The Weak Lefschetz Property for artinian Gorenstein algebras of low Sperner number

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Joint work with

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Remark (Banff, March 2024)

Artinian Gorenstein algebras

Definition

A graded artinian $\mathbb{K}$-algebra A is *Gorenstein* if it has a perfect pairing

$$A_i \times A_{d-i} \longrightarrow A_d, \quad \forall i$$

Macualay's Inverse Systems describe them. $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ acts on $S = \mathbb{K}[X_1, X_2, \ldots, X_n]$ by *differentiation*

$$x_i \circ F = \frac{\partial F}{\partial X_i}$$

Theorem (Macaulay)

If $A = R/I$ is a graded artinian algebra then

$$A = \bigoplus_{i=0}^{d} A_I \text{ is Gorenstein} \iff \exists F \in S_d : I = \operatorname{ann}_R(F)$$

The Lefschetz properties

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has maximal rank for all $i \geq 0$.

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has maximal rank for all $i \geq 0$.
2. A has the *Strong Lefschetz Property (SLP)* if $\exists \ell \in A_1$ such that
$$\times \ell^j \colon A_i \longrightarrow A_{i+j}$$
has maximal rank for all $i \geq 0$ and $j \geq 1$

General problem

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- Codimension 1 and 2 all have the SLP. (Briancon, 1972)
- Codimension 3 complete intersections all have the WLP (Harima, Migliore, Nagel and Watanabe, 2000)
- Codemension 3 socle degree at most 6 all have the SLP (B, Migliore, Nagel, Miró-Roig and Zanello, 2013)
- Codimension 3 and Sperner number at most 6 all have the SLP (Abdallah, Altafi, Iarrobino, Seceleanu and Yameogo, 2023)

Failure of WLP

Example (Stanley 1978)

Trivial extension of $B = k[x,y,z]/(x,y,z)^4$ with canonical module $A = B \ltimes \omega_B$.

B	1	3	6	10	0
ω_B	0	10	6	3	1
$A = B \ltimes \omega_B$	1	13	12	13	1

Failure of WLP

Example

Stanley's construction works for $B = k[x,y]/(x,y)^d$ to give

$$\begin{array}{c|cccccccc} B & 1 & 2 & 3 & 4 & \dots & d-1 & d & 0 \\ \omega_B & 0 & d & d-1 & d-2 & \dots & 3 & 2 & 1 \\ \hline B \ltimes \omega_B & 1 & d+2 & d+2 & d+2 & \dots & d+2 & d+2 & 1 \end{array}$$

which fails the WLP when $d \geq 3$.

Failure of WLP

Example

We can do this in codimension 4 to get

$$\begin{array}{c|ccccccccccc}
B & 1 & 2 & 3 & \dots & i+1 & i+2 & \dots & 2i & \dots & 4 & 2 & 0 \\
\omega_B & 0 & 2 & 4 & \dots & 2i & 2i-1 & \dots & i+1 & \dots & 3 & 2 & 1 \\
\hline
B \ltimes \omega_B & 1 & 4 & 7 & \dots & d+2 & d+2 & \dots & d+2 & \dots & 7 & 4 & 1
\end{array}$$

if $i = (d+1)/3 \in \mathbb{Z}$.

$$\begin{array}{r|cccccccccccc}
d=5 & 1 & 4 & 7 & 7 & 4 & 1 \\
d=7 & 1 & 4 & 7 & 9 & 9 & 7 & 4 & 1 \\
d=8 & 1 & 4 & 7 & 10 & 10 & 10 & 7 & 4 & 1 \\
d=9 & 1 & 4 & 7 & 10 & 11 & 11 & 10 & 7 & 4 & 1 \\
d=10 & 1 & 4 & 7 & 10 & 12 & 12 & 12 & 10 & 7 & 4 & 1 \\
d=11 & 1 & 4 & 7 & 10 & 13 & 13 & 13 & 13 & 10 & 7 & 4 & 1
\end{array}$$

Main result

These examples turn out to be minimal in the following sense.

Definition

The *Sperner number* of A is $\max_i h_i(A)$.

Theorem (B., Migliore,Miró-Roig, Nagel 2024)

Any artinian Gorenstein algebra A of socle degree d and Sperner number $\leq d+1$ satisfies the WLP.

An illustrative example

Example

If A has h-vector $(1,6,6,6,6,1)$ and fails the WLP we must have that $\times\ell\colon [A]_i \longrightarrow [A]_{i+1}$ has rank 5 for $i = 1,2,3$.

$$B = A/(0:\ell) \cong (\ell) \quad C = A/(\ell)$$

$$\begin{array}{rcccccc} B(-1): & 0 & 1 & 5 & 5 & 5 & 1 \\ A: & 1 & 6 & 6 & 6 & 6 & 1 \\ C: & 1 & 5 & 1 & 1 & 1 & 0 \end{array}$$

The short exact sequence

$$0 \longrightarrow B(-1) \longrightarrow A \longrightarrow C \longrightarrow 0$$

and the Snake Lemma show that B has to fail the WLP.

More generally

Example

If A has has socle degree $2m+1$, fails the WLP and $h_{m-1} = h_m = h_{m+1} = h_{m+2} = s$

$$B = A/(0:\ell) \cong (\ell) \quad C = A/(\ell)$$

deg	$m-1$	m	$m+1$	$m+2$
$B(-1)$	$s-a$	$s-1$	$s-1$	$s-1$
A	s	s	s	s
C	a	1	1	1

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deg	$m-1$	m	$m+1$	$m+2$
$B(-1)$	$s-a-1$	$s-1$	$s-1$	$s-1$
A	$s-1$	s	s	$s-1$
C	a	1	1	0

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- When the socle degree d is even or if $d = 2m+1$ with $h_m - h_{m-1} \leq 1$, we can argue by induction.
- We are left with socle degree $d = 2m+1$ and h-vector
  1. $(1,3,5,7,9,\ldots,2m+1,2m+1,\ldots,9,7,5,3,1)$
  2. $(1,3,6,8,10,\ldots,2m+2,2m+2,\ldots,10,8,6,3,1)$
  3. $(1,4,6,8,10,\ldots,2m+2,2m+2,\ldots,10,8,6,4,1)$

Subalgebra argument

Example

If A has h-vector $(1, 4, 6, 6, 4, 1)$ and there is $H = \langle \ell_1 . \ell_2, \ell_3 \rangle \subseteq [A]_1$ with $\dim_{\mathbb{K}} H^2 = 6$. Then

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- B has an artinian Gorenstein quotient C with h-vector $(1,3,6,6,3,1)$.
- We can deduce WLP for A from WLP for C.
- If $A = R/I$ has is no such subalgebra, the generic inititial monimials wrt the lexicographic order of $[I]_2$ are

$$x_1^2, x_1 x_2, x_1 x_3, x_2^2$$

Otherwise

Theorem

If $I = (f_1, f_2, f_3, f_4) \subset R = K[x_1, \ldots, x_4]$ *is generated by quadrics and* $gin_{lex}(I) \supseteq \{x_1^2, x_1x_2, x_1x_3, x_2^2\}$*, then up to isomorphism,*

(i) $I = (x_1x_3, x_1x_4, x_2x_3, x_2x_4)$;

(ii) $I = (x_1^2, x_2^2, x_3^2, x_1x_2 + x_1x_3 + x_2x_3)$;

(iii) $I = (x_1^2, x_2^2, x_3^2, x_1x_3 + x_2x_3)$;

(iv) $I = (x_1^2, x_1x_2, x_1x_3 - x_2^2, x_3^2)$;

(v) $I = (x_1^2, x_1x_2, x_1x_3 - x_2^2, x_2x_3)$;

(vi) $I = (x_1^2, x_1x_3, x_2^2, x_2x_3)$;

(vii) $I = (x_1^2, x_1x_2, x_1x_3, x_2x_3)$;

(viii) $I = (x_1^2, x_1x_2, x_2^2, q)$*, where* $q = x_3x_4$*,* $q = x_3^2 + x_2x_4$ *or* $q = x_3^2$*.*

(ix) $I = (x_1^2, x_1x_2, x_2^2, x_1x_4 - x_2x_3)$; *or*

(x) $I = (x_1^2, x_1x_2, x_2^2, x_1x_3)$.

Remark

The Hilbert function of I is

- $(1,4,6,6,6,\ldots)$ in cases (ii), (iii), (iv) and ($viii$),
- $(1,4,6,7,8,9,\ldots)$ in cases (v) and (vi), and
- $(1,4,6,8,10,\ldots)$ in cases (i), (vii), (ix) and (x).

Hence only (i), (vii), (ix) and (x) can occur for socle degree seven and higher. ($m \geq 3$)

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- In case (i), we have two skew lines, and A is the *connected sum* of two codimension two algebras.

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- In case (i), we have two skew lines, and A is the *connected sum* of two codimension two algebras.
- For the remaining cases, we can use explicit calculations with *catalecticant* matrices and *higher hessian* matrices.

Geometry of the quadrics

In proving Theorem 3, we use the existence of a rational map

$$\begin{array}{rcccccc} \Phi\colon \mathbb{P}^3 & = & \mathbb{P}([R]_1) & \longrightarrow & \mathbb{P}^3 & = & \mathbb{P}([I]_2). \\ & & H & \mapsto & q_H & & \end{array}$$

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- If the rank of q_H is generically one the image is a plane conic and the ideal contains $\langle \ell_1, \ell_2 \rangle^2$.