Lecture 1a

We'll go quickly through the 1st part of the Supplementary Notes. Make sure you try the exercises! The notes have a lot of detail that we'll gloss over or skip. See p 2 for a list of topics we'll assume you know (but we'll go over some). Attached papers give a lot of basic facts + connections.

Also see Juhnke-Kubitzke + Miró-Roig "List of Problems" (written independently of these notes) for list of open problems in the area. — arXiv: 2311.03081 v1

(not exercises!)

Background

- Assume k is a field. Characteristic zero + alg closed are often assumed. Interesting things sometimes happen when we change the characteristic (as we'll see in all 3 courses, probably).

- $R = k[x_1, \ldots, x_n]$ = polynomial ring in n variables

  R also has the structure of a homogeneous (or graded) ring

  $$R = \bigoplus_{t \geq 0} [R]_t \quad \text{where } [R]_t = \text{v.s. of homog polynomials of degree } t.$$

  Main point: any polynomial can be written in a unique way as a sum of homogeneous polynomials

  (Note $[R]_m \cdot [R]_p \subseteq [R]_{m+p}$)

  Def An element of $[R]_t$ is also called a form of degree t.

  Note - R is actually a standard graded k-algebra:

  R is generated in deg 1, each $[R]_t$ is a f d v s and $\dim [R]_t = \binom{n-1+t}{n-1}$.

- Any ideal $I \subset k[x_1, \ldots, x_n]$ is finitely generated (R is Noetherian):

  there exist $f_1, \ldots, f_t \in I$ such that

$I = \{$ lin. comb. of the f_i with coefficients in R $\}$

$= \{ g_1 f_1 + \dots + g_t f_t \mid g_i \in R \}$

- For a <u>homogeneous</u> ideal $I \subset k[x_0, x_1, \dots, x_n]$, the generators f_i can be chosen to be homogeneous.

It turns out that if I is homogeneous then $I = \bigoplus_{t \geq 0} [I]_t$ (where the first several direct summands may be 0). Note each $[I]_t$ is a finite dim'l vector subspace of $[R]_t$.

- The cosets of I in R form a standard graded k-algebra R/I:

$$R/I = \bigoplus_{t \geq 0} [R/I]_t \quad \text{where}$$

$$[R/I]_t \cong \left([R]_t / [I]_t \right)$$

<u>Key Definition</u>

The Hilbert <u>function</u> $h_{R/I}(t)$ is defined by

$$h_{R/I}(t) = \dim [R/I]_t$$
$$= \dim [R]_t - \dim [I]_t .$$

There are important theorems about which functions can be Hilbert functions of some R/I, and about what information $h_{R/I}(t)$ gives you about R/I.

<u>Example</u>/<u>Def</u> Let $R = k[x_1, \dots, x_n]$. Then $I \subseteq R$ is a <u>monomial</u> ideal if there exists a generating set consisting of monomials.

A basis for $[R/I]_t$ can be obtained by listing the monomials of degree t <u>not</u> in I. The Hilbert function of R/I can be found by counting these monomials in each degree.

E.g. $I = (x^2, xy, xz, yz^2, y^3, z^4) \subseteq k[x,y,z]$.

Degree	Monomials in $[I]_t$	Monomials not in $[I]_t$	$h_{R/I}(t)$
0	—	1	1
1	— omit	x, y, z	3
2	x^2, xy, xz	y^2, yz, z^2	3
3	$x^3, x^2y, x^2z, xy^2,$ xyz, xz^2, y^3, yz^2	y^2z, z^3	2
4	$x^4, x^3y, x^3z, x^2y^2, x^2yz,$ $x^2z^2, xy^3, xy^2z, xyz^2, xz^3,$ $y^4, y^3z, y^2z^2, yz^3, z^4$	none	0

Hilbert function is $(1, 3, 3, 2)$

Theorem (Macaulay)

Let $\underline{a} = (1, a_1, a_2, \ldots)$ be a sequence of positive integers. TFAE

(1) $\underline{a}$ is the Hilbert function of some standard graded algebra.

(2) $\underline{a}$ " " " " " " monomial algebra.

(3) $\underline{a}$ is an O-sequence, meaning each a_{i+1} is less than or equal to a very specific function of a_i and i. (See Theorem 4.10 of notes.)

Remarks

1. $\underline{a}$ is finite iff the algebras in (1) and (2) are artinian.

2. The definition of the Hilbert function of R/I extends to the HF of a graded module (finitely generated).

3. Let R/I be a standard graded k-algebra. Let L be a linear form.

Then $\times L : [R/I](-1) \longrightarrow R/I$ is an R-module homomorphism.

In particular, $\times L : [R/I]_{t-1} \longrightarrow [R/I]_t$ is a k-vector space homom.

4. Given a homogeneous ideal $I \subseteq R$, the saturation of I is

$$I^{sat} = \{ f \in R \mid \text{for each } 1 \le i \le n \text{ there exists } a_i \text{ such that } x_i^{a_i} f \in I \}$$

I is saturated if $I : m = I$ for $m = (x_1, -, x_n)$.

$\iff I = I^{sat}$

5. Fact: if $X \subseteq \mathbb{P}^{n-1}$ is a subset then

$$I(X) = \text{ideal generated by } \{ f \in R \mid f \text{ is homog. and } f(P) = 0 \ \forall P \in X \}$$

is a saturated homogeneous ideal.

E.g. $X = \{ [0,0,1] \} \subset \mathbb{P}^2$. $I = \mathbb{I}(X) = (x,y) \subseteq R = k[x,y,z]$

Note $[I]_2 = \langle x^2, xy, xz, y^2, yz \rangle \subset [R]_2$

vector space

If $J = (x^2, xy, xz, y^2, yz)$ then J is not saturated because

$x \cdot m \subseteq J$ and $y \cdot m \subseteq J$ but $x \notin J$, $y \notin J$.

In fact $I = J^{sat}$.

6. Fact:
$\operatorname{depth}(R/I) \ge 1$ iff $\times L$ is injective $\forall t$ (see Def 3.7 of notes)

This happens, for example, when I is a saturated ideal.

7. So when $\operatorname{depth}(R/I) = 0$ there's no hope that $\times L$ will always be injective! Best hope is that for each t, $\times L$ is either injective or surjective

Def The algebra R/I has the WLP if $\times L : [R/I]_{t-1} \to [R/I]_t$ has maximal rank (i.e. ...) for all t, for general $L \in [R]_1$

R/I has the SLP if $\times L^d : [R/I]_{t-d} \to [R/I]_t$ has max rank for all t and all d, for general $L \in [R]_1$.

Remark If $\operatorname{depth}(R/I) \geq 1$ then R/I (trivially) has WLP and SLP.

So for WLP to be interesting we need $\operatorname{depth}(R/I) = 0$

where do we look for depth 0 algebras?

We'll see two situations in a moment.

Remark (Simplifications) (useful)

1. To prove WLP for R/I it's enough to find one linear form that has max rank. (All characteristics - open property)
2. If R/I is monomial, R/I has WLP if and only if $L = x_1 + \dots + x_n$ has max rank.

   ($\Leftarrow$ follows from #1. $\Rightarrow$ is a special property for monomial algebras.) } Giuseppe will talk about this.
3. If R/I is Gorenstein (e.g. CI) then R/I has the WLP $\Leftrightarrow$ $\times L$ has maximal rank "in the middle" (more later)

Ex $R = k[x,y,z]$, $I = (x^3, y^3, z^3)$ $L = x+y+z$

$HF = (1, 3, 6, 7, 6, 3, 1)$ Let's see how the characteristic matters.

$\downarrow$ cont.

R/I has WLP

$\overset{\text{def}}{\Longleftrightarrow}$ $\times L : [R/I]_{t-1} \to [R/I]_t$ has max'l rk for gen'l $L \in [R]_1$ and all t

$\overset{\text{Fact 1}}{\Longleftrightarrow}$ $\times L : [R/I]_{t-1} \to [R/I]_t$ has max'l rk for some $L \in [R]_1$ and all t

$\overset{\text{Fact 2}}{\Longleftrightarrow}$ $\times L : [R/I]_{t-1} \to [R/I]_t$ has max'l rk for $L = x+y+z$ and all t

$\overset{\text{Fact 3}}{\Longleftrightarrow}$ $\times L : [R/I]_2 \to [R/I]_3$ is injective for $L = x+y+z$ (*)

$\underset{+\text{ duality}}{\overset{\text{Fact 3}}{\Longleftrightarrow}}$ $\times L : [R/I]_3 \to [R/I]_4$ is surjective for $L = x+y+z$

Use (*). Let $f = a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2 \in [R/I]_2$

$$Lf = x^2y(a_1+a_2) + x^2z(a_1+a_3) + xy^2(a_2+a_4) + xyz(a_2+a_3+a_5) + xz^2(a_3+a_6) + y^2z(a_4+a_5) + yz^2(a_5+a_6)$$

omit

$Lf = 0 \iff (a_1, a_2, a_3, a_4, a_5, a_6) = (a_1, -a_1, -a_1, a_1, -a_1, a_1)$ and $3a_1 = 0$

So injective iff $\mathrm{char}(k) \neq 3$. The characteristic matters!

<u>Remark</u> one way (not the most interesting, but important) for $\times L : [R/I]_i \to [R/I]_{i+1}$ to fail to be injective is if R/I has a <u>socle</u> element in degree i:

$f \in [R/I]_i$ is a socle element if $f \cdot x_i = 0$ in R/I $\forall i$

i.e. $f \in \dfrac{I : m}{I}$ where $m = (x_1, \dots, x_n)$.

We'll say more about socle later.

Note the difference:

- given f, $Lf = 0$ $\forall L$ $\leftrightarrow$ socle
- given L $\exists f$ s.t. $Lf = 0$ $\leftrightarrow$ fails injectivity