# A Gröbner basis for the ideal generated by $\{x_1^2, \ldots, x_n^2, (x_1 + \cdots + x_n)^k\}$

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# The main result

Over a field of characteristic zero, we determine, explicitly, a reduced minimal Gröbner basis for the ideal generated by

$$\{x_1^2, \ldots, x_n^2, (x_1 + \cdots + x_n)^k\}.$$

The Gröbner basis is only dependent upon the ordering of the variables, so it is e.g. the same for the degree lexicographical ordering and the degree reverse lexicographical ordering wrt $x_1 \succ x_2 \succ \cdots \succ x_n$.

# Some history on the squarefree algebra and the Lefschetz properties

As a PhD student, Ralf Fröberg was asked the following question by his advisor Christer Lech in the 70's: What is the Hilbert series of the quotient of the polynomial ring with an ideal in generated by $m$ general forms? He came up with his famous conjecture, below formulated in a Lefschetz type setting.

## Conjecture

*Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and let $m > 1$. For any positive integers $d_1, \ldots, d_m$, there exist forms $f_1, \ldots, f_m$ of degrees $d_1, \ldots, d_m$ such that for each $i$, the multiplication by $f_i$ map on $R/(f_1, ..., f_{i-1})$ has maximal rank in every degree.*

# History, Fröberg conjecture

Ralf gave a first nontrivial answer for the case $m = n + 1, d_i = 2$ by showing that the multiplication by $f$ map on $\mathbb{C}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ is either surjective or injective, where

$$f = \begin{cases} x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n & \text{if } n \text{ is even} \\ x_1x_2 + x_3x_4 + \cdots + x_{n-2}x_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

# History, Stanley and Watanabe

One year later, in 1980, Stanley proved the case $m = n + 1$ in general. Then came an independent proof by Junzo in 1987, and an elementary proof by Reid, Roberts, and Roitman in 1991. Although the method of proof was different in all the cases, they all proved that monomial CIs are SL.

# History, reduction to $d = 2$

In 2008, Masao Hara and Junzo gave yet another proof showing that one can view the SLP of the squarefree algebra as a corner stone for the SLP for all monomial CIs: They studied the determinant of the multiplication by $(x_1 + \cdots + x_n)^k$ map on $\mathbb{C}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ and then used a short projection argument to derive the result that all monomial CIs are SL.

Last year, 2023, Ho Phuong, Quang Hoa Tran gave a slightly different proof based on elementary linear algebra showing that $\mathbb{C}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ is SL.

# Short recap of Gröbner basis theory

Fix a term ordering $\prec$. Recall:

- $G$ is a Gröbner basis for an ideal $I$ wrt $\prec$ if it is a subset of $I$ and the leading terms of $G$ generate the initial ideal of $I$ wrt $\prec$.
- The set of monomials outside the initial ideal is a vector space basis for the quotient algebra. Thus, once we have a Gröbner basis, we can determine the Hilbert series of the algebra.

In almost all kinds of experiments in Computer algebra systems, there are Gröbner basis computations going on by means of variations of the Buchberger algorithm.

*But it is known that Gröbner bases are excellent in destroying symmetry, and often a mess.*

# Why care about this Gröbner basis computation?

- It gives an alternative proof (number 7?) that $\mathbb{C}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ has the SLP.
- It gives an example of an explicit description of a Gröbner basis for a non-trivial generic object.
- The Gröbner basis itself shows an interesting behavior.

# The Gröbner basis, degrees of the generators for $k = 2$

```
n = 12;
R = QQ[x_1..x_n];
I = ideal apply(gens R,i->i^2) + ideal (sum gens R)^2;
G = flatten entries gens gb I;
tally apply(G,degree)

      Tally{{2} => 13 }
            {3} => 2
            {4} => 5
            {5} => 14
            {6} => 42
            {7} => 132
```

# The Gröbner basis, of one the cubic elements for $k = 2$

$$
\begin{aligned}
&x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5 + x_2x_3x_6 + x_2x_4x_6 + x_3x_4x_6 + x_2x_5x_6 + x_3x_5x_6 + \\
&x_4x_5x_6 + x_2x_3x_7 + x_2x_4x_7 + x_3x_4x_7 + x_2x_5x_7 + x_3x_5x_7 + x_4x_5x_7 + x_2x_6x_7 + x_3x_6x_7 + \\
&x_4x_6x_7 + x_5x_6x_7 + x_2x_3x_8 + x_2x_4x_8 + x_3x_4x_8 + x_2x_5x_8 + x_3x_5x_8 + x_4x_5x_8 + x_2x_6x_8 + \\
&x_3x_6x_8 + x_4x_6x_8 + x_5x_6x_8 + x_2x_7x_8 + x_3x_7x_8 + x_4x_7x_8 + x_5x_7x_8 + x_6x_7x_8 + x_2x_3x_9 + \\
&x_2x_4x_9 + x_3x_4x_9 + x_2x_5x_9 + x_3x_5x_9 + x_4x_5x_9 + x_2x_6x_9 + x_3x_6x_9 + x_4x_6x_9 + x_5x_6x_9 + \\
&x_2x_7x_9 + x_3x_7x_9 + x_4x_7x_9 + x_5x_7x_9 + x_6x_7x_9 + x_2x_8x_9 + x_3x_8x_9 + x_4x_8x_9 + x_5x_8x_9 + \\
&x_6x_8x_9 + x_7x_8x_9 + x_2x_3x_{10} + x_2x_4x_{10} + x_3x_4x_{10} + x_2x_5x_{10} + x_3x_5x_{10} + x_4x_5x_{10} + x_2x_6x_{10} + \\
&x_3x_6x_{10} + x_4x_6x_{10} + x_5x_6x_{10} + x_2x_7x_{10} + x_3x_7x_{10} + x_4x_7x_{10} + x_5x_7x_{10} + x_6x_7x_{10} + x_2x_8x_{10} + \\
&x_3x_8x_{10} + x_4x_8x_{10} + x_5x_8x_{10} + x_6x_8x_{10} + x_7x_8x_{10} + x_2x_9x_{10} + x_3x_9x_{10} + x_4x_9x_{10} + x_5x_9x_{10} + \\
&x_6x_9x_{10} + x_7x_9x_{10} + x_8x_9x_{10} + x_2x_3x_{11} + x_2x_4x_{11} + x_3x_4x_{11} + x_2x_5x_{11} + x_3x_5x_{11} + x_4x_5x_{11} + \\
&x_2x_6x_{11} + x_3x_6x_{11} + x_4x_6x_{11} + x_5x_6x_{11} + x_2x_7x_{11} + x_3x_7x_{11} + x_4x_7x_{11} + x_5x_7x_{11} + x_6x_7x_{11} + \\
&x_2x_8x_{11} + x_3x_8x_{11} + x_4x_8x_{11} + x_5x_8x_{11} + x_6x_8x_{11} + x_7x_8x_{11} + x_2x_9x_{11} + x_3x_9x_{11} + x_4x_9x_{11} + \\
&x_5x_9x_{11} + x_6x_9x_{11} + x_7x_9x_{11} + x_8x_9x_{11} + x_2x_{10}x_{11} + x_3x_{10}x_{11} + x_4x_{10}x_{11} + x_5x_{10}x_{11} + \\
&x_6x_{10}x_{11} + x_7x_{10}x_{11} + x_8x_{10}x_{11} + x_9x_{10}x_{11} + x_2x_3x_{12} + x_2x_4x_{12} + x_3x_4x_{12} + x_2x_5x_{12} + \\
&x_3x_5x_{12} + x_4x_5x_{12} + x_2x_6x_{12} + x_3x_6x_{12} + x_4x_6x_{12} + x_5x_6x_{12} + x_2x_7x_{12} + x_3x_7x_{12} + \\
&x_4x_7x_{12} + x_5x_7x_{12} + x_6x_7x_{12} + x_2x_8x_{12} + x_3x_8x_{12} + x_4x_8x_{12} + x_5x_8x_{12} + x_6x_8x_{12} + x_7x_8x_{12} + \\
&x_2x_9x_{12} + x_3x_9x_{12} + x_4x_9x_{12} + x_5x_9x_{12} + x_6x_9x_{12} + x_7x_9x_{12} + x_8x_9x_{12} + x_2x_{10}x_{12} + \\
&x_3x_{10}x_{12} + x_4x_{10}x_{12} + x_5x_{10}x_{12} + x_6x_{10}x_{12} + x_7x_{10}x_{12} + x_8x_{10}x_{12} + x_9x_{10}x_{12} + x_2x_{11}x_{12} + \\
&x_3x_{11}x_{12} + x_4x_{11}x_{12} + x_5x_{11}x_{12} + x_6x_{11}x_{12} + x_7x_{11}x_{12} + x_8x_{11}x_{12} + x_9x_{11}x_{12} + x_{10}x_{11}x_{12}
\end{aligned}
$$

# The initial ideal (read from right to left) for $k = 2, n = 8$

$2: x_8^2,\ x_7^2,\ x_6^2,\ x_5^2,\ x_4^2,\ x_3^2,\ x_2^2,\ x_1x_2,\ x_1^2,$
$3: \ x_2x_3x_4,\ x_1x_3x_4,$
$4: \ x_3x_4x_5x_6,\ x_2x_4x_5x_6,\ x_1x_4x_5x_6,\ x_2x_3x_5x_6,\ x_1x_3x_5x_6,$
$5: \ x_4x_5x_6x_7x_8,\ x_3x_5x_6x_7x_8,\ x_2x_5x_6x_7x_8,\ x_1x_5x_6x_7x_8,\ x_3x_4x_6x_7x_8,\ x_2x_4x_6x_7x_8,$
$x_1x_4x_6x_7x_8,\ x_2x_3x_6x_7x_8,\ x_1x_3x_6x_7x_8,\ x_3x_4x_5x_7x_8,\ x_2x_4x_5x_7x_8,\ x_1x_4x_5x_7x_8,$
$x_2x_3x_5x_7x_8,\ x_1x_3x_5x_7x_8$

# Proof technique

- The computer experiments gives you a guess on the structure of the set.
- Next, show that each element belongs to the ideal.
- Then check that the Hilbert series agrees with what one should expect for $\mathbb{C}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ to have maximal rank.

# Proof challenges

- Not so hard to come up with an argument that the set of polynomials is a subset of the ideal.
- But harder to do this explicitly.
- Straightforward but tedious to count the number of monomials outside the initial ideal for the case $k = 2$. Eventually, we came up with a short lattice path argument that works for any $k$.

# Gröbner bases for generic objects

Since it holds that

$$\mathbb{C}[x_1, \ldots, x_n]/(x_1^{d_1}, \ldots, x_n^{d_n}, (x_1 + \cdots + x_n)^{d_{n+1}}) \cong$$

$$\mathbb{C}[x_1, \ldots, x_n]/(\ell_1^{d_1}, \ldots, \ell_{n+1}^{d_{n+1}}),$$

with the $\ell_i$ being general linear forms, we can argue that *we have found a Gröbner basis for a non-trivial generic object*. Question: Would it be possible to do something similar for other classes of generic objects?

# Gröbner bases for generic objects, continued

The question is related to a conjecture by Moreno-Socías on the structure of the initial ideal with respect to the degree reverse lexicographical ordering of an ideal generated by generic forms.
Pardue (2010) has shown that Moreno-Socias' conjecture implies the longstanding Fröberg conjecture. Nenashev showed Moreno-Socias' conjecture when giving an alternative proof of Anick's proof of the Fröberg conjecture for $n = 3$.

# Natural next step

Determine the Gröbner basis for $\{x_1^d, \ldots, x_n^d, (x_1 + \cdots + x_n)^k\}$ when $d \geq 3$.

# A problem

Let $R = \mathbb{C}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2, (x_1 + \cdots + x_n)^2)$. Find, for each $n$, a quadratic form $f$ that has the maximal rank property. This would give a proof of the Fröberg conjecture for $d = 2, m = n + 2$.
Notice that $R$ does not have the WLP except for some small $n$ (noticed by Cruz and Iarrobino, proved by Sturmfels and Xu), so we cannot use the square of a general linear form for $f$.

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