partition $P_{A,\ell}$ is $\mathrm{Sperner}(A)$, the minimum possible given $H(A)$ (Corollary 1.7).

Proof. The proof of Lemma 2.2(A) under the hypothesis is a bit subtle see [IMM2, Prop. 2.10]. For Lemma 2.2(B), the proof of B(i) $\Leftrightarrow$ B(ii) is straightforward from the definitions; the proof of B(ii) $\Leftrightarrow$ B(iii) follows from decomposing A as a direct sum of *strings* (Lemma 1.3). $\square$

2.2 Higher Hessians and mixed Hessians.

Graded Artinian Gorenstein algebras are determined by a single polynomial in the Macaulay dual ring, by a result of F.H.S. Macualay [Mac2]. Let $A = R/\operatorname{Ann} F$ be an Artinian Gorenstein algebra with dual generator $F \in \mathcal{E}_j = \mathsf{k}[X_1, \dots, X_r]_j$, where $\operatorname{Ann} F$ is the ideal generated by all the forms $g \in R$ such that $g \circ F = 0$. T. Maeno and J. Watanabe [MW] introduced higher Hessians associated to the dual generator F and provided a criterion for Artinian Gorenstein algebras having the SLP.

We first briefly recall the Macaulay duality [Mac3], see [Ei, §21.2], [I2]; the recent emendation by J. O. Kleppe and S. Kleiman gives a geometric view consistent with studying deformation [KlKl]. We let $R = \mathsf{k}[x_1, \dots, x_r]$ act on $\mathcal{E}$ by contraction[2] where for $u \geq k$, $x_i^k \circ X_j^u = \delta_{i,j} X_i^{u-k}$ (we will call this $\partial^k/\partial X_i^k \circ X_j^u$) and extending this multilinearly to an action of $h \in R$ on $F \in \mathcal{E}$.

$$h \circ F = h(\partial/\partial X_1, \dots, \partial/\partial X_r) \circ F, \tag{2.1}$$

so taking $F = X_1^3 X_2^2 + X_1 X_2^4$ we have $x_1 x_2^2 \circ F = X_1^2 + X_2^2$.

Definition 2.3. [MW, Definition 3.1] Let F be a polynomial in $\mathcal{E}$ and $A = R/\operatorname{Ann} F$ be its associated Gorenstein algebra. Let $\mathcal{B}_k = \{\alpha_i^{(k)}\}_i$ be an ordered k-basis of A_k. The entries of the k-th Hessian matrix of F with respect to $\mathcal{B}_k$ are

$$\operatorname{Hess}^k(F) = (\alpha_u^{(k)} \alpha_v^{(k)} F)_{u,v}.$$

Note that when $k = 1$, $\operatorname{Hess}^1(F)$ coincides with the usual Hessian. P. Gordan and M. Noether proved that the (first) Hessian of every homogeneous form F in at most 4 variables has non-zero determinant unless F defines a cone [GorNo]. This is no longer the case in polynomial rings with at least 5 variables: a family of forms that do not define a cone and for which the Hessian has zero determinant was provided by [GorNo] and [Per], they are called Perazzo forms.

Up to non-zero constant multiple, $\det \operatorname{Hess}^k(F)$ is independent of the choice of basis $\mathcal{B}_k$. For every $0 \leq k \leq \lfloor \frac{j}{2} \rfloor$ and a linear form $\ell = a_1 x_1 + \cdots + a_r x_r$ the rank of $\times \ell^{j-2k} : A_k \to A_{j-k}$ is equal to the rank of $\operatorname{Hess}^k_\ell(F)$; i.e. the Hessian matrix evaluated at the point $P_\ell = (a_1, \dots, a_r)$ – see Theorem 2.6 below. For now we state,

Theorem 2.4. *[MW, Theorem 3.1],[Wat2]. An AG algebra $A = R/\operatorname{Ann} F$ with socle degree j has the SLP if and only if there exists linear form $\ell \in R_1$ such that*

$$\det \operatorname{Hess}^k_\ell(F) \neq 0,$$

for every $k = 0, \dots, \lfloor \frac{j}{2} \rfloor$.

As mentioned above, for Perazzo forms F the determinant of the first order Hessian, $\operatorname{Hess}^1(F)$, is identically zero. So by the above theorem the associated AG algebra of a Perazzo form fails to have the SLP. The WLP and Jordan types of Perazzo algebras in 5 variables have been studied in [FMM] and [A-N].

[2] When char $\mathsf{k} = 0$ or char $\mathsf{k} > j$ we may use the usual differentiation action, see [IKa, Appendic A].

For an AG algebra for which all higher Hessians have non-vanishing determinants, the above theorem shows that for a general enough linear form ℓ all the multiplication maps $\ell^{j-2k} : A_k \to A_{j-k}$ have maximal rank. It is natural to ask: if an AG algebra A has at least one Hessian with vanishing determinant, which multiplication maps have maximal rank and which ones do not? R. Gondim and G. Zappalà [GoZ2] introduced mixed Hessians that generalize the higher Hessians.

Definition 2.5. [GoZ2, Definition 2.1] Let $A = R/\operatorname{Ann} F$ be the AG algebra associated to $F \in \mathcal{E}_j$. Let $\mathcal{B}_k = \{\alpha_i^{(k)}\}_i$ and $\mathcal{B}_u = \{\beta_i^{(u)}\}_i$ be k-basis of A_k and A_u respectively. The entries of the *mixed Hessian* matrix of order (k,u) for F with respect to $\mathcal{B}_k$ and $\mathcal{B}_u$ is given by

$$\operatorname{Hess}^{(k,u)}(F) = (\alpha_u^{(k)} \beta_v^{(u)} F)_{u,v}.$$

Notice that this generalizes the definition of higher Hessians and we have $\operatorname{Hess}^k(F) = \operatorname{Hess}^{(k,k)}(F)$.

Theorem 2.6. *[GoZ2, Theorem 2.4]. Let A be an AG standard graded algebra. Then the rank of the mixed Hessian matrix of order (k,u) evaluated at the point $P_\ell = (a_1, \dots, a_r)$ is the same as the rank of the multiplication map $\ell^{u-k} : A_k \to A_u$ for $\ell = a_1x_1 + \cdots + a_rx_r$.*

The method of higher Hessians and mixed Hessians has been used to study the Lefschetz properties for graded AG algebras in the literature, for instance see [Al1, Go, FMM]. The ranks of higher and mixed Hessians together at the point P_ℓ completely determine the ranks of multiplication maps by different powers of the linear form ℓ in all degrees, and hence, when the Hilbert function $H(A)$ is unimodal, the Jordan degree type of (ℓ, A) (Proposition 2.19). B. Costa and R. Gondim in [CGo] determined the Jordan types for general linear forms of AG algebras having low codimension and low socle degree in terms of the ranks of the associated mixed Hessians.

The first and second authors with L. Khatami classified [AlIK] all partitions that can occur as Jordan types of linear forms for AG algebras in codimension two (these are exactly complete intersection algebras by [Mac1]) having a fixed Hilbert function. It has been shown that in codimension two, the Jordan types of linear forms of AG algebras are completely determined by the rank of higher Hessians. In fact, they are uniquely determined by the sets of higher Hessians that have vanishing determinants.

Theorem 2.7. *[AlIK, Theorem 3.8] Assume that $H = \big(1, 2, 3, \dots, d^k, \dots, 3, 2, 1\big)$, is a Hilbert function of some complete intersection algebra for $d \geq 2$ and $k \geq 2$ ($k = 1$, respectively). Let P be a partition that can occur as the Jordan type of a linear form and an Artinian complete intersection algebra having Hilbert function H. Then the following are equivalent.*

(i). $P = P_{\ell,A}$ for a linear form $\ell \in R_1$ and an Artinian complete intersection algebra $A = R/\operatorname{Ann} F$, and there is an ordered partition $n = n_1 + \cdots + n_c$ of an integer n satisfying $0 \leq n \leq d$ (or $0 \leq n \leq d-1$, respectively) such that $\det \operatorname{Hess}_\ell^{n_1+\cdots+n_i-1}(F) \neq 0$, for each $1 \leq i \leq c$, and the remaining Hessians are zero;

(ii). P satisfies

$$P = \big(p_1^{n_1}, \dots, p_c^{n_c}, (d-n)^{d-n+k-1}\big), \tag{2.2}$$

where $p_i = k - 1 + 2d - n_i - 2(n_1 + \cdots + n_{i-1})$, for $1 \leq i \leq c$.

The above theorem shows that given H there are exactly 2^d (when $k > 1$), or 2^{d-1} (when $k = 1$) possible Jordan types for Artinian complete intersection algebras of codimension two with Hilbert function H. These correspond to the vanishing subsets of higher Hessians, $\mathrm{Hess}^i(F)$ for $i = 0, \ldots, d-1$.

In codimension two, the ranks of mixed Hessians $\mathrm{Hess}^{(k,u)}(F)$ for every $k \neq u$ are determined by the ranks of higher Hessians $\mathrm{Hess}^k(F)$ for $0 \leq k \leq d-1$ [AlIK, Proposition 3.12]. This is no longer true in higher codimensions. The first author in [Al2] introduced rank matrices associated to AG algebras and linear forms which provide same information as the ranks of mixed Hessians. The first and second authors with N. Abdallah and J. Yameógo in a work in progress study all partitions that can occur as Jordan types for AG algebras of codimension three and low Sperner numbers (maximum value of the Hilbert function) [AAIY].

2.3 Jordan degree type.

If A is a graded algebra, quotient of the polynomial ring R, and M is a graded module over A (possibly non-standard graded), we can consider a refinement of Jordan type, called *Jordan degree type* (see [IMM2, Definition 2.28], where several equivalent ways of defining it are presented and different notations are proposed).

Definition 2.8. Let A be a graded algebra and let M be a finite graded module over A. Let $\ell \in A_k$, $k \geq 1$ be a homogeneous element. If $\mathcal{B}$ is a Jordan basis for the multiplication by ℓ in M as in (1.1) and all the elements of $\mathcal{B}$ are homogeneous then the *Jordan degree type* of ℓ in M is the sequence of pairs of integers

$$\mathcal{S}_{\ell,M} = \big((p_1, \nu_1), \ldots, (p_s, \nu_s)\big),$$

where ν_k is the degree of the first bead z_k in the string $S_k = \{z_k, \ell z_k, \ldots, \ell^{p_k-1} z_k\}$, and, by reordering the strings if necessary, we require that if $p_k = p_{k+1}$ for some k then $\nu_k \leq \nu_{k+1}$.

We may write the Jordan degree type in list manner:

$$\mathcal{S}_{\ell,A} = \sum_p \sum_\nu (p, \nu)^{\eta(p,\nu)}, \tag{2.3}$$

where $\eta(p, \nu)$ is the multiplicity of (p, ν), and the sum is over distinct pairs (p, ν).

For convenience we may write the pair (p, ν) as p_ν, and abbreviate a list of pairs having consecutive degrees $(p, \nu), (p, \nu + 1), \ldots, (p, \nu + e)$ as $(p \uparrow_\nu^{\nu+e})$ in listing Jordan degree type – e.g. the list $(5, 2), (5, 3), (5, 4)$ could be written as $(5 \uparrow_2^4)$.

Example 2.9. Consider the family of graded Artinian algebras $\big(C(t)^*\big)_{t\in\mathsf{k}}$ associated to $C(t)$ from Example 1.9iv. For $t \neq 0$, the Jordan degree type of any linear form $\ell = ax + by + cz$, with $ab \neq 0$, satisfies $\mathcal{S}_{\ell,C^*} = \big((5, 0), (3, 1), (2, 1)\big)$, which is the generic JDT of $C(t)^*$, and is the JDT associated with the Hilbert function $H = (1, 3, 3, 2, 1)$ (see Note 2.16). If $t = 0$, $C^* = C(0)^*$ has generic JDT $\mathcal{S}_{\ell,C^*} = \big((5, 0), (3, 1), (1 \uparrow_1^2)\big)$. A few examples of special Jordan degree types are $\mathcal{S}_{x,C(t)^*} = \big((4 \uparrow_0^1), (1 \uparrow_1^2)\big)$ and $\mathcal{S}_{y,C(t)^*} = \big((3, 0), (2 \uparrow_1^3), (1, 1)\big)$, for any $t \in \mathsf{k}$, and $\mathcal{S}_{z,C(t)^*} = \big((2 \uparrow_0^1), (1 \uparrow_1^4), (1 \uparrow_2^3)\big)$, for $t \neq 0$, while $\mathcal{S}_{z,C(0)^*} = \big((2, 1), (1 \uparrow_1^3), (1 \uparrow_2^4), (1 \uparrow_1^2)\big)$.

Jordan degree type also enjoys a semi-continuity property. To be able to state it, we need to define a Hilbert function associated with a Jordan degree type and a partial order among Jordan degree types sharing the same Hilbert function. We will adopt here purely combinatorial definitions, without requiring that a given sequence is the Jordan degree type of some linear form in a module.

Definition 2.10. Let $\mathcal{S} = \big((p_1,\nu_1),\dots,(p_s,\nu_s)\big)$ be a sequence of pairs of non-negative integers satisfying

$$p_1 \geq \cdots \geq p_s \qquad \text{and} \qquad \nu_k \leq \nu_{k+1} \text{ whenever } p_k = p_{k+1}. \tag{2.4}$$

Then

(i.) The *partition associated to* $\mathcal{S}$ is $P = (p_1,\dots,p_s)$.

(ii.) The *Hilbert function associated to* $\mathcal{S}$ is that defined by

$$H(\mathcal{S})_i = \#\{k : \nu_k \leq i < \nu_k + p_k\}.$$

(iii.) For each $i \geq 0$, we define the *truncation* $\mathcal{S}_{\leq i}$ as the sequence we obtain from $\mathcal{S}$ by omitting the pairs for which $\nu_k > i$, then replacing each other pair (p_k,ν_k) in $\mathcal{S}$ by the pair $(\min\{p_k, i+1-\nu_k\},\nu_k)$.

(iv.) We say that two pairs (p_k,ν_k), (p_l,ν_l), can be *combined* or *concatenated* if $\nu_l = \nu_k + p_k$. In this case, the result of the concatenation is the pair (p_k+p_l,ν_k).

Remark 2.11. Note that if $\mathcal{S}$ is the Jordan degree type of a linear form ℓ in a finite graded module M over an Artinian algebra A then $H(\mathcal{S}) = H(M)$. Moreover, the truncation $\mathcal{S}_{\leq i}$ is the Jordan type of ℓ in the module $M/\mathfrak{m}^{i+1}M$.

The following definition is adapted from [IMM2, Definition 2.28 (iii) and (vi)].

Definition 2.12 (Concatenation partial order and dominance partial order)**.** Let $\mathcal{S}$ and $\mathcal{S}'$ be two sequences of pairs of non-negative integers satisfying (2.4), with $H(\mathcal{S}) = H(\mathcal{S}')$.

(i.) Concatenation partial order: we say that $\mathcal{S} \geq_c \mathcal{S}'$ if $\mathcal{S}$ can be obtained from $\mathcal{S}'$ by a sequence of concatenations.

(ii.) Dominance order for JDT: we say that $\mathcal{S} \geq \mathcal{S}'$ if for each $i \geq 0$, the partition associated to $\mathcal{S}_{\leq i}$ is greater or equal to that associated to $\mathcal{S}'_{\leq i}$, in the dominance partial order (Definition 1.5).

In [IMM2, Definition 2.28 (vi)], the dominance partial order is defined only for Jordan degree types sharing the same partition. Here we have adopted an extension of this order, comparing sequences which may have different associated partitions.

Remark 2.13. Note that if $\mathcal{S} \geq_c \mathcal{S}'$ then $\mathcal{S} \geq \mathcal{S}'$. To see this, suppose that $\mathcal{S}$ is obtained from $\mathcal{S}'$ by one concatenation, of the pairs (p_k,ν_k) and (p_l,ν_l). If $i < \nu_l$ then $\mathcal{S}_{\leq i} = \mathcal{S}'_{\leq i}$. If $i \geq \nu_l$ then the pair $(\min\{p_k+p_l, i+1-\nu_k\},\nu_k)$ occurs in $\mathcal{S}_{\leq i}$, while the pairs (p_k,ν_k) and $(\min\{p_l, i+1-\nu_l\},\nu_l)$ occur in $\mathcal{S}'_{\leq i}$, and this is the only difference between $\mathcal{S}_{\leq i}$ and $\mathcal{S}'_{\leq i}$. In any case, we see that the partition associated to $\mathcal{S}_{\leq i}$ is greater or equal to the one associated to $\mathcal{S}'_{\leq i}$. The result follows by transitivity.

Example 2.14. Let $\mathcal{S} = \big((4,0),(2,1)\big)$ and $\mathcal{S} = \big((3,0),(3,1)\big)$. Then $\mathcal{S} \geq \mathcal{S}'$, but $\mathcal{S}$ and $\mathcal{S}'$ are not comparable under the concatenation partial order. This shows that the dominance partial order is a proper extension of the concatenation partial order.

Lemma 2.15. *[IMM2, Lemma 2.29] Fix $\ell \in R_1$. Let $\big(A(w)\big)_{w\in W}$ be a family of graded Artinian algebras with constant Hilbert function $H\big(A(w)\big) = H$, and constant Jordan type $P_{\ell,A(w)} = P_\ell$. Let $w_0 \in W$ be such that for $w \in W \setminus \{w_0\}$, also the Jordan degree type is constant, $\mathcal{S}_{\ell,A(w)} = \mathcal{S}_\ell$. Then $\mathcal{S}_\ell \geq \mathcal{S}_{\ell,A(w_0)}$ in the dominance partial order of Definition 2.12(ii).*

Note 2.16. We may assign a contiguous Jordan type and JDT(H) to a Hilbert function: we just regard H as a bar graph, and list the rows of the bar graph with their initial degrees [IMM2, Definition 2.8ii]. In [IMM2, Prop. 2.32] is shown a concatenation inequality JDT$(H) \geq_c \mathcal{S}_{\ell,A}$ for a graded A of Hilbert function H.

However, using our extended Definition 2.12ii. above of dominance of JDT, we have also that for A graded with $H(A) = H$,

$$\text{JDT }(H) \geq \mathcal{S}_{\ell,A}, \tag{2.5}$$

in the dominance partial order. This follows from the inequality $(H_{\leq i})^\vee \geq P_{\ell,A/\mathfrak{m}_A^{i+1}}$, the Jordan type of $A/\mathfrak{m}_A^{i+1}$, for each $i \in [0,j]$, and the equivalence between JDT and the sequential Jordan type (Lemma 3.7).

Remark 2.17. Note that Lemma 2.15 implies an analogous result for the Jordan type in a family of graded algebras of fixed Hilbert function. This is generalized in Proposition 3.11 to families of possibly non-graded algebras.

The Jordan degree type for a graded Artinian algebra is a finer invariant than the Jordan type [IMM2, Example 3.1]. Taking $B = \mathsf{k}[x,y,z]/(yz, x^2y, xy^2, z^3, x^4, y^4)$ and $A = \mathsf{k}[x,y,z]/(y^2, x^2z, x^2y, z^3, x^6)$, we have $H(A) = H(B) = (1,3,5,4,2,1)$ and $P_{z,A} = P_{z,B} = (3^4, 1^4)$; but $\mathcal{S}_{z,A} = \big((3\uparrow_0^2), 3_1, (1\uparrow_2^5)\big)$ and $\mathcal{S}_{z,B} = \big((3\uparrow_0^3), (1\uparrow_1^3), 1_2\big)$.

The first author in [Al3, Example 4.2] provided AG algebras A and B, quotients of R, having Hilbert function $(1,3,6,9,9,9,6,3,1)$ and a linear form $\ell \in R_1$ for which the Jordan types of pairs (A,ℓ) and (B,ℓ) are equal but their Jordan degree types are different. It is stated in the same paper that this Hilbert function has the smallest socle degree in codimension three, for which there is a pair of AG algebras having the same property. However, in codimension greater than three there are pairs of such AG examples of smaller socle degree [Al3, Example 4.4].

However, in codimension two, the Jordan degree type of an Artinian graded algebra is determined by the Jordan type, a result that follows from J. Briançon's vertical strata in [Bri].

Lemma 2.18 (Codimension two JDT)**.** *[IMM2, Lemma 2.30] When A is a standard graded algebra of codimension two, and A has Jordan type $P_{\ell,A} = (p_1, p_2, \dots, p_s)$, $p_1 \geq \cdots \geq p_s$ with respect to an element $\ell \in A_1$ and* char $\mathsf{k} = 0$ *or* char $\mathsf{k} > j$*, the socle degree of A, then the Jordan degree type satisfies*

$$\mathcal{S}_{\ell,A} = ((p_1,0),(p_2,1),\dots,(p_k,k-1),\dots,(p_s,s-1))\,.$$

2.4 Avatars of Jordan degree type.

The information in the Jordan degree type $\mathcal{S}_{\ell,A}$ is the same as the numerical information in the central simple modules with respect to ℓ of T. Harima and J. Watanabe [HW1, HW2], (see [IMM2, Lemma 2.34]). We now show this is equivalent to the information in the ranks of the maps by powers of ℓ from the graded pieces of A. Recall from Equation (2.3) that $\eta(p,\nu)$ is the multiplicity of the string (p,ν) in the JDT $\mathcal{S}$.

Proposition 2.19. *Let A be a standard graded Artinian algebra of socle degree j. Then the Jordan degree type $\mathcal{S}_{\ell,A}$ is equivalent to knowing the ranks of each of the multiplication maps $\ell^{u-k}: A_k \to A_u$, for $0 \le k < u \le j$.*

In particular, if A is graded Artinian Gorenstein the JDT $\mathcal{S}_{\ell,A}$ is determined by the set of ranks of all the mixed Hessians, and vice-versa.

Proof. The second statement concerning mixed Hessians follows from the first and Theorem 2.6. Now, given the JDT, we determine the ranks of maps $\ell^{u-k}: A_k \to A_u$ first by order of k, then u. For $k = 0$ the rank of $\ell^u : A_0 \to A_u$ is just the number of strings beginning in degree 0 and ending in degree at least u. For $k > 0$, we have

$$\mathrm{rk}(\ell^{u-k}: A_k \to A_u) = \#\{(p,\nu) \in \mathcal{S}_{\ell,A} \mid \nu \le k, p > u - \nu\}.$$

For the converse, it is easy to see that for $0 \le \nu \le j$ and $1 \le p \le j + 1 - \nu$ the component (p,ν) appears $\eta(p,\nu)$ many times in $\mathcal{S}_{\ell,A}$ where

$$\begin{aligned}\eta(p,\nu) &= \mathrm{rk}(\ell^{p-1}: A_\nu \to A_{p+\nu-1})\\ &\quad - \left[\mathrm{rk}(\ell^{p}: A_{\nu-1} \to A_{p+\nu-1}) + \mathrm{rk}(\ell^{p}: A_{\nu} \to A_{p+\nu}) - \mathrm{rk}(\ell^{p+1}: A_{\nu-1} \to A_{p+\nu})\right].\end{aligned}$$

□

For AG algebras the JDT is symmetric - this was first shown by T. Harima and J. Watanabe in the context of properties of central simple modules [HW1], then noted by B. Costa and R. Gondim [CGo, Lemma 4.6]; for a proof see also [IMM2, Proposition 2.38]. This property is useful in determining the possible JDT given a Hilbert function, and is a strong reason for using JDT instead of just JT for graded AG algebras in codimension at least three.[A-N, AAIY].

Proposition 2.20. *(Symmetry of JDT) Let A be a graded AG algebra and $\ell \in A_1$, and write in list manner $\mathcal{S}_{\ell,A} = \sum_p \sum_\nu (p,\nu)^{\eta(p,\nu)}$ (Equation (2.3)). Then we have for $\nu \le j/2$*

$$\eta(p,\nu) = \eta(p, j + 1 - \nu - p). \tag{2.6}$$

Remark 2.21. (Using dual strings to calculate JDT) The JDT, like the JT for a graded Artinian algebra may be calculated by considering dual strings in $A^\vee = \mathrm{Hom}(A, \mathsf{k}) = R \circ F$. Here $R = \mathsf{k}[x, y, z]$.

If A is standard graded, and ℓ nilpotent, then $A^\vee$ has a decomposition as a direct sum

$$A^\vee = \sum_i k[\ell] \circ v_i \text{ of simple } k[\ell] \text{ modules.}$$

We set $p_i = \dim_{\mathsf{k}} k[\ell] \circ v_i$ and recall that the Jordan degree type $\mathcal{S}(A) = \{(p_i, \deg v_i)^{\eta(p_i, \deg v_i)}\}$, where the pairs $(p_i, \deg v_i)$ are distinct and $\eta(p_i, \deg v_i)$ is the multiplicity.

Let $A = R/I$ be a standard graded Artinian $\mathsf{k}[x]$-module, $R = \mathsf{k}[x_1, \dots, x_r]$ and let the action of x have degree one. Denote by $\overrightarrow{s} : (p, v)$ a string-simple $k[x]$-module (p, v). If $\overrightarrow{s} \subset A$, we may write it as $(p, v) = \{v, xv, \dots, x^{p-1}v\}$ in degrees $(\deg v, \dots, \deg v + p - 1)$; we let $\overrightarrow{s}^\vee$ denote an abstract dual string $(w, x \circ w, \dots, x^{p-1} \circ w)$, in degrees $\deg v + p - 1, \dots, \deg v$ in the $\mathsf{k}[x]$ module $\mathsf{k}_{DP}[X_1, \dots, X_r]$, upon which x acts as a contraction lowering degree.

Lemma 2.22. *(Using ℓ-strings in $A^\vee$ to find JDT) Using the notation above let the graded Artinian $\mathsf{k}[x]$-module A satisfy the isomorphism $A \cong \oplus_i(\overrightarrow{s}_i)$ where $\overrightarrow{s}_i$ are the simple graded dual $k[x]$-modules. Then we have that* $\operatorname{Hom}(A, k) \cong \oplus_i \overrightarrow{s}_i^\vee$*, a direct sum of graded simple $\mathsf{k}[x]$-modules (strings of the same length). Assume $\mathcal{S}(A) = \oplus_i\{(p_i, n_i)^{\eta(p_i,n_i)}\}$. Then we have*

$$\mathcal{S}^\vee(A) = \{(p_i, n_i + (p_1 - 1))^{\eta(p_i,n_i)}\}. \tag{2.7}$$

Proof. Writing A as a finite direct sum of simple $\mathsf{k}[x]$-modules, we have $\operatorname{Hom}(A, \mathsf{k})$ is the analogous finite direct sum of their duals. $\square$

Example 2.23. The Artinian algebra $A = \mathsf{k}[x, y]/(x^3, y^3, x^2y^2)$ has Hilbert function $H = (1, 2, 3, 2, 0)$ and x-strings $\{1, x, x^2\}$, $\{y, yx, yx^2\}$, and $\{y^2, y^2x\}$ so A has SJT $(3_0, 3_1, 2_2)$. Here $A = R/I$, $I = \operatorname{Ann}(X^2Y, XY^2)$, whose x-strings are $\{X^2Y, XY, Y\}$, $\{X^2, X, 1\}$, and $\{XY^2, Y^2\}$ so $\mathcal{S}^\vee(A) = (3_3, 3_2, 2_3)$ which for socle degree 3 corresponds to $\mathcal{S}(A)$, by Equation (2.7).

Example 2.24. [AAIY] Let $j \geq 4$, take $F = X^j + X^2Y^{j-2} + XZY^{j-2}$, let $A = R/\operatorname{Ann} F$. Then $\operatorname{Ann} F = I = (z^2, x^2z, x^2y - xyz, y^{j-1}, x^{j-1} - y^{j-2}z)$ and the Hilbert function $H(A) = (1, 3, 5^{j-3}, 3, 1)$.[3] We take $\ell = x$ and claim that $\mathcal{S}_{x,A} = \big((j + 1)_0, (3 \uparrow_1^{j-3}), 2_{1,j-2}, (1 \uparrow_2^{j-2})\big)$. There are corresponding strings in $A^\vee$ with (dual) generators as $\mathsf{k}[x]$-modules, $(j + 1)_0$ corresponds to F. For every $1 \leq i \leq j - 3$ we have $y^i \circ F = X^2Y^{j-i-2} + XZY^{j-i-2}$ which explains $(3 \uparrow_1^{j-3})$. Now let $(y^{j-2} - x^{j-2}) \circ F = XZ$ and $z \circ F = XY^{j-2}$ so there are strings of length two in degrees 1 and $j - 2$; i.e. $2_{1,j-2}$ exists in $\mathcal{S}_{x,A}$. That leaves $(1 \uparrow_2^{j-2})$ to be explained. The difference $H(A)$ minus the HF of the Jordan type so far is just $(0, 0, 1, 1, \dots, 1_{j-2})$. Since there can be no further strings in $A^\vee$ of length 2 (which require an $X \cdot h(Y, Z)$ dual generator), the symmetry of JDT forces that $(1 \uparrow_2^{j-2})$ is the last set of $j - 3$ strings.

Note that the limit of a family of graded Gorenstein algebras need not be Gorenstein, and need not have symmetric Jordan degree type, even for simple Hilbert function as $H = (1, 3, 3, 1)$ ([AEIY, Example 4.16].

Question 2.25. Can we use Lemma 2.15 to show the existence of irreducible components in the constant Jordan type strata of an algebra A, or in a family of Artinian algebras?

[3]To verify the Hilbert function $H(A)$ it helps that according to [BuEi, St, Di] (see [IKa, Thm. 5.25]) that a codimension three Gorenstein ideal of order κ can have at most $2\kappa + 1$, generators, the given ideal is in $\operatorname{Ann} F$, and the first three generators of I define a length 5 scheme, of Hilbert function $(1, 3, \overline{5})$.

Warning: Dual strings are not usually obtained by just changing variables from $x, y, \dots$ to $X, Y, \dots$!

3 Non-graded algebras, and further invariants.

One nice application of Jordan type is to show that certain families of non-graded Artinian Gorenstein algebras with given Hilbert function have several irreducible components (Section 3.1). Let H be a *Gorenstein sequence*, i.e. a sequence of integers that is the Hilbert function of some AG algebra (not necessarily graded). Let $\operatorname{Gor}(H)$ be the family of AG algebras whose Hilbert function is H. Since Jordan type is semi-continuous, if A and B are two algebras in $\operatorname{Gor}(H)$ whose generic Jordan types satisfy $P_A > P_B$ then A cannot lie on the border of an irreducible component of $\operatorname{Gor}(H)$ whose general element has Jordan type P_B. On the other hand, the symmetric decomposition of the Hilbert function of an AG algebra provides another semi-continuous invariant, namely $N_{i,b} = \dim_{\mathsf{k}}\big(\mathfrak{m}^i/\big(\mathfrak{m}^i \cap (0 : \mathfrak{m}^b)\big)\big)$ (see [I2, Lemma 4.1A and Lemma 4.2A]). Combining both invariants, in [IM2] the second and third authors of the present survey obtain results on the reducibility of certain families of AG algebras, including the following two, Theorems 3.3 and 3.4 in Section 3.1 showing that there are infinite collections of such families.

Symmetric decomposition

We recall the symmetric decomposition of the associated graded algebra to an AG algebra, from [I2, IM1]. Given an AG algebra A, of socle degree j, with associated graded algebra A^* we define an ideal $C_A(a)$ componentwise, by (here ρ is the projection to A_i)

$$C_A(a)_i = \rho\left(\mathfrak{m}_A^i \cap (0 : \mathfrak{m}_A^{j-a-i})/(\mathfrak{m}_A^{i+1} \cap (0 : \mathfrak{m}_A^{j+1-a-i}))\right).$$

Lemma 3.1. *Let A be an AG algebra over a field k, of socle degree j. The A^*, its associated graded algebra, has a filtration by ideals*

$$A^* = C_A(0) \supset C_A(1) \subset \cdots \supset C_A(j-2), \text{ such that } Q_A(a) = C_A(a)/C_A(a+1)$$

is a reflexive k-module satisfying

$$Q_A(a)_i \cong Q_A(a)^\vee_{j-a-i} = \operatorname{Hom}_{\mathsf{k}}(Q(a)_{j-a-i}, \mathsf{k}).$$

Let a Macaulay dual generator of A be $F_A = F_j + F_{j-1} + \cdots$. Then $Q(0)$ is a socle degree j graded Artinian algebra, whose Macaulay dual generator is F_j, and $Q(a)$ is determined by $(F_A)_{\geq j-a} = F_j + \cdots + F_{j-a}$.[4]

Let $H_A(a) = H(Q_A(a))$. Then $H_A(a)$ is symmetric about $(j-a)/2$, and the Hilbert function $H(A)$ satisfies

$$H(A) = \sum_a H_A(a).$$

Example 3.2. (i). Let $R = \mathsf{k}[x, y, z, w]$ and $F = Y^6 + X^4Z + W^3$; then the Artinian Gorenstein algebra $A = R/\operatorname{Ann} F$ satisfies

$A = (xy,\ xw,\ yz,\ yw,\ zw,\ z^2,\ w^3-x^4z,\ w^3-x^5,\ w^3-y^6)$ and $Q(0) = R/(x, z, w, y^7) \cong \mathsf{k}[y]/(y^7)$,

[4]Warning: when $a \geq 1$ the relation between $Q(a)$ and $(F_A)_{\geq j-a}$ is subtle: see [IM1, §1.3].

The Macaulay dual is

$$A^\vee = R \circ F = \langle F, \{Y^i, 0 \in [0,5]\}, \{X^i Z, i \in [0,3]\}, \{X^i, i \in [1,4]\}, W, W^2\rangle,$$

and we can identify $Q(1)^\vee = \langle X, Z, X^2, XZ, X^3, X^2Z, X^4, X^3Z\rangle$, $Q(2) = 0$, $Q(3)^\vee = \langle W, W^2\rangle$. The symmetric Hilbert function decomposition $\mathcal{D}(A)$ of $H(A) = (1,4,4,3,3,1,1)$ is thus

$$\mathcal{D}(A) = \big(H(0) = (1^6),\ H(1) = (0,2,2,2,2,0),\ H(3) = (0,1,1,0)\big)$$

It is not hard to show that this is the unique symmetric decomposition possible for $H(A)$.

(ii). In contrast, there are two symmetric decompositions for $H(C) = (1,3,3,2,1)$ of Example 1.9. For both C and its deformation $C(t)$ of the Example we have

$$H(0) = (1,2,2,2,1),\ H(1) = (0,1,1,0).$$

But for $G + (X+Y)^4 = X^3Y + (X+Y)^4 + Y^2Z$ we have $H(0) = (1,2,3,2,1)$, $H(1) = 0$ and $H(2) = (0,1,0)$.[5]

The generic Jordan type for $\mathcal{R}/\operatorname{Ann}\big(G + (X+Y)^4\big)$ is $(5,3,1,1)$, the same as for G.

Note that in the first dual generator $G + (X+Y)^4 + Y^2Z$ in (ii) above Y^2Z is an *exotic summand*, since the multiplier Y^2 of the new variable Z is a partial of $G + (X+Y)^4$; but here this does not hide Z enough to change the Hilbert function. Exotic summands are defined and discussed in [IM1, §2.2] and [BJMR], we will not treat them here.

Symmetric decompositions of AG algebras, discovered in 1985 ([I1]), have seen increasing appllications recently, particularly to issues of the scheme or cactus length of forms -a study begun by Alessandra Bernardi and Kristen Ranestad in their much-cited 2013 [BR], see also [BJMR]. They have been applied also to classification of Gorenstein local algebras, as [JeMR].

3.1 Families $\operatorname{Gor}(H)$ with multiple irreducible components.

As mentioned earlier the following two results depend on a comparison of the dominance partial order on Jordan types $P_{(\ell,A)}$ with a contrasting partial order arising from the semicontinuity of certain invariants $N_{i,b} = \dim_{\mathsf{k}}\big(\mathfrak{m}^i/\big(\mathfrak{m}^i \cap (0 : \mathfrak{m}^b)\big)\big)$ of symmetric decompositions. Families $\operatorname{Gor}(H)$ with multiple irreducible components had been previously exhibited using the semicontinuity of symmetric decompositions and other arguments, such as dimension, as [I2, Theorem 4.3] for $H = (1,3,3,2,1,1)$.

Theorem 3.3. *[IM2, Theorem 3.3]. Let $k \geq 2$ and consider the Gorenstein sequence $H(k) = (1,3,4^k,3,2,1)$. Then $\operatorname{Gor}\big(H(k)\big)$ has exactly three irreducible components, each corresponding to a symmetric decomposition of the Hilbert function $H(k)$.*

Theorem 3.4. *[IM2, Theorem 3.6]. Let $k \geq 1$ and $s \geq 2$, and consider the Gorenstein sequence $H(k,s) = (1,3,4^k,3^s,2,1)$. Then $\operatorname{Gor}\big(H(k)\big)$ has at least two irreducible components, each corresponding to a symmetric decomposition of the Hilbert function $H(k,s)$.*

[5]Note, $G(t) + (X+Y)^4$, $t \neq 0$ has larger HF $(1,3,4,2,1)$, with decomposition $(1,2,3,2,1) + (0,1,1,0)$.

3.2 Sequential Jordan type, Löewy Jordan type, Double sequential Jordan type.

Jordan type is defined for all Artinian algebras; however Jordan degree type does not have a natural extension to non-graded algebras: Chris McDaniel showed that it is not even possible to find a Jordan basis for the multiplication by x on the AG algebra $A = \mathsf{k}[x,y]/I$, $I = (x^2 - xy^2, y^4)$ that is consistent with the Hilbert function $H(A) = (1,2,2,2,1)$, so a Jordan order type for non-graded Artinian algebras generalizing JDT for graded Artinians is not possible ([IM2, Example 2.15]). We define several refinements of Jordan type introduced in [IMS], that have some desirable deformation properties. They each depend upon a filtration of an Artinian algebra A by $\mathfrak{m}$-adic quotients $A/\mathfrak{m}^i$. or by Löewy quotients $A/(0:\mathfrak{m}^i)$ or by ideals obtained by combining the two.

Definition 3.5 (Sequential, Löewy and Double Jordan type)**.** Let A be an Artinian local algebra of socle degree j, let $\mathfrak{m}$ be its maximal ideal, and let $\ell \in \mathfrak{m}$.

(i). The Sequential Jordan type (SJT) of (ℓ, A) is given by the sequence

$$(P_{,\ell,A/\mathfrak{m}^i}),\ i \in \{1,\dots,j\}$$

of Jordan types of successive quotients of A by powers of the maximal ideal.

(ii). The Löewy Sequential Jordan type (LJT) of (ℓ, A) is given by the sequence

$$(P_{\ell,A/(0:\mathfrak{m}^{j-k})}),\ k \in \{1,\dots,j\}$$

of Jordan types of successive quotients of A by the Löewy ideals.

(iii). The Double Sequential Jordan type (DSJT) for a pair $(A,\ \ell) \in \mathfrak{m}$, is given by the table whose (a,i) entry is the partition

$$P_{\ell,B_{a,i}},\ \text{where } B_{a,i} := A/\big(\mathfrak{m}^i \cap (0:\mathfrak{m}^{j+1-a-i})\big),\ 0 \le a \le j,\ 0 \le i \le j+1-a$$

giving the Jordan type of the quotient of A by intersections of a Löewy ideal with a power of the maximal ideal.

Example 3.6. Let $\mathcal{R} = \mathsf{k}\{x,y,z\}$ and $C = \mathcal{R}/\operatorname{Ann} G$, where $G = X^3Y + Y^2Z$, an AG algebra with socle degree $j = 4$, as in Example 1.9. Then the double sequential Jordan type of an element $\ell = ax + by + cz + h$, with $h \in \mathfrak{m}_C^2$, and $ab \neq 0$ is

$a,\ i$	$i=0$	1	2	3	4
$a=0$		(1)	(2, 1, 1)	(3, 2, 1, 1)	(4, 3, 1, 1)
1	(1)	(2, 1)	(3, 2, 1)	(4, 3, 1, 1)	
2	(2, 1)	(3, 2, 1)	(4, 3, 1, 1)		
3	(3, 2, 1)	(4, 3, 1, 1)			
4	(4, 3, 1, 1)				$P_{\ell,C} = (5,3,1,1)$

$$\text{DSJT: } P_{\ell,C_{a,i}},\ C_{a,i} := C/\big(\mathfrak{m}_C^i \cap (0:\mathfrak{m}_C^{j+1-a-i})\big)$$

Note that for $a=0$, since $\mathfrak{m}_C^i \subseteq (0:\mathfrak{m}_C^{j+1-i})$, the first row in this table gives us the sequential Jordan type, and for $i=0$, since $\mathfrak{m}_C^0 = C$, the first column shows us the Loewy sequential Jordan type. Also, if $a+i=j+1$, we have $\mathfrak{m}_C^i \cap (0:\mathfrak{m}_C^{j+1-a-i}) = 0$, so the Jordan type is that of the pair (ℓ, C). To save space, we write this Jordan type at the lower right-hand corner of the table. Here, also the diagonal $a+i=j$ is constant. This is always the case for an AG algebra, because $(0:\mathfrak{m}_C) = \mathfrak{m}_C^j$.

Lemma 3.7. *[IMS, Prop. 2.2] When A is standard graded, both the Sequential Jordan type and the Löewy Sequential Jordan type are equivalent to the Jordan degree type.*

Lemma 3.8. *[IMS, Theorem 2] We have that $JT \geq SJT \geq DSJT$ and $JT \geq LSJT \geq DSJT$ are true refinements.*

We do not know if $SJT + LSJT$ to $LSJT$ is a true refinement. Given their definitions, and the dominance order (Definition 1.5) for Jordan type, we can define natural dominance orders for SJT, LSJT, and DSJT when the total lengths of the algebras compared are the same. For example an SJT S dominates S' if for every degree i, the partition $S_{\leq i} \geq S'_{\leq i}$ in the dominance order of Definition 1.5. Recall that we take $\mathcal{R} = \mathsf{k}[x_1, \ldots, x_r]$ with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_r)$. We need a preparatory result, adapted from [IMM2, Corollary 2.44], that will imply the deformation results we wish to show for these invariants.

Lemma 3.9 (Semicontinuity of Jordan type)**.** *(i). Let $M(\tau)$ for $\tau \in \mathfrak{Z}$ be a family of constant length R-modules over a parameter space $\mathfrak{Z}$ and let $\ell \in \mathfrak{m}_R$. Then for a neighborhood $U_0 \subset \mathfrak{Z}$ of τ_0, we have that $\tau \in U_0 \Rightarrow P_{\ell, M(\tau)} \geq P_{\ell, M(\tau_0)}$ in the dominance partial order of Definition 1.5.*

(ii). Let M be a finite module over an Artinian algebra A and $\{\ell(t), t \in \mathfrak{X}\}$, $\mathfrak{X}$ a curve, be a family of linear forms or of elements of $\mathfrak{m}$ (according to whether A is graded or local). Assume that $P_{\ell,M} = P$ is constant for $\ell \in \mathcal{U} \subset \mathfrak{X}$, an open dense in $\mathfrak{X}$; and let $\ell_0 \in \mathfrak{X} \backslash U$. Then $P \geq P_{\ell_0, M}$ in the dominance partial order.

Since an Artinian algebra is a deformation of its associated graded algebra, we have

Corollary 3.10. *for any Artinian A, and any $\ell \in \mathfrak{m}_A$, the Jordan type $P_{\ell,A}$ of A dominates the Jordan type P_{ℓ,A^*} of the associated graded algebra of A.*

See Examples 1.8 and 1.9.

Proposition 3.11 (Deformations)**.** *[IMS, Prop. 2.12] Let $A(\tau), \tau \in \mathfrak{Z}$ be a flat (constant length) family of Artinian algebra quotients of R, let $\tau_0 \in \mathfrak{Z}$, and fix $\ell \in \mathcal{R}$.*

(i). (SJT) Assume that the Hilbert function $H(A(\tau))$ is constant. Then there is an open neighborhood $\mathcal{U}$ of τ_0 in $\mathfrak{Z}$ such that $\tau \in \mathcal{U}$ implies that the SJT $\mathcal{S}(\ell, \tau)$ of the pair $(\ell, A(\tau))$ dominates the SJT $\mathcal{S}(\tau_0)$ of $(\ell, A(\tau_0))$

(ii). (LSJT) Assume that the dimensions of the Löewy ideals $(0:\mathfrak{m}^i_{A(\tau)})$ are constant along the family $\mathfrak{Z}$. Then there is an open neighborhood $\mathcal{U}$ of τ_0 such that $\tau \in \mathcal{U}$ implies that the LSJT for the pair $(\ell, A(\tau))$ dominates the LSJT for $(\ell, A(\tau_0))$.

(iii). (DSJT) Assume that for each pair (i,k) the dimensions of the ideals $\mathfrak{m}^i_{A(\tau)} \cap (0:\mathfrak{m}^k_{A(\tau)})$ are constant along the family $\mathfrak{Z}$. Then there is an open neighborhood $\mathcal{U}$ of τ_0 such that $\tau \in \mathcal{U}$ implies that the DSJT for the pair $(\ell, A(\tau))$ dominates the DSJT for $(\ell, A(\tau_0))$.

Proof. The proof is immediate from the semicontinuity of the appropriate invariant on the given locus. $\square$

Question 3.12. There are many other ways for an Artinian Gorenstein to use the symmetric decomposition to make up a Jordan type related invariant. For example, we could consider the Jordan degree type of each component $Q_A(a)$ of the symmetric decomposition of A^*, $a \in [0, j-2]$, where j is the socle degree. These JDT will satisfy appropriate deformation properties, for example if one fixes the symmetric decomposition of the Hilbert function. Which of these can we use to study irreducible components of symmetric decomposition strata $\mathcal{D} = \big(H(Q(0)), H(Q(1)), \dots, H(Q(j-2))\big)$ or of $\mathrm{Gor}(H)$, the family of Gorenstein local algebras of a given Hilbert function?

4 Open questions.

Question 4.1. Can we use SJT, LJT, DSJT to help show that certain families of local Artinian algebras of, say, fixed Hilbert function, have several irreducible components?

Question 4.2. What is the relation between JT and Betti diagrams? See recent articles of N. Abdallah, and N. Abdallah and H. Schenck [Ab, AbSc].

Question 4.3. Consider Briançon's vertical cells (Jordan cells) for the family of codimension two local Artinian algebra of Hilbert function H. Answer analogous questions to those of [AlIK] involving numbers of generators, and also concerning the symmetric decomposition.

Question 4.4. The following is related to Section 1.1 Problem viii. The Jordan block decomposition for a (similarity class) of matrices - the Jordan normal form (JNF) has been long known. But only relatively recently - 2008 - had there been work on which pairs of partitions can occur of JNF of two commuting $n \times n$ matrices.[6] This problem reduces to considering two nilpotent matrices, where JNF is just a partition of n. A partition is called *stable* if its parts differ pairwise by at least two. One perhaps surprising fact, shown by T. Košir and P. Oblak with help from others is that a commuting pair cannot consist of two different stable partitions: the result has to do with the Hilbert functions of complete intersection quotients of $k[x, y]$. The problem of finding the maximum (in dominance order) Jordan type (it is stable) commuting with a given partition has attracted some interest, in particular a conjecture of P. Oblak answering this has recently proved by R. Basili. The problems in this area seem quite difficult, they include study of certain graph associated to a partition; it is as if there is some hidden structure lurking behind what we know. See [JeS], and the recent survey by Leila Khatami [Kh] and the references cited there - we have not included references here on this rich topic.

[6] Early researchers in this area seemed more interested in the maximum dimension of a vector space of commuting matrices.

References

[Ab] N. Abdallah: *A note on the structure of some codimension four Gorenstein Artin algebras*, (2023) arXiv:math.AC/2302.14797.

[A-N] N. Abdallah, N. Altafi, P. De Poi, L. Fiorindo, A. Iarrobino, P. Macias-Marques, E. Mezzetti, R. M. Miró-Roig, L. Nicklasson: *Hilbert functions and Jordan type of Perazzo Artinian algebras*, (2023) arXiv:math.AC/2303.16768.

[AAIY] N. Abdallah, N. Altafi, A. Iarrobino, and J. Yaméogo: *Jordan type for codimension three Gorenstein algebras*, preprint, (2023).

[AEIY] N. Abdallah, J. Emsalem, A. Iarrobino, and J. Yaméogo: *Limits of graded Gorenstein algebras of Hilbert function* $(1, 3^k, 1)$, (2023) arXiv:math.AC/2302.00287.

[AbSc] N. Abdallah and H. Schenck: *Free resolutions and Lefschetz properties of some Artin Gorenstein rings of codimension four*, (2022) arXiv:math.AC/2208.01536.

[Al1] N. Altafi: *Hilbert functions of Artinian Gorenstein algebras with the strong Lefschetz property*, Proc. Amer. Math. Soc. 150 (2022), no. 2, 499-513.

[Al2] N. Altafi: *Jordan types with small parts for Artinian Gorenstein algebras of codimension three,* Linear Algebra Appl. 646 (2022), 54-83.

[Al3] N. Altafi: *A note on Jordan types and Jordan degree types*, (2023) arXiv:math.AC/2303.13673.

[AlIK] N. Altafi, A. Iarrobino, and L. Khatami: *Complete intersection Jordan types in height two*, J. Algebra 557 (2020), 224–277.

[AlIKY] N. Altafi, A. Iarrobino, L. Khatami, and J. Yaméogo: *Number of generators of ideals in Jordan cells of graded Artinian algebras of height two*, Journal of Pure and Applied Algebra 227 (2023) 107419, 43p. (also arXiv:math.AC/2006.11794 v.3).

[Ar] M. Artin: *Algebra*, 2nd edition, (2011), Prentice Hall.

[BCR] C. Bertone, F. Cioffi, M. Roggero: *Smoothable Gorenstein points via marked schemes and double-generic initial ideals,* Exp. Math. 31 (2022), no. 1, 120-137.

[BR] A. Bernardi and K. Ranestad: *On the cactus rank of cubics forms*, J. Symbolic Comput. 50 (2013), 291–297.

[BJMR] A. Bernardi, J. Jelisiejew, P. Macias Marques, K. Ranestad: *On polynomials with given Hilbert function and applications,* Collect. Math. 69 (2018), no. 1, 39–64.

[BMMN] M. Boij, J. Migliore, R. M. Miró-Roig, U. Nagel: *The non-Lefschetz locus*, J. Algebra 505 (2018), 288-320.

[Bri] J. Briançon: *Description de* $Hilb^n C[x, y]$, [Preprint, Univ. Nice, 1972], Invent. Math. 41 (1977), 45-89.

[BuEi] D. Buchsbaum and D. Eisenbud: *Algebra structures for finite free resolutions, and some structure theorems for codimension three*, Amer. J. Math. **99** (1977), 447–485.

[CaJN] G. Casnati, J. Jelisiejew, and R. Notari: *Irreducibility of the Gorenstein loci of Hilbert schemes via ray families,* Algebra Number Theory 9 (2015), no. 7, 1525-1570.

[CeGIM] A. Cerminara, R. Gondim, G. Ilardi, F. Maddaloni: *Lefschetz properties for higher order Nagata idealizations*, Adv. in Appl. Math. 106 (2019), 37-56.

[CM] D. Collingwood and W. McGovern: *Nilpotent Orbits in Semisimple Lie algebras*, Van Nostrand Reinhold (New York), (1993).

[CoV] A. Conca and G. Valla: *Hilbert functions of powers of ideals of low codimension,* Math. Zeitschrift 230 (1999), #4, 753–784.

[CGo] B. Costa and R. Gondim: *The Jordan type of graded Artinian Gorenstein algebras*, Adv. in Appl. Math. 111 (2019), 101941, 27 pp.

[Di] S. J. Diesel: *Some irreducibility and dimension theorems for families of height 3 Gorenstein algebras*, Pacific J. Math. **172** (1996), 365–397.

[DIV] R. Di Gennaro, G. Ilardi, J. Vallès: *Singular hypersurfaces characterizing the Lefschetz properties,* J. Lond. Math. Soc. (2) 89 (2014), no. 1, 194-212.

[Ei] D. Eisenbud: *Commutative Algebra, with a view toward Algebraic Geometry*, Springer GTM #150, (1995), Springer-Verlag, New York.

[FMM] L. Fiorindo, E. Mezzetti, and R. M. Miró-Roig: *Perazzo 3-folds and the weak Lefschetz property,* J. Algebra 626 (2023), 56?81.

[Go] R. Gondim: *On higher Hessians and the Lefschetz properties*, J. Algebra 489 (2017), 241–263.

[GoZ1] R. Gondim and G. Zappalà: *Lefschetz properties for Artinian Gorenstein algebras presented by quadrics,* Proc. Amer. Math. Soc. 146 (2018), no. 3, 993-1003.

[GoZ2] R. Gondim and G. Zappalà: *On mixed Hessians and the Lefschetz properties,* J. Pure Appl. Algebra 223 (2019), no. 10, 4268–4282.

[GorNo] P. Gordan and M. Noether: *Über die algebraischen Formen deren Hesse'sche Determinante identisch verschwindet*, Math. Ann. 10 (1876), 547–568.

[GS] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at `http://www.math.uiuc.edu/Macaulay2/`

[HScSe] B. Harbourne, H. Schenck, A. Seceleanu: *Inverse systems, Gelfand-Tsetlin patterns and the weak Lefschetz property*, J. Lond. Math. Soc. (2) 84 (2011), no. 3, 712–730.

[Ha1] T. Harima: *Characterization of Hilbert functions of Gorenstein Artin algebras with the weak Stanley property*, Proc. Amer. Math. Soc. 123 (1995), no. 12, 3631–3638.

[Ha2] T. Harima: *A note on Artinian Gorenstein algebras of codimension three,* J. Pure Appl. Algebra 135 (1999), no. 1, 45–56.

[HMNW] T. Harima, J. Migliore, U. Nagel, and J. Watanabe: *The weak and strong Lefschetz properties for Artinian K-algebras,* J. Algebra 262 (2003), no. 1, 99–126.

[H-W] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, and J. Watanabe: *The Lefschetz properties.* Lecture Notes in Mathematics, 2080. Springer, Heidelberg, 2013. xx+250 pp. ISBN: 978-3-642-38205-5.

[HW1] T. Harima and J. Watanabe: *The central simple modules of Artinian Gorenstein algebras*, J. Pure Appl. Algebra 210 (2007), no. 2, 447–463.

[HW2] T. Harima and J. Watanabe: *The strong Lefschetz property for Artinian algebras with non-standard grading*, J. Algebra 311 (2007) 511–537.

[Hui] M. Huibregtse: *An elementary construction of the multigraded Hilbert scheme of points* Pacific J. Math. 223 (2006), no. 2, 269-315.

[I1] A. Iarrobino: *The Hilbert function of a Gorenstein Artin algebra*, Commutative algebra (Berkeley, CA, 1987), 347–364, Math. Sci. Res. Inst. Publ., 15, Springer, New York, 1989.

[I2] A. Iarrobino: *Associated graded algebra of a Gorenstein Artin algebra*, Mem. of the Amer. Math. Soc., Vol. 107 (1994), no. 514, viii+115pp.

[IKa] A. Iarrobino and V. Kanev: *Power Sums, Gorenstein Algebras, and Determinantal Varieties*, Appendix C by Iarrobino and Steven L. Kleiman *The Gotzmann Theorems and the Hilbert scheme*, Lecture Notes in Mathematics, 1721. Springer-Verlag, Berlin, 1999. xxxii+345 pp. ISBN: 3-540-66766-0 (1999), 345+xxvii p.

[IM1] A. Iarrobino and P. Macias Marques, *Symmetric Decomposition of the Associated Graded Algebra of an Artinian Gorenstein Algebra*, J. Pure and Applied Algebra, Vol. 225 #3 (2021) 106496, 49p., https://doi.org/10.1016/j.jpaa.2020.106496, (arXiv:math.AC/1812:03856 v.2 (2020).)

[IM2] A. Iarrobino and P. Macias Marques, *Reducibility of a family of local Artinian Gorenstein algebras*, (2021) arXiv:math.AC/212:14664 v.1.

[IMM1] A. Iarrobino, P. Macias Marques, and C. McDaniel, *Free extensions and Jordan type,* J. Algebra 549 (2020), 346-364.

[IMM2] A. Iarrobino, P. Macias Marques, and C. McDaniel: *Artinian algebras and Jordan type*, J. Commut. Algebra 14 (3), 365-414, (2022).

[IMM3] A. Iarrobino, P. Macias Marques, and C. McDaniel: Artinian Gorenstein algebras that are free extensions over $k[t]/(t^n)$, and Macaulay duality. J. Commut. Algebra 14 (2022), no. 4, 553-569.

[IMS] A. Iarrobino, P. Macias Marques, and J. Steinmeyer: *Sequential Jordan type of an Artinian algebra*, 18p. (in progress, 2023).

[IY] A. Iarrobino and J. Yaméogo: *The family G_T of graded Artinian quotients of $k[x,y]$ of given Hilbert function*, Special issue in honor of Steven L. Kleiman, Commun. Algebra 31 (8) (2003) 3863–3916.

[Je] J. Jelisiejew: *Classifying local Artinian Gorenstein algebras,* Collect. Math. 68 (2017), no. 1, 101–127.

[JeMR] J. Jelisiejew, S. Masuti, and M. Rossi: *On the Hilbert functions of Artinian local complete intersections of codimension three*, (2022) arXiv:math.AC/2202.04021.

[JeS] J. Jelisiejew and K. Sǐvic: *Components and singularities of Quot schemes and varieties of commuting matrices*, J. Reine Angew. Math. 788 (2022), 129-187, DOI 10.1515/crelle-2022- 0018. MR4445543

[KKRSSY] G. Kapustka, M. Kapustka, K. Ranestad, H. Schenck, M. Stillman, and B. Yuan: *Quarternary quartic forms and Gorenstein rings*, (2021) arXiv:math.AC/2111.05817.

[Kh] L. Khatami: *Commuting Jordan types: a survey*, (2023) arXiv:math.AC/2304.08550.

[Kl] J. O. Kleppe: *The smoothness and the dimension of PGor(H) and of other strata of the punctual Hilbert scheme*, J. Algebra 200 (1998), no. 2, 606-628, DOI 10.1006/jabr.1997.7226. MR1610672

[KlKl] J.O. Kleppe and S. L. Kleiman: *Macaulay duality and its geometry,* (2022) arxiv:math.AG/2210.10934

[Mac1] F.H.S. Macaulay: *On a method of dealing with the intersections of plane curves*, Trans. Amer. Math. Soc. 5 (1904), no. 4, 385–410.

[Mac2] F.H.S. Macaulay: *Some Properties of Enumeration in the Theory of Modular Systems*, Proc. London Math. Soc. 2 (1927), no. 26, 531–555.

[Mac3] F.H.S. Macaulay: *The algebraic theory of modular systems*, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1916. Reissued with an introduction by P. Roberts, 1994. xxxii+112 pp. ISBN: 0-521-45562-6.

[MW] T. Maeno and J. Watanabe: *Lefschetz elements of Artinian Gorenstein algebras and Hessians of homogeneous polynomials*, Illinois J. Math. 53 (2009) 593–603.

[MCIM] C. McDaniel, S. Chen, A. Iarrobino, P. Macias Marques: *Free extensions and Lefschetz properties, with an application to rings of relative coinvariants*, Linear Multilinear Algebra 69 (2021), no. 2, 305-330.

[MeMO] E. Mezzetti, R.M. Miró-Roig, G. Ottaviani: *Laplace equations and the weak Lefschetz property,* Canad. J. Math. 65 (2013), no. 3, 634-654.

[MiNa] J. Migliore, U. Nagel: *Survey article: a tour of the weak and strong Lefschetz properties*, J. Commut. Algebra 5 (2013), no. 3, 329-358.

[MR-S] R. M. Miró-Roig, M. Salat: *On the classification of Togliatti systems*, Comm. Algebra 46 (2018), no. 6, 2459-2475.

[OCV] K. C. O'Meara, J. Clark, and C. I. Vinsonhaler: *Advanced topics in linear algebra: Weaving matrix problems through the Weyr form*, Oxford University Press, 2011.

[OW] K.C. O'Meara, and J. Watanabe: *Weyr structures of matrices and relevance to commutative finite-dimensional algebras* Linear Algebra Appl.532(2017), 364-386.

[PaSh] J. P. Park, Y.-S. Shin: *Modular Jordan type for* $k[x,y]/(x^m,y^n)$ *for* $m=3,4$*,* J. Korean Math. Soc.57(2020), no.2, 283-312.

[Per] U. Perazzo, *Sulle varietà cubiche la cui hessiana svanisce identicamente*, G. Mat. Battaglini **38** (1900), 337–354.

[Sh] H. Shapiro: *The Weyr characteristic*, Amer. Math. Monthly106(1999), no.10, 919-929.

[St] R. Stanley: *Hilbert functions of graded algebras*, Advances in Math. 28 (1978), 58-73.

[Wat1] J. Watanabe: *The Dilworth number of Artin Gorenstein rings*, Advances in Math. 76 (1989), 194-199.

[Wat2] J. Watanabe: *A remark on the Hessian of homogeneous polynomials*, in A. Geramita et al: The curves seminar at Queens, vol. XIII. Queen's Papers in Pure and Applied Mathematics, vol. 119, p. 171-178, Queen's University, Kingston, (2000).

[We] S. Weintraub: *Jordan canonical form, theory and practice*, Synthesis Lectures on Mathematics and Statistics, Lecture #6, Morgan and Claypool (2006), republished Springer Nature, Switzerland (2022) ISBN 978-3-031-01270-9.

[Y1] J. Yaméogo: *Décomposition cellulaire de variétés paramétrant des idéaux homogènes de C[[x,y]]. Incidence des cellules. I,* Compositio Math. 90 (1) (1994), 81–98.

[Y2] J. Yaméogo: *Décomposition cellulaire de variétés paramétrant des idéaux homogènes de C[[x,y]]. Incidence des cellules. II,* J. Reine Angew. Math. 450 (1994), 123–137.