Lecture 36.

Connection between Lefschetz properties and Weddle loci. (Strictly in the context of $Z = 6$ pts in LGP, continuing example.)

Along the way we'll recover the fact that the Weddle locus of Z is a surface of degree 4.

→ this is based on POLITUS 1 — see that or see the notes for this course.

$Z = \{P_1, \dots, P_6\} = 6$ pts in $\mathbb{P}^3$ in LGP

$\mathcal{W}$ = set of pts P from which $\pi_P(Z)$ lies on a conic. (Weddle)

Let's focus on $\mathcal{W}$. Some notation:

$R = k[x,y,z,w]$

Let $\wp_i = \mathbb{I}(P_i)$

$\wp = \mathbb{I}(P)$ (point from which we project)

Each point in $\mathbb{P}^3$ is dual to a linear form. For example

$P = [a,b,c,d] \longleftrightarrow ax + by + cz + dw \in [R]_1$

Similarly

$P_1 \longleftrightarrow \ell_1$

$\vdots$

$P_6 \longleftrightarrow \ell_6$

$P \longleftrightarrow \ell$

We want to know:

Question For which a,b,c,d is it true that for $P = [a,b,c,d]$,

$\dim [\wp_1 \cap \dots \cap \wp_6 \cap \wp^2]_2 \neq 0$ (equiv. to projecting to a conic)

Fact from Macaulay duality:

$$\dim\left[\wp_1 \cap \cdots \cap \wp_6 \cap \wp^2\right]_2 = \dim\left[R/(\ell_1^2, \ldots, \ell_6^2, \ell)\right]_2$$

So we focus on that. Recall $\ell = ax+by+cz+dw$. New formulation:

For which a,b,c,d is it true that $\dim\left[R/(\ell_1^2, \ldots, \ell_6^2, \ell)\right]_2 \neq 0$?

We consider the following familiar exact sequence

$$\left[R/(\ell_1^2, \ldots, \ell_6^2)\right]_1 \xrightarrow{\times \ell} \left[R/(\ell_1^2, \ldots, \ell_6^2)\right]_2 \longrightarrow \left[R/(\ell_1^2, \ldots, \ell_6^2, \ell)\right]_2 \longrightarrow 0$$

We want to describe which $\ell = ax+by+cz+dw$ make this map fail to have max. rank.

Represent $\times\ell$ by a matrix A whose entries depend on a,b,c,d.

What does this matrix look like?

1. Size

Clearly $\dim\left[R/(\ell_1^2, \ldots, \ell_6^2)\right]_1 = \dim[R]_1 = 4$

Also $\underbrace{\dim\left[R/(\ell_1^2, \ldots, \ell_6^2)\right]_2 = \dim\left[\wp_1 \cap \cdots \cap \wp_6\right]_2}_{\text{Macaulay duality}} = \dim\left[I(Z)\right]_2$

But Z imposes indep conditions on forms of degree 2 so

$$\dim\left[I(Z)\right]_2 = \dim[R]_2 - |Z| = 10 - 6 = 4$$

So A is a 4×4 matrix

2. Entries of A:

Choose bases for $[R/(l_1^2,\dots,l_6^2)]_1 = [R]_1$

and for $[R/(l_1^2,\dots,l_6^2)]_2$.

we know $l = ax+by+cz+dw$

The linear form x is represented by a 4×4 matrix A_1 of scalars

$\vdots$

The linear form w is represented by a 4×4 matrix A_4 of scalars

$\Rightarrow$ l is represented by the 4×4 matrix

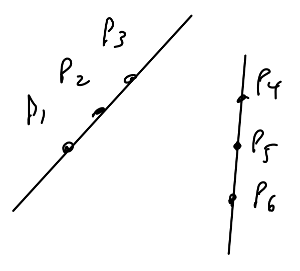
$$A = aA_1 + \dots + dA_4$$

where each entry of A is a linear form in a,b,c,d

So the determinant of A gives a poly. of degree 4 which gives the locus of points W where projection lies on a conic.

Conclusion: W is both the Weddle locus for Z and the non-Lefschetz locus for $R/(l_1^2,\dots,l_6^2)$!

Example

$Z =$ 

this Z also imposes indep conditions on quadrics

As above, we consider the algebra $R/(l_1^2,\dots,l_6^2)$.

The non-Lefschetz locus = locus of points where projection sends Z to points on a conic = all $\mathbb{P}^3$. (The 4×4 determinant is identically zero.)

$\Rightarrow$ $R/(l_1^2,\dots,l_6^2)$ fails the WLP! Purely from the geometry!