Binary Trees of ideals

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Junzo Watanabe

Joint work with T.Harima and S.Isogawa

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Lemma (Ikeda, 1996)

(1) Suppose A **is a graded Artinian algebra and** (A, l) **is SL. Then** $(\tilde{A} = A[z]/(z^2), l + z)$ **is SL. (**z **is a new variable.)**

Corollary

Suppose (A, l) **is SL. Then**

(2) $(A[z_1, \ldots, z_m]/(z_1^2, \ldots, z_m^2), l + z_1 + \cdots + z_m)$ **is SL.**

(3) $(A[y]/(y^{m+1}), l + y)$ **is SL.**

(4) Monomial ci (complete intersections) are SL.

Proof. (1) can be proved by a diagram chasing. (2) is immediate from (1). (3) follows from (2) by Subring Thm:

$$A[y]/(y^{m+1}) \hookrightarrow A[z_1, \ldots, z_m]/(z_1^2, \ldots, z_m^2).$$

$$y \mapsto z_1 + \cdots + z_m$$

Both have the same socle degree. To prove (3) $\Rightarrow$ **(4) induct on** n**.**

We can extend this theorem as follows:

Proposition

Suppose $A = K[x_1, \ldots, x_n]/I$ **is a complete intersection. Suppose a power of a linear element can be a minimal basis of** I**. I.e.,**

$$I = (f_1, \ldots, f_{n-1}, l^m), l \in R_1$$

Let z **be the image of** l **in** A**. Then we have:** $A/(z)$ **is SL** $\Rightarrow$ A **is SL. I.e., to prove SLP of** A **it is enough to prove SLP of** $A/(z)$**.** z **is the image of** l **in** A**.**

Proof. We can use the flat extension theorem, because we have the coexact sequence

$$K[y]/(y^m) \hookrightarrow A \twoheadrightarrow A/(z).$$

The flat extension theorem itself follows from

$$G = Gr_{(z)}(A) \stackrel{\text{def}}{=} A/(z) \oplus (z)/(z^2) \oplus \cdots \oplus (z^{m-1})/(z^m)$$
$$\cong \big(A/(z)\big)[y]/(y^m).$$

G **is SL by Ikeda's Lemma.**

Theorem

(Harima–Watanabe 2007)

"A is at least as good as $Gr_{(z)}(A)$"

Suppose $A = \oplus_{i=0}^{c} A_i$ is an Artinian Gorenstein ring. Suppose $z \in A_1$. If

$$G = \mathrm{Gr}_{(z)}(A) = A/(z) \oplus (z)/(z^2) \oplus (z^2)/(z^3) \oplus \cdots \oplus (z^m)/(z^{m+1})$$

has SLP, then A has SLP. Lefschetz element of G is $\overline{L} + z^*$.

I am trying to find complete intersections A for which $G = \mathrm{Gr}_{(z)}(A)$ can be computed and prove that it has SLP.

Examples of ci in which a power of a linear element can be a minimal basis of a ci:

Examples: l^k can be a basis element of I.

1. $I = (x_1^{d_1}, \ldots, x_n^{d_n})$**.**

2. $I = (e_1, \ldots, e_n)$**,** e_k **the elementary symmetric polynomial.**

3. $I = (h_d, h_{d_1}, \ldots, h_{d+n-1})$**.** h_k **the complete symmetric polynomial.**

4. $I = (e_1(y), e_2(y), \ldots, e_n(y))$**,** **where** $(y_1, \ldots, y_n) = (x_1^s, \ldots, x_n^s)$**.**

5. $I = (e_1(y), e_2(y), \ldots, e_{n-1}(y), (x_1 \cdots x_n)^r)$**, same** $e_k(y)$ **for** $k < n$**.**

Proof. (1) $l = x_n$**.**

(2) e_n **can be replaced by** x_n^n**.** $I + x_n = (\overline{e}_1, \ldots, \overline{e}_{n-1}, x_n)$**.**

(3) h_{d+n-1} **can be replaced by** x_n^{d+n-1}**.** $I + x_n = (\overline{h}_d, \ldots, \overline{h}_{d+n-2}, x_n)$**.**

(4) $e_n(y)$ **can be replaced by** x_n^{sd}**.**

(5) Similar to (4).

In these examples we can induct on n **and use flat extension thm.**

$\boxed{\text{Generalize these examples}}$

Drop the condition "l^k can be a minimal basis element" in this proposition but try to use the implication

$$G = Gr_{(z)}(A) \textbf{ is SL} \Rightarrow A \textbf{ is SL.}$$

What condition do I need on A and $\times z : A \to A$.

To understand

$$G = G_z(A) \stackrel{\text{def}}{=} A/(z) \oplus (z)/(z^2) \oplus (z^2)/(z^3) \oplus \cdots$$

we have to understand the sequence

$$\cdots \hookrightarrow A/(0:z^3) \hookrightarrow A/(0:z^2) \hookrightarrow A/(0:z) \hookrightarrow A$$

because

$$A/(0:z) \cong (z)$$
$$A/(0:z):z = A/(0:z^2) \cong (z^2)$$
$$A/((0:z):z):z = A/(0:z^3) \cong (z^3)$$
$$\vdots$$
$$\vdots$$

This sequence is close to

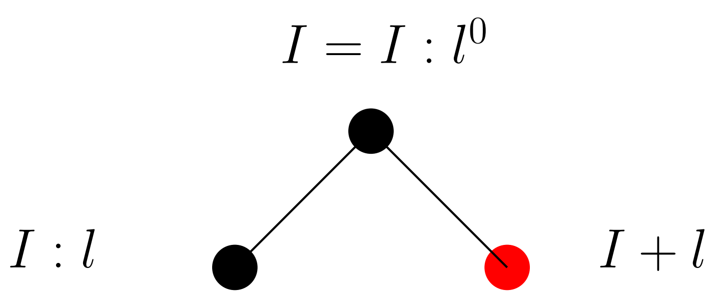
$$0 \subset (z^m) \subset \cdots \subset (z^2) \subset (z) \subset A.$$

(There is a shift of degrees.) If we know the dimensions of these vector spaces, we know the Jordan type of $z : A \to A$.

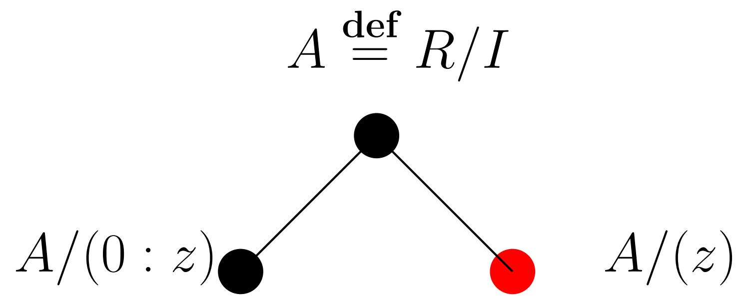
It is convenient to denote the short exact sequence

$$0 \to R/(I : l) \to R/I \to R/(I + l) \to 0$$

by the diagram:



or



z **is the image of** $l \in R$ **in** A**.**

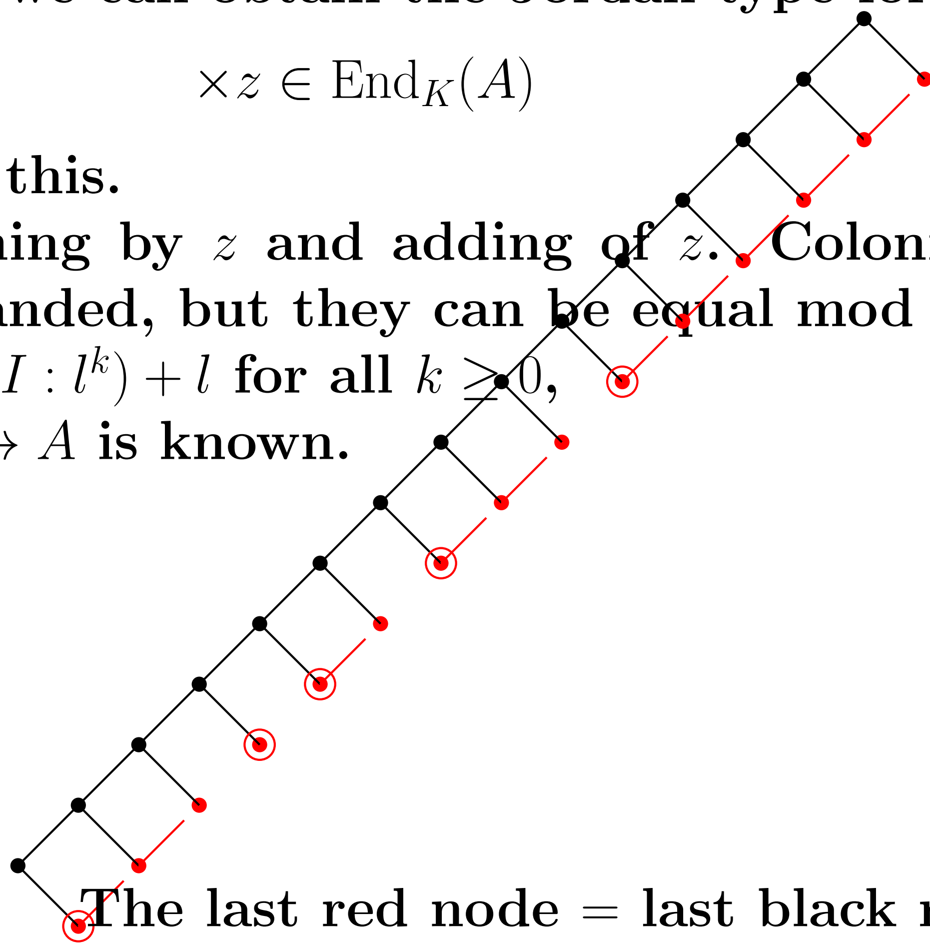
We may think of nodes as ideals or Artinin algebras.

In a general set up, we can obtain the Jordan type for

$$\times z \in \mathrm{End}_K(A)$$

by the diagram like this.

This depicts coloning by z **and adding of** z**. Coloning by** z **the ideal is strictly expanded, but they can be equal mod** z**. If Hilbert seires is known for** $(I : l^k) + l$ **for all** $k \geq 0$**, Jordan type** $\times l : A \to A$ **is known.**



The last red node = last black node.

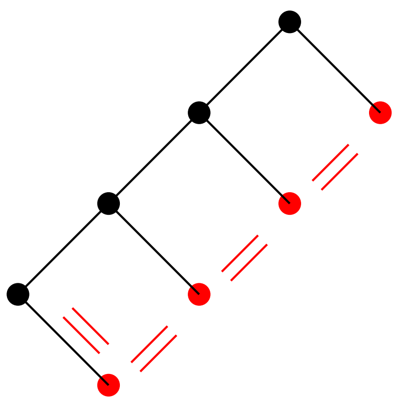
Note $(A/(0:z))\,[1] \cong (z)$ **and**

$$H(A) = H(A/z) + H(A/0:z)[T]$$

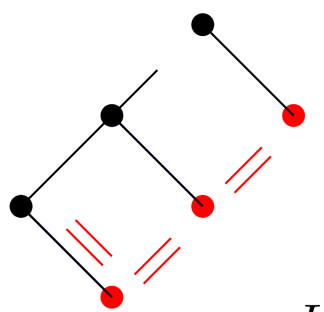
So the diagram can be used to find the Hilbert series of A**, if we know the Hilbert series of** $A/((0:z^k) + z)$ **for all** k**.**

We consider $A = K[x_1, x_2, x_3]/(x_1^3, x_2^3, x_3^4)$**.** $B = K[x_1, x_2, x_3]/(x_1^3, x_2^3, x_3^3)$**,** $z = x_3$**.**

$$\begin{aligned} H(A) =& 1 + 3T + 6T^2 + 8T^3 + 8T^4 + 6T^5 + 3T^6 + T^7 && (1) \\ =& (1 + 2T + 3T^2 + 2T^3 + T^4)(1 + T + T^2 + T^3) && (2) \\ =& 1 + 2T + 3T^2 + 2T^3 + T^4 && (3) \\ & + T + 2T^2 + 3T^3 + 2T^4 + T^5 && (4) \\ & + T^2 + 2T^3 + 3T^4 + 2T^5 + T^6 && (5) \\ & + T^3 + 2T^4 + 3T^5 + 2T^6 + T^7 && (6) \end{aligned}$$



$$H(B) = 1 + 3T + 6T^2 + 7T^3 + 6T^4 + 3T^5 + T^6.$$



$$H(B/(x_3)) = 1 + 2T + 3T^2 + 2T^3 + T^4.$$

$$H(B/(0 : x_3^1) + x_3) = 1 + 2T + 3T^2 + 2T^3 + T^4.$$

$$H(B/(0 : x_3^2) + x_3) = 1 + 2T + 3T^2 + 2T^3 + T^4.$$

Generally, the Jordan type of $\times z : A \to A$ is

$$\times z = J_1 \oplus J_2 \oplus \cdots \oplus J_r, \qquad J_i\textbf{, a Jordan block}.$$

In a flat extension the Jordan block for $\times z$ is the conjugate of

$$\big(\underbrace{|A/(z)|, \cdots, |A/(z)|}_{m}\big).$$

I.e.,

$$\big(\underbrace{m, m, \cdots, m}_{|A/(z)|}\big).$$

The number of Jordan blocks is $|A/(z)|$, the size is m.

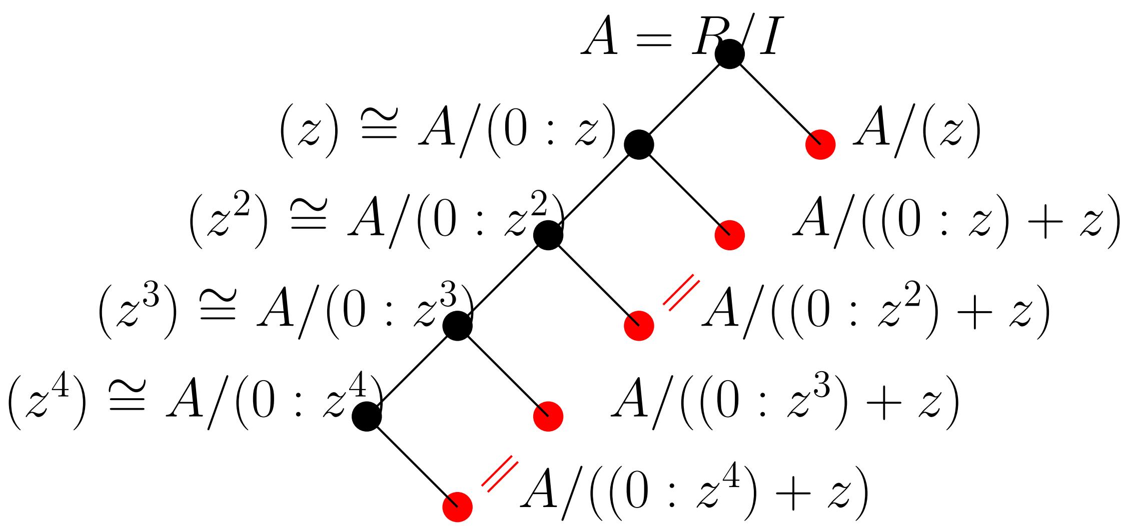
$$\{1_A, z, z^2, \cdots, z^{m-1}\} \subset A, \quad \textbf{is a Jordan block}$$

A lift of an element in $A/(z)$ behaves, more or less, in the same way, but it will give us a smaller block.

In the next diagram we consider $A = K[x_1, x_2, x_3]/(x_1^3, x_2^3, x_3^3)$, $z = x_2 + x_3$. It will be shown that the h-vector can be decomposed adjusted at the middle.

This depicts coloning by l and adding of l.

$$A = R/I,\ \ I = (x_1^3, x_2^3, x_3^3),\ \ l = x_2 + x_3.\text{, } z = l \bmod I.$$



$$|A/(z)| = 9, |A/(0:z)+z| = 6$$

This shows $\times z : A \to A$ has 9 blocks, $\times z : A/(0:z) \to A/(0:z)$ has 6 blocks. Hence $\times z : A \to A$ has 3 blocks of size 1. We can proceed by induction.

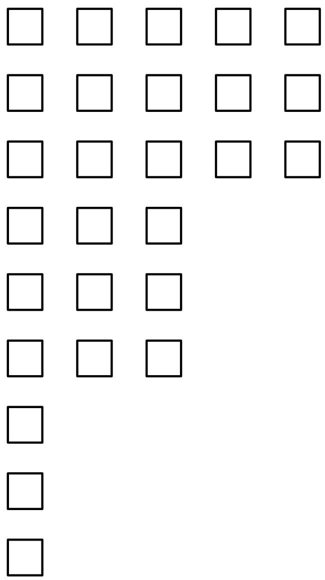
$$|H(A/(0:z^2)+z)| = 6,$$

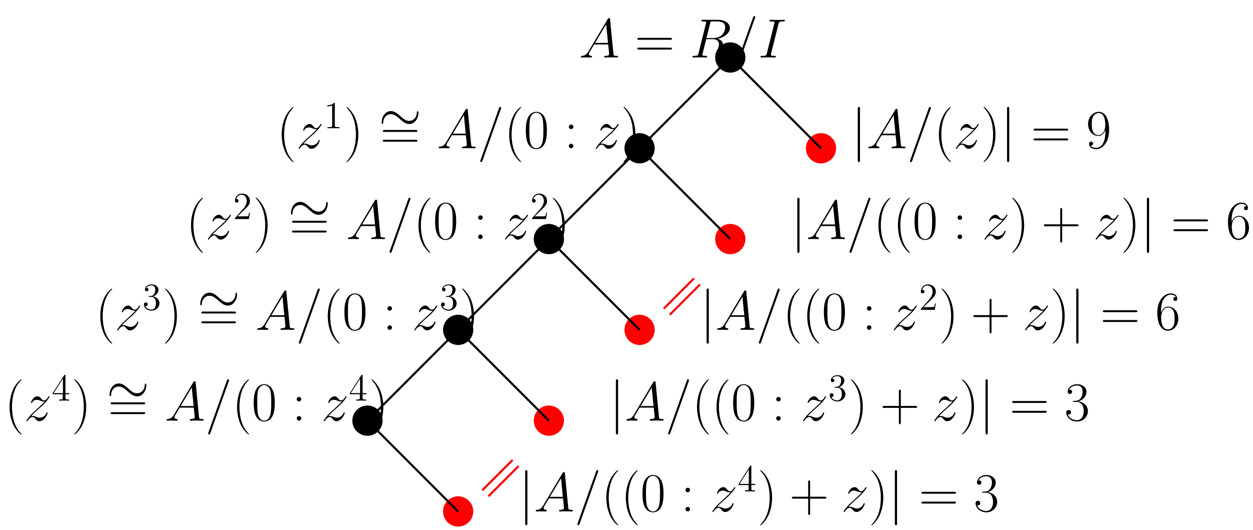
$$|H(A/((0:z^3)+z))| = |H(A/((0:z^3)+z))| = 3.$$

So the Young diagram for this map $\times z : A \to A$

$$A = R/I,\ \ I = (x_1^3, x_2^3, x_3^3),\ \ l = x_2 + x_3.,\ \ z = l \bmod I.$$

is





$$|A/(z)| = 9, |A/(0:z)+z| = 6$$

This shows $\times z : A \to A$ **has** 9 **blocks,** $\times z : A/(0:z) \to A/(0:z)$ **has 6 blocks. Hence** $\times z : A \to A$ **has** 3 **blocks of size** 1**. We can proceed by induction.**

$$|H(A/(0:z^2)+z)| = 6,$$
$$|H(A/((0:z^3)+z))| = |H(A/((0:z^3)+z))| = 3.$$

Each box has a degree. So we can put its degree in each box.

$$
\begin{array}{lllll}
\boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \\
\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} \\
\boxed{2} & \boxed{3} & \boxed{4} & \boxed{6} & \boxed{6} \\
\boxed{1} & \boxed{2} & \boxed{3} & & \\
\boxed{2} & \boxed{3} & \boxed{4} & & \\
\boxed{3} & \boxed{4} & \boxed{5} & & \\
\boxed{2} & & & & \\
\boxed{3} & & & & \\
\boxed{4} & & & &
\end{array}
\qquad\qquad
\begin{array}{lllllll}
\boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & & \\
 & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} & \\
 & & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{6} & \boxed{6} \\
 & \boxed{1} & \boxed{2} & \boxed{3} & & & \\
 & & \boxed{2} & \boxed{3} & \boxed{4} & & \\
 & & & \boxed{3} & \boxed{4} & \boxed{5} & \\
 & & \boxed{2} & & & & \\
 & & & \boxed{3} & & & \\
 & & & & \boxed{4} & &
\end{array}
$$

$$1 \quad 3 \quad 6 \quad 7 \quad 6 \quad 3 \quad 1$$

Proposition

It is possible to decompose the graded vector space A into smaller spaces with symmetric h-vectors adjusted at the center. (We need the assumption A has a symmetric h-vector.)

What condition do I need to conclude A has SLP, provided we know $A/((0:z_n^j)+z_n)$ for all $i=0,1,2,\cdots$?

Remark

All

$$\frac{(0:z^k)+z}{(0:z^{k-1})+z} \text{ for } k=0,1,2,\ldots$$

are $A/(z)$-modules, symmetric h-vector. In general these are not algebra, but sometimes they have an algebra structure!

So this was a motivation for central simple modules. Let me explain the "central modules" for $\times z : A \to A$.

Central simple modules for $\times z : A \to A$

A block in the Jordan decomposition is an $n_1 \times n_1$ matrix like $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. $(\times z) = J_1 \oplus J_2 \oplus \cdots \oplus J_r$ can be described as a list of sizes.

The Jordan type can be expressed as a decreasing sequence of integers:

$$\underbrace{n_1, \ldots, n_1}_{m_1}, \underbrace{n_2, \ldots, n_2}_{m_2}, \underbrace{n_3, \ldots, n_3}_{m_3}, \underbrace{n_4, \ldots, n_4}_{m_4},$$

where $n_1 > n_2 > \cdots > n_4$. A Young diagram

$$\begin{array}{l} \square\square\square\square\square\square \\ \square\square\square\square\square\square \\ \square\square\square \end{array}$$

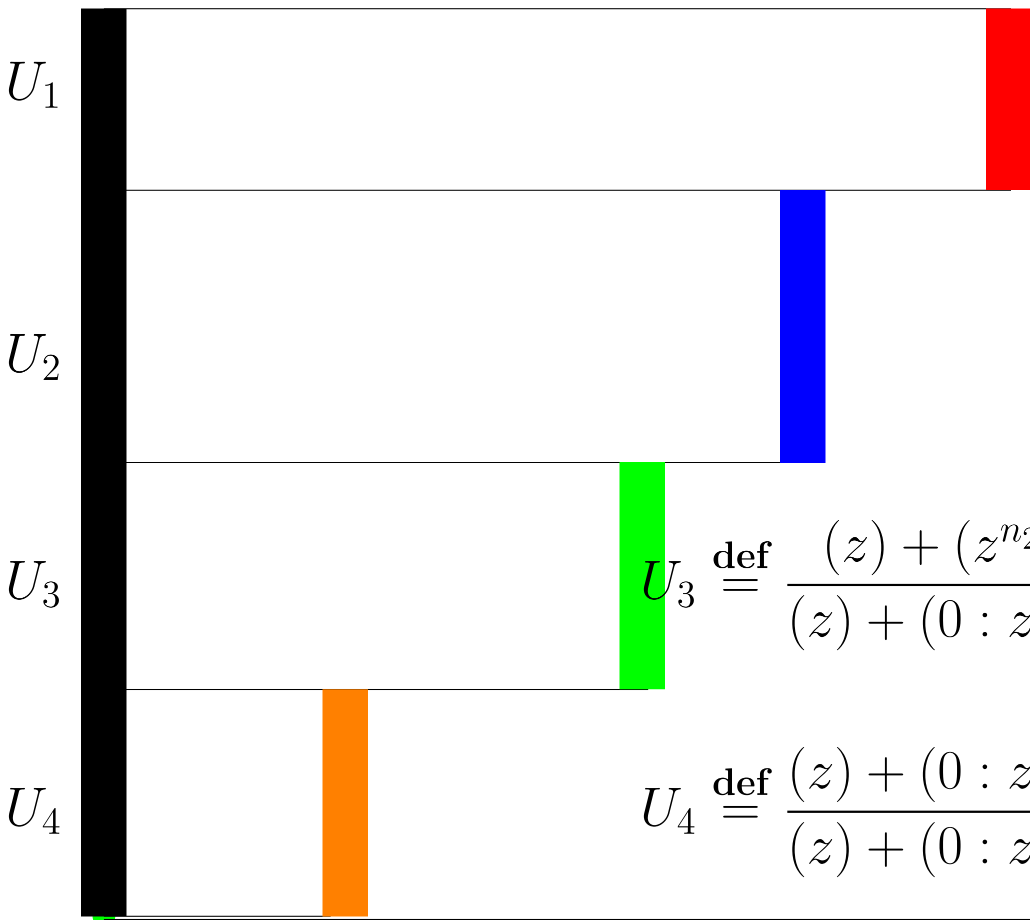
can be used to depict the map $\times z : A \to A$. $\square$ is a basis element of

A **and** $\times z$ **sends a box to the next box on its right.**

The number of blocks in the Jordan decomposition of $\times z : A \to A$ **is equal to** $|A/(z)| = |0 : z|$**.**

So to find a Jordan decomposition, choose $a_1, \ldots, a_r \in A$ **such that** $A = \langle a_1, \ldots, a_r \rangle + (z)$**, apply** $\times z$ **to these elements and see when they reach zero.**

This is shown in the next picture.



$$V_1 \overset{\textbf{def}}{=} (0:z) \cap (z^{n_1})$$

$$V_2 \overset{\textbf{def}}{=} \frac{(0:z) \cap (z^{n_2})}{(0:z) \cap (z^{n_1})}$$

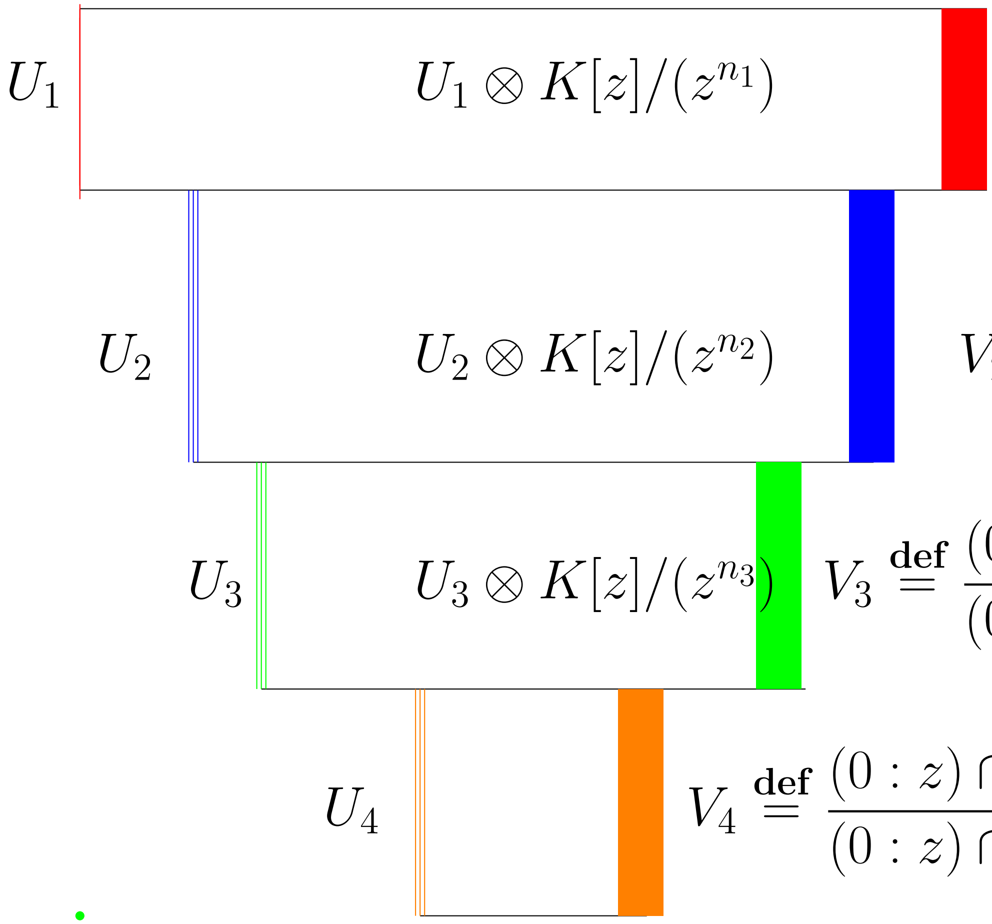
$$U_3 \overset{\textbf{def}}{=} \frac{(z) + (z^{n_2})}{(z) + (0:z^{n_3})} \qquad V_3 \overset{\textbf{def}}{=} \frac{(0:z) \cap (z^{n_3})}{(0:z) \cap (z^{n_2})}$$

$$U_4 \overset{\textbf{def}}{=} \frac{(z) + (0:z^{n_3})}{(z) + (0:z^{n_4})} \qquad V_4 \overset{\textbf{def}}{=} \frac{(0:z) \cap (z^{n_4})}{(0:z) \cap (z^{n_3})}$$

$$\boxed{\textbf{Central modules}}\ U_k \overset{\textbf{def}}{=} V_k^* = \frac{(0:z^{n_k}) + (z)}{(0:z^{n_{k-1}}) + (z)}$$

$$A \supsetneq 0:z^{n_4} \supsetneq 0:z^{n_3} \supsetneq 0:z^{n_2} \supsetneq 0:z^{n_1} \supsetneq (0)$$

$$A \supsetneq (z^{n_4}) \supsetneq (z^{n_3}) \supsetneq (z^{n_2}) \supsetneq (z^{n_1}) \supsetneq (0)$$



$$V_1 \overset{\textbf{def}}{=} (0:z) \cap (z^{n_1})$$

$$V_2 \overset{\textbf{def}}{=} \frac{(0:z) \cap (z^{n_2})}{(0:z) \cap (z^{n_1})}$$

$$V_3 \overset{\textbf{def}}{=} \frac{(0:z) \cap (z^{n_3})}{(0:z) \cap (z^{n_2})}$$

$$V_4 \overset{\textbf{def}}{=} \frac{(0:z) \cap (z^{n_4})}{(0:z) \cap (z^{n_3})}$$

Central modules $U_k \overset{\textbf{def}}{=} V_k^* = \dfrac{(z) + (0:z^{n_{k-1}})}{(z) + (0:z^{n_k})}$

Theorem

Suppose A **is a graded Gorenstein algebra.** $z \in A_1$**. Pick up non-zero terms from** $\frac{(0:z^{i-1})+z}{(0:z^i)+z}$ **and** $U_j = \frac{(0:z^i)+z}{(0:z^{i-1})+z}$, $j = 1, \ldots, s$**. If all** U_j **have SLP, then** $Gr_{(z)}(A)$ **has the SLP.**

Proof. Consider $Gr_{(z)}(A) \stackrel{\mathbf{def}}{=} A/(z) \oplus (z)/(^2) \oplus (z^2)/(z^3) \cdots \oplus (z^p)/(z^{p+1})$**. This is close to the sum**

$$\bigoplus U_k \otimes K[z]/(z_k^{a_k}).$$

SLP of $Gr_{(z)}(A)$ **follows from SLP of** U_k**.**

Before we state our main result, make two remarks.

Lemma

If I is a ci and if $I : l$ is a ci for a linear form in R, then $I + (l)$ is a ci.

Proof. This is not quite obvious, but this can be proved.

Lemma

Suppose J/I is a principal module. Then $(J/I) \cong R/(I : f)$ for some $f \in R$.

Proof. $J = I + f$ for some f. Hence $J/I = (I + f)/I \cong R/I : f$.

Definition(Tentative) "Bianry tree of ci's"

1. $\mathcal{F} = \bigcup_{\nu=1}^{\infty} \mathcal{F}_\nu$
2. $\mathcal{F}_\nu = \{I \subset K[x_1, \ldots, x_\nu] \mid I$ **is an Artinian ci.**$\}$.
3. **For** $I \in \mathcal{F}_\nu$, $I : x_\nu \in \mathcal{F}$ **as long as** $x_\nu \notin I$.
4. **For any** $I \in \mathcal{F}_\nu$ **there exists** $J \in \mathcal{F}_{\nu-1}$ **such that** $I + x_n = JR_\nu + x_n$.
5. **All central modules** $U_j = \frac{(I:x_\nu^k)+(x_\nu)}{(I:x_\nu^{k-1})+(x_\nu)}$ **are principal (so it is isomorphic to** $R_{\nu-1}/J$ **for some** $J \in \mathcal{F}_{\nu-1}$**).**

Theorem

All members of $\mathcal{F}$ **have SLP.**

Remark

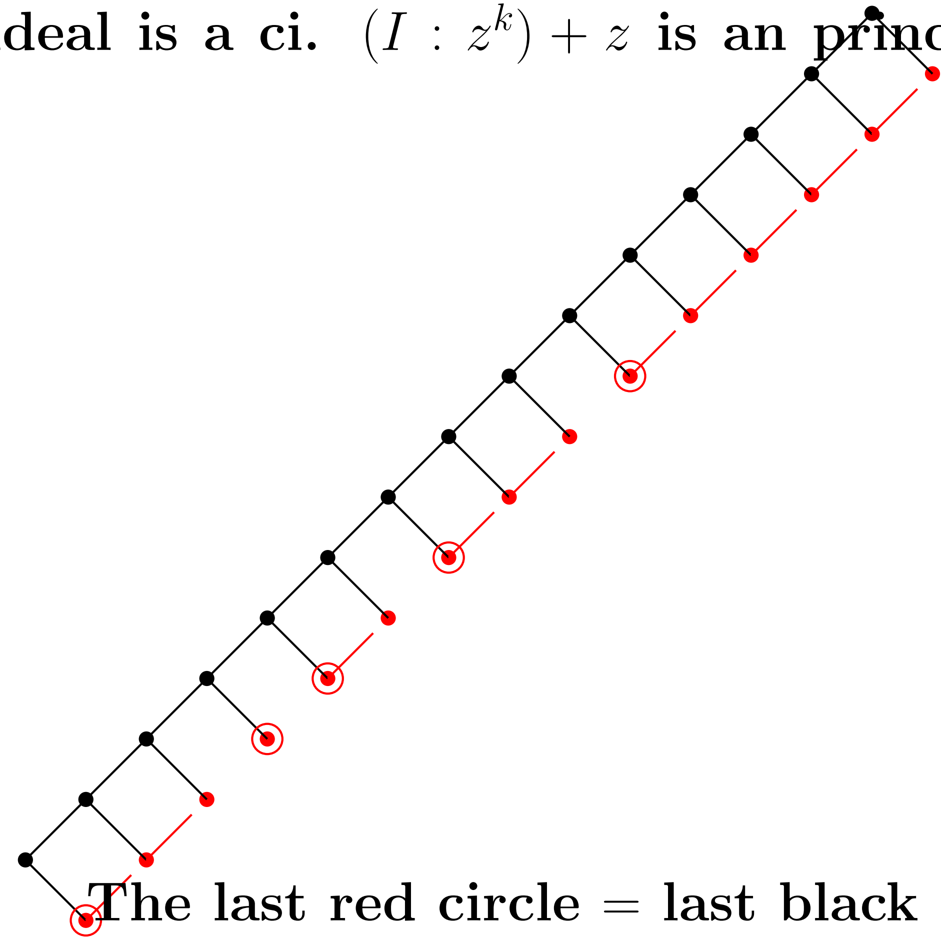
1. **Simplify the definition. For example (5) may follow from other conditions.**
2. **If $\mathcal{F}'$ and $\mathcal{F}''$ satisfy (1)–(5). Then $\mathcal{F}' \bigcup \mathcal{F}''$ satisfy (1)–(5). So what is the largest such $\mathcal{F}$.**
3. **$\mathcal{F}_2$ consists of all ci in $K[x_1, x_2]$.**
4. **$\mathcal{F}_3$ contains all ci I in $K[x_1, x_2, x_3]$ such that $I : x_3^k$ is a ci for all $k = 1, 2, \ldots$.**

At least this gives us a new proof all ci in $K[x_1, x_2]$ are SL.

For $\nu = 3$, if I is a ci and if there is a linear element $l \in R$ such that $I : l^k$ is a ci for all k, then I has the SLP.

We can use the same picture.

Consider this diagram for $A = K[x_1, x_2]/(F_1, F_2)$. z is a linear element. Coloning by z the ideal is a ci. $(I : z^k) + z$ is an principal ideal in $K[x]$.



The last red circle = last black circle.

$$A \supset (0 : l^p) \supset (0 : l^{p-1}) \supset \cdots (0 : l^2) \supset (0 : l) \supset 0_A.$$

Add l

$$A \supset (0 : l^p) + l \supset (0 : l^{p-1}) + l \supset \cdots (0 : l^2) + l \supset (0 : l) + l \supset (l).$$

Take quotient

$$\frac{(0 : l^j) + (l)}{(0 : l^{j-1}) + (l)}$$

Equivalently,

$$A \supset (l) \supset (l^2) \supset (l^3) \supset \cdots \supset (l^m) = (0_A).$$

Intersect with $0 : l$

$$(0 : l) \supset (l) \cap (0 : l) \supset (l^2) \cap (0 : l) \supset (l^3) \cap (0 : l) \supset \cdots \supset (l^m) \cap (0 : l) = (0).$$

Take quotient

$$\frac{(l^{j-1}) \cap (0 : l)}{(l^j) \cap (0 : l)} = \left(\frac{(0 : l^j) + l}{(0 : l^{j-1}) + l} \right)^*$$

$$\boxed{\textbf{It remains to find examples where this theorem is applicable.}}$$

(generalized) Newton's identity

Newton's identity consists of two parts.

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} w_1 & 0 & 0 & \cdots & 0 \\ f_1 & w_2 & 0 & \cdots & 0 \\ f_2 & f_1 & w_3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ f_{n-1} & f_{n-2} & \cdots f_2 & f_1 & w_n \end{pmatrix} \begin{pmatrix} e_1 \\ -e_2 \\ e_3 \\ \vdots \\ (-1)^{n-1} e_n \end{pmatrix}$$

$$f_k = \sum_{j=1}^{n} (-1)^{j-1} f_{k-j} e_j, \qquad k > n.$$

Recall that

$$f_k = \begin{cases} \sum_{j=1}^{n} x_j^k & \textbf{if } w_1, w_2, \cdots, w_n = \mathbf{(1,2,3,\ \ldots,\ n)} \\ \sum_{k_1+\cdots+k_n=k} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} & \textbf{if } w_1, w_2, \ldots, w_n = \mathbf{(1,1,\ \ldots,\ 1)} \end{cases}$$

These are power sum and complete symmetric polynomials.

Proposition

Suppose $n = 3$**. Then** $(f_d, f_{d+1}, f_{d+2}) : x_3^k$ **for all** $k = 0, 1, 2, \cdots$, $(f_d, f_{d+1}, f_{d+2}) : x_3^k \in \mathcal{F}_3$ **for all** $k = 0, 1, 2, \cdots$ **, as long as it is not** 0**.**

Proposition

Suppose $n \geq 4$**. If** $w_i = \frac{1-q^i}{1-q}$ **for some** $q \in K$**, then** $(f_d, f_{d+1}, \ldots, f_{d+n-1}) : x_n^k$ **for all** $k = 0, 1, 2, \cdots$, $(f_d, f_{d+1}, \ldots, f_{d+n-1}) : x_n^k \in \mathcal{F}_n$ **for all** $k = 0, 1, 2, \cdots$ **, as long as it is not** 0**.**

Remark

If $q = 0$**, then** $(w_1, \ldots, w_n) = (1, 1, \ldots, 1)$**. If** $q = 1$**, then** $(w_1, \ldots, w_n) = (1, 2, \ldots, n)$**.**

Suppose f_k satisfies (generalized) Newton's identity with arbitrary weights $w = (w_1, \dots, w_n)$. $w_1 w_2 \cdots w_n \neq 0$.

Theorem

If $n = 3$,

1. $(f_d, f_{d+1}, f_{d+2}) \in \mathcal{F}_3$ **for all** $d \geq 2$.

2. $(x_1x_2, f_d, f_{d+1}) \in \mathcal{F}_3$.

3. $(x_1 + x_2, x_1x_2, f_d) \in \mathcal{F}_3$. **($f_d$ could be replaced by x_3^d.)**

For $n \geq 4$ it is not true that

$$(f_1, \dots, f_n) : x_n^k \text{ is a ci for all } k.$$

But if $w = (w_1, \dots, w_n)$ is of the form

$$(w_1, \dots, w_n) = \left(\frac{1-q}{1-q}, \frac{1-q^2}{1-q}, \dots, \frac{1-q^{n-1}}{1-q}\right)$$

this is true and the above theorem generalizes.

$$(w_1, w_2, \dots, w_n) = (1, 1, \dots, 1), (1, 2, \dots, n)$$

Suppose f_k satisfies the Newtons's identity with weights $w = (w_1, \ldots,$ $\overline{e}_i$ **is the elementary symmetric polynomial in $x_1, \cdots, x_{n-1}$. We have the following theorem.**

Suppose $n \geq 4$. $(w_1, \ldots, w_n) = (1, \frac{1-q^2}{1-q}, \ldots, \frac{1-q^i}{1-q})$.

Theorem

(0) $(f_d, f_{d+1}, \ldots, f_{d+n-1}) \in \mathcal{F}_n$ **for all $d \geq 2$.**

(1) $(\overline{e}_{n-1}, \underbrace{f_d, \ldots, f_{d+n-2}}_{n-1}) \in \mathcal{F}_n$.

(2) $(\overline{e}_{n-2}, \overline{e}_{n-1}, \underbrace{f_d, \ldots, f_{d+n-2}}_{n-2}) \in \mathcal{F}_n$.

(k) $(\underbrace{\overline{e}_{n-k}, \cdots, \overline{e}_{n-1}}_{k}, \underbrace{f_d, f_{d+1}, \cdots, f_{d+n-k}}_{n-k}) \in \mathcal{F}_n$.

(n-1) $(\underbrace{\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_{n-1}}_{n-1}, f_d) \in \mathcal{F}_n$. **($f_d$ can be replaced by x_n^d.)**

(n) $(\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_{n-1}, x_n) \in \mathcal{F}_n$.

In particular $(f_1, \ldots, f_n) : x_n^k$ are ci for all $k = 0, 1, 2, \ldots,$.

This is the end of my talk.

Thank you for listening.