lecture 26

Last time: Some important areas of research on WLP:

(a) do all artinian Gorenstein algebras have WLP? (any # of variables)

(b) do all CI's have WLP? (any # of variables)

(c) Monomial algebras

(d) Ideals generated by powers of linear forms

We talked about (a).

We've said some things about (c) but we won't say more in this course.

But I'll repeat the first interesting example (see Exercise 51)

came in a paper by Brenner & Kaid in 2007

$I = (x^3, y^3, z^3, xyz) \subset k[x,y,z] = R$

$HF = (1, 3, 6, 6, 3)$

R/I is level (ie ---) but fails WLP.

We'll touch on (d) later.

So for now just say a word about (b).

(b) (complete intersections) are known in char. 0 for $n \leq 3$.

The proof is in HMNW (attached) and introduced the use of rank 2 vector bundles and the Grauert-Mülich theorem. Give overview in a moment.

For CI's of higher codimension (ie $n \geq 4$) only partial results are known...

theorem (HMNW)

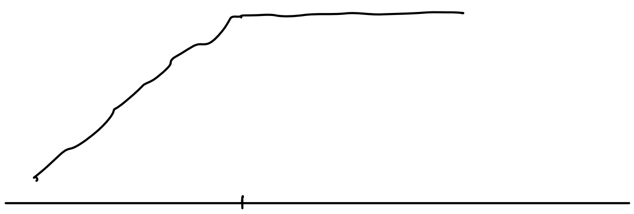
Every CI in $k[x,y,z]$ over a field of char. 0 has the WLP.

Let's give the main ideas of the proof. Let (f_1, f_2, f_3) be a CI, $\deg f_i = d_i$

- $I = (f_1)$ or $I = (f_1, f_2)$

  R/I is CM with depth = dim > 0 so $\times L$ is injective in all degrees $\Rightarrow$ WLP holds

  for (f_1, f_2) the HF looks like



$d_1 + d_2 - 2$

reaches top value here

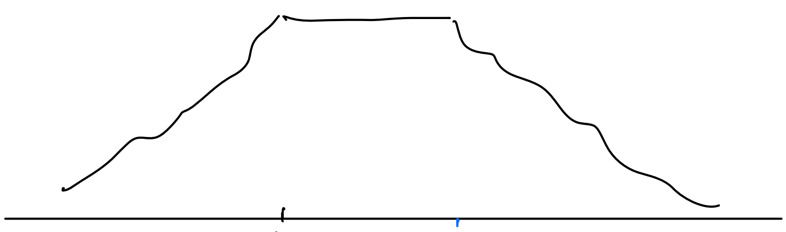
- $I = (f_1, f_2, f_3)$ with $d_3 \geq d_1 + d_2 - 1$

  e.g. $(d_1, d_2, d_3) = (2, 3, 7)$

  HF

  $(1, 3, 5, 6, 6, 6, 6, 5, 3, 1)$

In general the HF looks like



(symmetric)

$d_1 + d_2 - 2$ $\quad$ d_3

same as (f_1, f_2) $\Rightarrow$ injective here

isom $\Rightarrow$ isom here

surjective by self-duality

$\Rightarrow$ WLP holds

• $I = (f_1, f_2, f_3)$ with $d_3 \le d_1 + d_2 - 2$, let L be a general linear form.

WTS $[R/I]_{t-1} \xrightarrow{\times L} [R/I]_t$ has max rank $\forall t$

Set up comm. diagram: let $\mathbb{F} = R(-d_1) \oplus R(-d_2) \oplus R(-d_3)$

$\mathbb{G} = R(-d_1-d_2) \oplus R(-d_1-d_3) \oplus R(-d_2-d_3)$

Consider the koszul resol.

$$0 \to R(-d_1-d_2-d_3) \to \mathbb{G} \to \mathbb{F} \to R \to R/I \to 0$$

Break into two short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & R(-d_1-d_2-d_3) \arrow[r] & \mathbb{G} \arrow[rr] \arrow[dr] & & \mathbb{F} \arrow[r] & R \arrow[r] & R/I \arrow[r] & 0 \\
 & & & E \arrow[ur] \arrow[dr] & & & & \\
 & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

E is the syzygy module. We get two exact sequences:

$$0 \to R(-d_1-d_2-d_3) \to \mathbb{G} \to E \to 0$$

$$0 \to E \to \mathbb{F} \to R \to R/I \to 0$$

Make a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & E(-1) \arrow[r] & \mathbb{F}(-1) \arrow[r] \arrow[d, "{\begin{bmatrix} L & & \\ & L & \\ & & L \end{bmatrix}}"] & R(-1) \arrow[r] \arrow[d, "L"] & R/I(-1) \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r] & \mathbb{F} \arrow[r] & R \arrow[r] & R/I \arrow[r] & 0
\end{tikzcd}$$

Let $\bar{R} = R/(L) \cong k[x,y]$

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & E(-1) \arrow[r] & \mathbb{F}(-1) \arrow[r] \arrow[d, "\left[\begin{smallmatrix} L & & \\ & L & \\ & & L \end{smallmatrix}\right]"] & R(-1) \arrow[r] \arrow[d, "L"] & R/I\,(-1) \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/I \arrow[r] & 0 \\
 & & \overline{\mathbb{F}} \arrow[r] \arrow[d] & \overline{R} \arrow[d] & & \\
 & & 0 & 0 & &
\end{tikzcd}
$$

Put in kernel + cokernel of bottom map

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & E(-1) \arrow[r] & \mathbb{F}(-1) \arrow[r] \arrow[d, "\left[\begin{smallmatrix} L & & \\ & L & \\ & & L \end{smallmatrix}\right]"] & R(-1) \arrow[r] \arrow[d, "L"] & R/I\,(-1) \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/I \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] & \overline{\mathbb{F}} \arrow[r] \arrow[d] & \overline{R} \arrow[r] \arrow[d] & B \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}
$$

Snake lemma:

$$0 \to E(-1) \to E \to A \to R/I\,(-1) \xrightarrow{\times L} R/I \to B \to 0$$

Look in degree t

$$0 \to [E(-1)]_t \to [E]_t \to [A]_t \to [R/I\,(-1)]_t \xrightarrow{\times L} [R/I]_t \to [B]_t \to 0$$

(*) It's enough if we show that $\forall t$ either $[A]_t = 0$ or $[B]_t = 0$.

Sheafify (sheaves on $\mathbb{P}^2$). Let $\mathcal{E} = \tilde{E}$ locally free sheaf of rk 2 on $\mathbb{P}^2$

$$\mathcal{F} = \widetilde{\mathbb{F}}$$
$$\overline{\mathcal{F}} = \widetilde{(\overline{\mathbb{F}})}$$
$$\mathcal{O}_{\mathbb{P}^2} = \tilde{R}$$
$$\mathcal{O}_{\mathbb{P}^1} = \widetilde{(\overline{R})}$$
$$\mathcal{A} = \tilde{A}$$

$\widetilde{R/I}$ and $\tilde{B}$ are 0 because of finite length.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E}(-1) \arrow[r] & \mathcal{F}(-1) \arrow[r] \arrow[d] & \mathcal{O}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A} \arrow[r] & \overline{\mathcal{F}} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Note $\oplus H^1(\mathbb{P}^2, \mathcal{E}(t)) = R/I$
$\oplus H^1(\mathbb{P}^1, \mathcal{A}(t)) = B$
$\oplus H^0(\mathbb{P}^1, \mathcal{A}(t)) = A$

Snake Lemma gives us blue arrows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E}(-1) \arrow[r] \arrow[d] & \mathcal{F}(-1) \arrow[r] \arrow[d] & \mathcal{O}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & \overline{\mathcal{F}} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$\Rightarrow \mathcal{A} = \mathcal{E}|_{\mathbb{P}^1}$$

It turns out that for $d_3 \leq d_1 + d_2 - 2$, $\mathcal{E}$ is semistable

$$\overset{G-M}{\Longrightarrow} \quad \mathcal{E}|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b) \quad \text{with } |a-b| \leq 1$$

$\Longrightarrow$ at most one of $h^0(\mathcal{A}(t))$ and $h^1(\mathcal{A}(t))$ can be nonzero

$\Longrightarrow$ $(*)$ holds.