Lefschetz-type properties of cohomology rings

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- We end with some questions.

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- $H^k(\mathbb{C}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$

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- Recursively, $H^i(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}[Y]^{i/2} & \text{if i even} \\ 0 & \text{otherwise} \end{cases}$

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- Therefore, the cohomology ring

$$H^*(X,\mathbb{Q}) = \bigoplus_{i \geq 0} H^i(X,\mathbb{Q})$$

has a natural graded Artinian ring structure with multiplication operator given by cup-product.

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- Let X be a smooth, projective curve of genus g. Then,
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- $H^1(X, \mathbb{Q})$ is a symplectic vector space under cup-product (i.e., vector space equipped with a non-degenerate alternating bilinear form).
- We can find a symplectic basis $e_1, e_2, ..., e_{2g}$ of $H^1(X, \mathbb{Q})$ i.e.,

$$e_i \cup e_{i+g} = -f, \text{ and } e_i \cup e_j = 0 \text{ for } j \neq i+g \text{ and } f \in H^2(X, \mathbb{Z})$$

  is a positive generator.

Examples: Contd...

- In this case,

$$H^*(X,\mathbb{Q}) \xrightarrow{\sim} \frac{\mathbb{Q}[t_0, t_1, t_2, ..., t_{2g}]}{(t_i t_{i+g} + t_0, t_i t_j, t_0^2)},$$

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- Hence, we have a ring isomorphism $(H^2(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z}\{p\})$:

$$\mathbb{Q}[t_1, t_2]/(t_1^2, t_2^2) \xrightarrow{\sim} H^*(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q}), \text{ sending } t_1 \mapsto \{p\} \oplus 0$$

and $t_2 \mapsto 0 \oplus \{p\}$, where $p \in \mathbb{P}^1$ is a closed point.

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  • Therefore, C satisfies the strong Lefschetz property. In particular, C satisfies the weak Lefschetz property.

Counterexample to NC-WLP

- **Theorem (-)**: Product of smooth, projective curves do not always satisfy the weak Lefschetz property. In particular, take a smooth rational curve X and a smooth curve Y of genus at least 1. Then, $X \times Y$ does not satisfy the WLP.

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- By the Künneth decomposition, we have:

$$H^1(X \times Y, \mathbb{Q}) \cong q^* H^1(Y, \mathbb{Q}), \text{ and}$$
$$H^2(X \times Y, \mathbb{Q}) \cong p^* H^2(X, \mathbb{Q}) \oplus q^* H^2(Y, \mathbb{Q}) \cong \mathbb{Q}^{\oplus 2}.$$

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H^1(Y, \mathbb{Q}) \arrow[r, "0 \oplus (\cup \zeta')"] \arrow[d, "q^*"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & H^2(X, \mathbb{Q}) \oplus H^2(Y, \mathbb{Q}) \arrow[d, "\cong"', "p^* \oplus q^*"] \\
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- Therefore, $\cup\zeta$ is neither injective, nor surjective. So, $X \times Y$ does not satisfy WLP.

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- Restriction of $\mathbb{H}^i$ to the closed (resp. generic point) is canonically isomorphic to

$$\mathbb{H}^i_k \cong H^i(\mathcal{X}_k, \mathbb{Q}) \text{ and } \mathbb{H}^i_K \cong H^i(\mathcal{X}_K, \mathbb{Q}).$$

Specialization of local systems

- Every element $\zeta_0 \in H^1(\mathcal{X}_k)$ (resp. $\zeta_{\mathrm{gen}} \in H^1(\mathcal{X}_K)$) uniquely extends to a section of $\zeta \in \Gamma(R^1\pi_*\mathbb{Q})$.

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- Similarly, if $\cup\zeta_{\mathrm{gen}}$ is injective (resp. surjective), then so is $\cup\zeta_0$ i.e., X_K WLP $\Rightarrow X_k$ WLP.

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- Let X be a smooth, projective variety.

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- **Example:** $H^{*}_{\mathrm{Hdg}}(\mathbb{P}^n) = \mathbb{Q}[t]/(t^{n+1})$.

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- What happens in the quasi-projective setting?

Quasi-projective case

- **Lefschetz hyperplane theorem:** Let Y be a smooth, projective variety of dimension n and $X \subset Y$ a very ample non-singular divisor. Then, the natural restriction map

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- Let's look at an example.....

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- Hence, Lefschetz hyperplane theorem and Hard Lefschetz fail for $U \hookrightarrow \mathbb{P}^4 \backslash x_0$.

What about WLP and SLP?

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- Then, for a general $\zeta \in H^2_{\mathrm{Hdg}}(U)$,

$$\begin{aligned} \cup\, \zeta : H^2_{\mathrm{Hdg}}(U) \to H^4_{\mathrm{Hdg}}(U) & \text{ is surjective} \\ (\cup \zeta)^2 : H^2_{\mathrm{Hdg}}(U) \to H^6_{\mathrm{Hdg}}(U) = 0 & \text{ is trivially surjective} \\ \cup\, \zeta : H^4_{\mathrm{Hdg}}(U) \to H^6_{\mathrm{Hdg}}(U) = 0 & \text{ again, trivially surjective} \end{aligned}$$

What about WLP and SLP?

- We computed: $H^i(U, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 0, 4 \\ \mathbb{Q}^{\oplus 2} & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$
- Then, for a general $\zeta \in H^2_{\mathrm{Hdg}}(U)$,

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- Hence, $H^*_{\mathrm{Hdg}}(U)$ satisfies WLP and SLP.

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- Ciliberto-Harris-Miranda showed that if $n = 1$, then the Noether-Lefschetz locus is analytically as well as Zariski dense in $U_{d,1}$.

Geometry of the Noether-Lefschetz locus

- Let L be an irreducible component of $\mathrm{NL}_{d,1}$. Then,

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- **Harris conjecture:** There are only finitely many special components.
- **Theorem (-):** If $\mathrm{NL}_{d,1}$ is equipped with the natural scheme structure (as Hodge loci), then there are infinitely many special components (of codimension $2d-6$) for all $d \geq 6$.

Questions

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- Then, we use arguments similar to the specialization of the weak / strong Lefschetz property mentioned earlier.
- **Question 2:** Let Y be a smooth, projective variety and $X \subset Y$ be a non-singular, very ample divisor in Y. If Y satisfies weak / strong Lefschetz property, then does X satisfy the same property? Conversely, if X satisfies weak / strong Lefschetz property, then does Y satisfy the same property?

Questions contd...

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- Remaining to check for $i = n-1$.

Thank you for your attention !