Graphs giving algebras with the SLP

Filip Jonsson Kling

June 24, 2024

Stockholm University

Algebras from graphs

Let G be a finite simple graph on n vertices labelled $\{1, \ldots, n\}$ having edge set $E(G)$.

Associated algebra $A(G) = k[x_1, \ldots, x_n]/(I(G) + (x_1^2, \ldots, x_n^2))$ where $I(G) = (x_i x_j \mid \{i, j\} \in E(G))$ and k is some field.

Algebras from graphs

Let G be a finite simple graph on n vertices labelled $\{1, \dots, n\}$ having edge set $E(G)$.

Associated algebra $A(G) = k[x_1, \dots, x_n]/(I(G) + (x_1^2, \dots, x_n^2))$ where $I(G) = (x_i x_j \mid \{i, j\} \in E(G))$ and k is some field.

Question

Given some numbers n of vertices and μ of edges that are possible for a simple graph, is there a simple graph G with that many vertices and edges such that $A(G)$ has the SLP?

Algebras from graphs

Let G be a finite simple graph on n vertices labelled $\{1, \ldots, n\}$ having edge set $E(G)$.

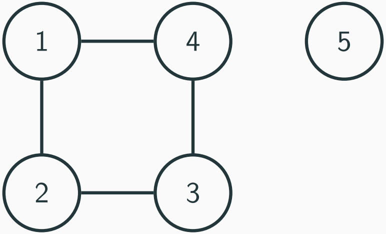
Associated algebra $A(G) = k[x_1, \ldots, x_n]/(I(G) + (x_1^2, \ldots, x_n^2))$ where $I(G) = (x_i x_j \mid \{i, j\} \in E(G))$ and k is some field.

Question

Given some numbers n of vertices and μ of edges that are possible for a simple graph, is there a simple graph G with that many vertices and edges such that $A(G)$ has the SLP?

Altafi and Lundqvist classified all cases for which WLP is forced.

Examples



Here
$A(G) = k[x_1, x_2, x_3, x_4, x_5]/(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1x_2, x_2x_3, x_3x_4, x_1x_4)$
has the SLP.

Examples

Here $A(P_7) = k[x_1, \ldots, x_7]/(x_1^2, \ldots, x_7^2, I(P_7))$ has the SLP.

Examples

Here $A(P_7) = k[x_1, \dots, x_7]/(x_1^2, \dots, x_7^2, I(P_7))$ has the SLP.

But $A(P_8) = k[x_1, \dots, x_8]/(x_1^2, \dots, x_8^2, I(P_8))$ fails the WLP.

The construction

Definition

Fix a number of vertices n. The graph $G_{i,j}$ is then obtained as a complete graph on $j - 1$ vertices connected to an additional vertex with i extra edges.

The construction

Definition

Fix a number of vertices n. The graph $G_{i,j}$ is then obtained as a complete graph on $j-1$ vertices connected to an additional vertex with i extra edges.

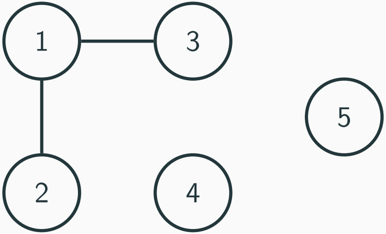
$G_{1,2}$

The construction

Definition

Fix a number of vertices n. The graph $G_{i,j}$ is then obtained as a complete graph on $j-1$ vertices connected to an additional vertex with i extra edges.

$G_{1,3}$

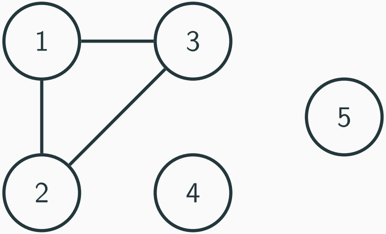


The construction

Definition

Fix a number of vertices n. The graph $G_{i,j}$ is then obtained as a complete graph on $j - 1$ vertices connected to an additional vertex with i extra edges.

$G_{2,3}$

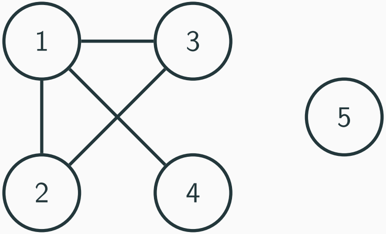


The construction

Definition

Fix a number of vertices n. The graph $G_{i,j}$ is then obtained as a complete graph on $j-1$ vertices connected to an additional vertex with i extra edges.

$G_{1,4}$

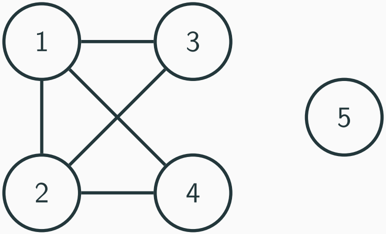


The construction

Definition

Fix a number of vertices n. The graph $G_{i,j}$ is then obtained as a complete graph on $j-1$ vertices connected to an additional vertex with i extra edges.

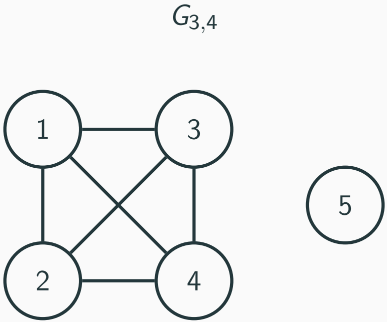
$G_{2,4}$



The construction

Definition

Fix a number of vertices n. The graph $G_{i,j}$ is then obtained as a complete graph on $j - 1$ vertices connected to an additional vertex with i extra edges.

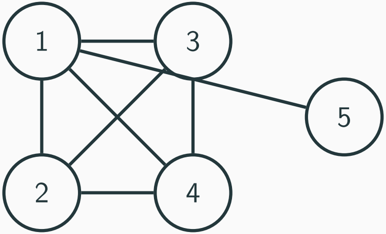


The construction

Definition

Fix a number of vertices n. The graph $G_{i,j}$ is then obtained as a complete graph on $j-1$ vertices connected to an additional vertex with i extra edges.

$G_{1,5}$



The main theorem

Theorem

Let k be a field of characteristic zero. For any $1 \leq i < j \leq n$, $A(G_{i,j})$ has the SLP.

The main theorem

Theorem

Let k be a field of characteristic zero. For any $1 \leq i < j \leq n$, $A(G_{i,j})$ has the SLP.

Works also for k with characteristic larger than n.

The main theorem

Theorem

Let k be a field of characteristic zero. For any $1 \leq i < j \leq n$, $A(G_{i,j})$ has the SLP.

Works also for k with characteristic larger than n.

Corollary

For any n, μ, possible numbers of vertices and edges of a finite simple graph, there is a simple graph G with that many vertices and edges such that $A(G)$ has the SLP.

Hilbert series

Proposition

The Hilbert series of $A(G_{i,j})$ is given by

$$\mathrm{HS}(A(G_{i,j}); t) = (1+t)^{n-j}(1 + jt + (j-i-1)t^2).$$

Hilbert series

Proposition

The Hilbert series of $A(G_{i,j})$ is given by

$$\mathrm{HS}(A(G_{i,j}); t) = (1+t)^{n-j}(1 + jt + (j-i-1)t^2).$$

Corollary

The Hilbert series $\mathrm{HS}(A(G_{i,j}); t)$ is a real-rooted polynomial. In particular, its coefficients form a log-concave and unimodal sequence.

Mid-heavy

Definition

A sequence of positive integers $(a_i)_{i=0}^{n}$ is *mid-heavy* if for $0 \leq i < j \leq n$, we have $a_i \leq a_j$ gives $a_{i-1} \leq a_{j+1}$ and $a_i \geq a_j$ gives $a_{i-1} \geq a_{j+1}$.

Mid-heavy

Definition

A sequence of positive integers $(a_i)_{i=0}^{n}$ is *mid-heavy* if for $0 \leq i < j \leq n$, we have $a_i \leq a_j$ gives $a_{i-1} \leq a_{j+1}$ and $a_i \geq a_j$ gives $a_{i-1} \geq a_{j+1}$.

Definition

Let D be the socle degree of an artinian algebra A. We say that $\mathrm{HS}(A;t) = \sum_{i=0}^{D} h_i t^i$ is in the class $\mathcal{H}$ if $h_{i-1} \leq h_{D-i} \leq h_i$ for all $1 \leq i \leq D/2$ or $h_{D-i+1} \leq h_i \leq h_{D-i}$ for all $1 \leq i \leq D/2$.

Mid-heavy

Definition

A sequence of positive integers $(a_i)_{i=0}^n$ is *mid-heavy* if for $0 \leq i < j \leq n$, we have $a_i \leq a_j$ gives $a_{i-1} \leq a_{j+1}$ and $a_i \geq a_j$ gives $a_{i-1} \geq a_{j+1}$.

Definition

Let D be the socle degree of an artinian algebra A. We say that $\mathrm{HS}(A; t) = \sum_{i=0}^D h_i t^i$ is in the class $\mathcal{H}$ if $h_{i-1} \leq h_{D-i} \leq h_i$ for all $1 \leq i \leq D/2$ or $h_{D-i+1} \leq h_i \leq h_{D-i}$ for all $1 \leq i \leq D/2$.

$A = k[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ is both, but $k[x]/(x^3)$ is only of class $\mathcal{H}$.

Lindsey

Theorem (Lindsey, 11)

Let A be an algebra over a field k of characteristic zero with the SLP. Then $\mathrm{HS}(A; t)$ is in the class $\mathcal{H}$ if and only if $A \otimes_k k[y]/(y^m)$ has the SLP for all $m \geq 0$.

Lindsey

Theorem (Lindsey, 11)

Let A be an algebra over a field k of characteristic zero with the SLP. Then $\mathrm{HS}(A;t)$ is in the class $\mathcal{H}$ if and only if $A \otimes_k k[y]/(y^m)$ has the SLP for all $m \geq 0$.

Lemma

If $\mathrm{HS}(A;t) = \sum_{i=0}^{D} a_i t^i$ with (a_i) mid-heavy, then $\mathrm{HS}(A;t)$ is in the class $\mathcal{H}$.

Lindsey

Theorem (Lindsey, 11)

Let A be an algebra over a field k of characteristic zero with the SLP. Then $\mathrm{HS}(A; t)$ is in the class $\mathcal{H}$ if and only if $A \otimes_k k[y]/(y^m)$ has the SLP for all $m \geq 0$.

Lemma

If $\mathrm{HS}(A; t) = \sum_{i=0}^{D} a_i t^i$ with (a_i) mid-heavy, then $\mathrm{HS}(A; t)$ is in the class $\mathcal{H}$.

Lemma

If $p(t)$ is the generating function of a mid-heavy sequence, then so is $(1 + t)p(t)$.

The proof

Corollary

The Hilbert series

$$\mathrm{HS}(A(G_{i,j}); t) = (1+t)^{n-j}(1 + jt + (j-1-i)t^2)$$

is in the class $\mathcal{H}$.

The proof

Corollary

The Hilbert series

$$\mathrm{HS}(A(G_{i,j}); t) = (1+t)^{n-j}(1 + jt + (j-1-i)t^2)$$

is in the class $\mathcal{H}$.

Theorem

$A(G_{i,j})$ has the SLP.

The proof

Corollary

The Hilbert series

$$\mathrm{HS}(A(G_{i,j});t) = (1+t)^{n-j}(1+jt+(j-1-i)t^2)$$

is in the class $\mathcal{H}$.

Theorem

$A(G_{i,j})$ has the SLP.

Proof.

True if $j = n$. Else $\overline{A} = A(G_{i,j} \setminus \{n\})$ is another graph in our family. Then $\overline{A}$ has the SLP and a Hilbert series of class $\mathcal{H}$. Hence $\overline{A} \otimes_k k[x_n]/(x_n^2) \cong A(G_{i,j})$ has the SLP. □

Questions

Conjecture

Fix $n, d \geq 3$ and μ in the interval $[n, \binom{n+d-1}{d}]$. Then there is a monomial ideal I of $k[x_1, \ldots, x_n]$ minimally generated by μ monomials of degree d such that R/I has the SLP.

Thank you for listening!