$I \subseteq \mathbb{C}[\mathbb{P}^n]$

$X \subseteq \mathbb{P}^n$

X imposes r independent conditions on $[I]_t$

if $\dim [I \cap I_X]_t = \dim [I]_t - r$

X is a set of "d" distinct points in $\mathbb{P}^n$
reduced

$I = R$

$$\dim [R \cap I_X]_t = \dim [R]_t - r$$

$$\begin{gathered} \dim [R \cap I_X]_t \\ \| \\ \dim [I_X]_t \end{gathered}$$

$$r = \dim R_t - \dim [I_X]_t = h_X(t)$$

Exercise 56. Prove that in order to show that a finite set of points Z imposes independent conditions on $[R]_t$, it is enough to show that for *each* $P \in Z$ there is a form of degree t vanishing on $Z \backslash \{P\}$ but not vanishing at P.

$d-1 \qquad Z = \{P_1, \ldots, P_d\}$

$R_t \qquad F_1, \ldots, F_d : \quad F_i(P_i) \neq 0$

$F_i(P_j) = 0 \quad i \neq j$

$[R/I_X]_t \simeq \mathbb{K}^d$

$h_X(t) = d \qquad t = d-1$

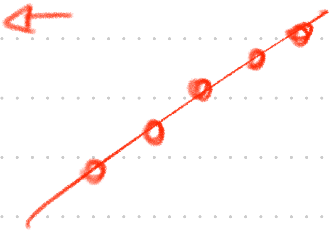
Exercise 42.

(a) Prove that five points in $\mathbb{P}^2$ fail to impose independent conditions on plane cubics (i.e. forms of degree 3 in $\mathbb{C}[x_0, x_1, x_2]$) if and only if they all lie on a line.

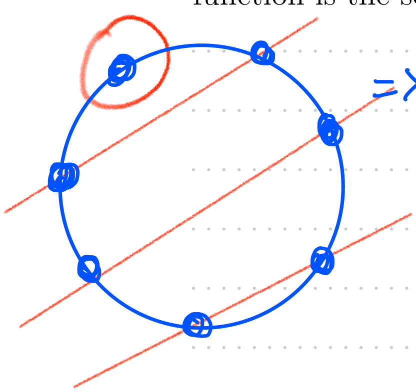
0	1	2	3			
1	2	3	4	5	5	----
1	3	4	5	5	5	
1	3	5	5	5	---	

If X has $= h_X \Rightarrow X$ fails.

$\Rightarrow \dim [I_X]_1 = 1$



(b) If V is a set of seven points lying on an irreducible conic in $\mathbb{P}^2$, prove that its Hilbert function is the sequence $(1, 3, 5, 7, 7, 7, \dots)$.



$= X$

6 ?

1 3 4

$\dim [I_X]_2 = 2$

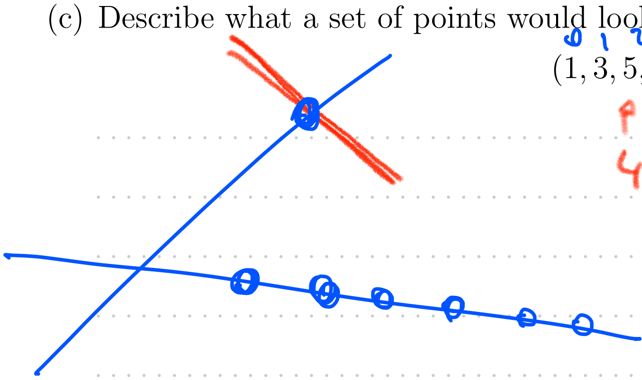
0	1	2	3	4	
1	3	5	6	7	---
1	3	5	7	7	---

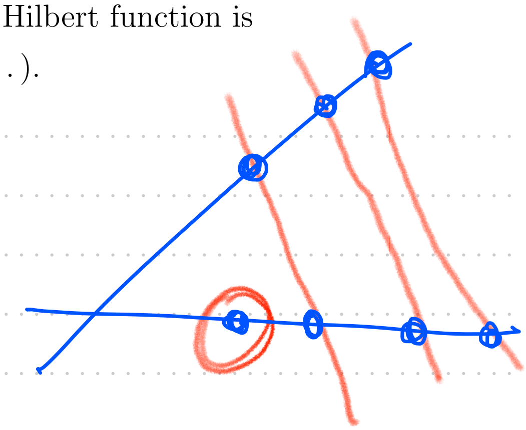
(c) Describe what a set of points would look like if its Hilbert function is

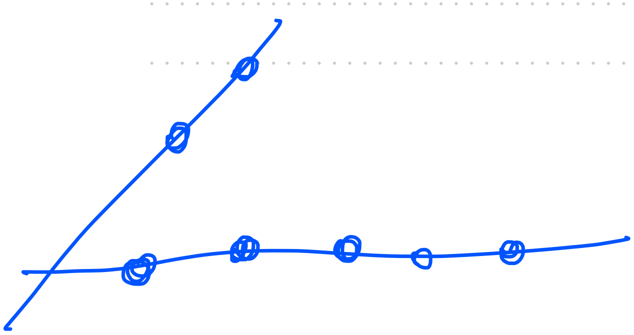
0 1 2 3

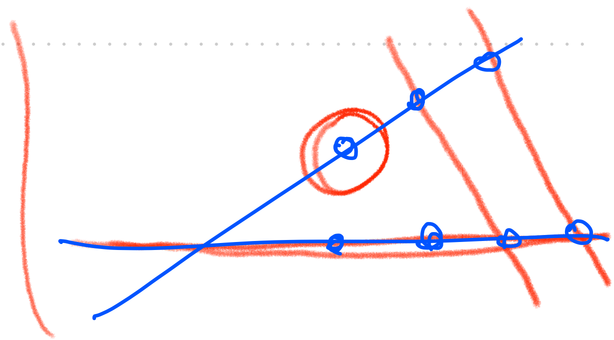
$$(1, 3, 5, 6, 7, 7, 7, \dots).$$

4







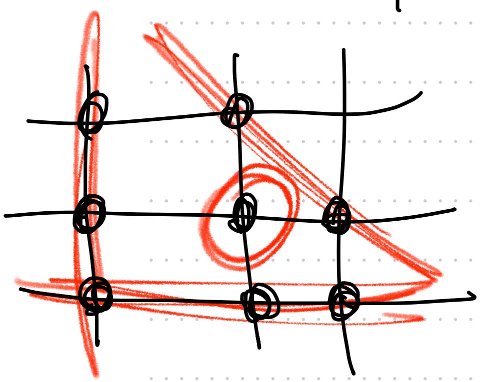


Exercise 58. Let X be the following set of 8 points in $\mathbb{P}^2$

$$X = \left\{ \begin{array}{ccc} [-1,1,1] & [0,1,1] & \\ [1,0,1] & [0,0,1] & [-1,0,1] \\ [1,-1,1] & [0,-1,1] & [-1,-1,1] \end{array} \right\}$$

(a) Sketch this set of points in the affine space given by $z \neq 0$, noting the collinearities.
(b) Compute the Hilbert function of X.

$$\begin{array}{lccccc} & 0 & 1 & 2 & 3 & 4 \\ h_X = & 1 & 3 & 6 & 7 & 8 \; \ldots \\ & 1 & 3 & 6 & 8 & 8 \; \ldots \end{array}$$



$1 \quad 3 \quad \underline{6 \quad 6} \quad 6 \quad 6 \; \ldots$

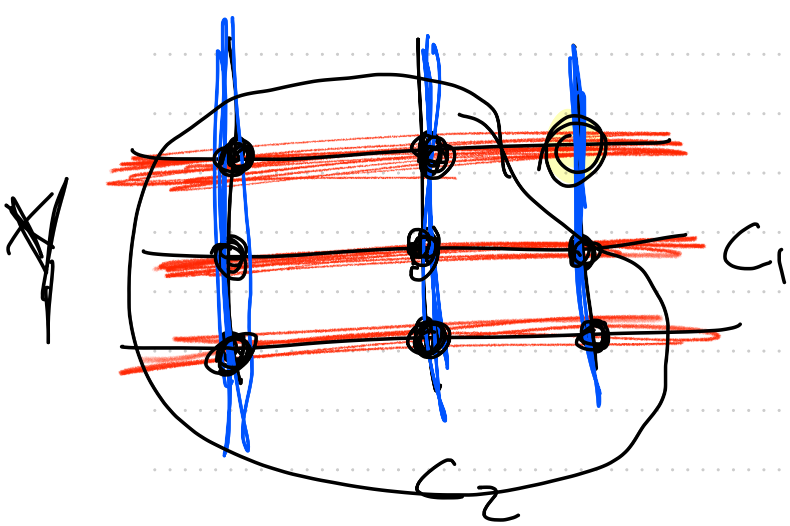
Exercise 59. Let X be the following set of 8 points in $\mathbb{P}^2$

$$X = \left\{ \begin{array}{ccc} [-1,1,1] & [0,1,1] & \\ [1,0,1] & [0,0,1] & [-1,0,1] \\ [-1,-1,1] & [0,-1,1] & [1,-1,1] \end{array} \right\}$$

Compute the number of conditions imposed by $P = [1,1,1]$ on $[I_X]_3$. How many conditions does the point $P' = [1,0,0]$ impose on $[I_X]_3$?

$$\begin{array}{lccccc} & 0 & 1 & 2 & 3 & 4 \\ h_X = & 1 & 3 & 6 & 8 & 8 \end{array}$$

$\dim [I_X]_3 = 2$



$$\begin{array}{lcccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ h_Y = & 1 & 3 & 6 & 8 & 9 & 9 \end{array}$$

$C = a\,C_1 + b\,C_2$