## The Strong Lefschetz Property for Modules over Clements-Lindström Rings

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- Let k be a field of characteristic zero.
- $S = k[x_1, \ldots, x_n]$  standard graded polynomial ring
- $A = S/I = \bigoplus_{i=0}^{c} A_i$  standard graded Artinian k-algebra
- $M = \bigoplus_{i=a}^{b} M_i$  finitely generated A-module
- We denote the Hilbert function of M by  $i \mapsto h_i := \dim_k(M)$  and the Hilbert series by  $H_M(t)$

## The Lefschetz Properties (for Modules)

### Definition

*M* has the weak Lefschetz property (WLP) if there exists a linear form  $\ell \in A_1$  such that the multiplication map

$$\times \ell \colon M_i \to M_{i+1}$$

has maximal rank for all *i*.

### Definition

*M* has the **strong Lefschetz property** (SLP) if there exists a linear form  $\ell \in A_1$  such that the multiplication map

$$\times \ell^d \colon M_i \to M_{i+d}$$

has maximal rank for all d > 0 and all *i*.

## Modules over Rings with the WLP or SLP

- Much less is known about the Lefschetz properties for modules than for algebras. Are there interesting classes of modules which do or do not have the WLP or SLP?
- If A = S/I is an Artinian *k*-algebra which has the WLP or SLP, which *A*-modules have the WLP or SLP?
- The first case we consider for modules is that of ideals in Clements-Lindström rings in two variables. A **Clements-Lindström** ring has of the form  $S/(x_1^{a_1}, \ldots, x_n^{a_n})$  with  $a_1 \leq \cdots \leq a_n \leq \infty$  (where  $x_i^{\infty} = 0$ ). In this context, we may assume all Clements-Lindström rings are Artinian, i.e., complete intersections.
- Question: Given a monomial ideal *I* ⊂ *S*/*J*, where *S* = *k*[*x*, *y*] and *J* = (*x<sup>a</sup>*, *y<sup>b</sup>*), does *I* have the SLP as an *S*/*J*-module? Equivalently, does *I*/*J* have the SLP as an *S*-module?

Even in codimension two, where any k-algebra A is guaranteed to have the SLP, the WLP can fail for A-modules which are monomial ideals in A.

### Example

Let  $J = (x^5, y^5)$  be an ideal in S = k[x, y]. Consider the generic initial ideal in degree reverse lexicographic order, gin $(J) = \text{Lex}(J) = (x^5, x^4y, x^3y^3, x^2y^5, xy^7, y^9)$ . Then S/gin(J) has the SLP. However, every monomial ideal I which does not contain x or y fails the SLP as an S/gin(J)-module. On the other hand, I does have the SLP as an S/J module (we prove this later).

Additionally, tools used to prove the WLP/SLP for k-algebras can fail when we attempt to use them for modules (or just don't apply).

The Lindström-Gessel-Viennot Lemma gives a combinatorial interpretation for the determinant of a matrix of binomial coefficients as a count of non-intersecting lattice paths.

In two variables, the matrix representing the map of multiplication by  $\times(x + y)$  on any k[x, y]-module is a matrix of binomial coefficients.

### Corollary (Gessel-Viennot, 1985)

Let  $a_1 < a_2 < \cdots < a_k$  and  $b_1 < b_2 < \cdots < b_k$  be two sets of integers. Then the determinant of the matrix  $C = (c_{ij})$  of binomial coefficients  $c_{ij} = \begin{pmatrix} a_i \\ b_j \end{pmatrix}$  is nonzero if and only if  $\begin{pmatrix} a_i \\ b_i \end{pmatrix} \neq 0$  for each *i*.

### Theorem

Let I be a monomial ideal in S = k[x, y] and  $J = (x^a, y^b)$  an Artinian ideal in S such that  $J \subset I$ . Then I/J has the SLP as a k[x, y]-module

The proof relies on the LGV Lemma. As an immediate corollary, we get the useful result:

### Corollary

Let  $I = (x^{\alpha}, y^{\beta})$  and  $J = (x^{a}, y^{b})$  be ideals in S where  $0 \le \alpha \le a$  and  $0 \le \beta \le b$ . Then I/J has the SLP as a k[x, y]-module.

### In Which We Put the Lemma to Good Use

**Sketch of proof:** Matrix representing the multiplication map  $\times (x + y)^d : S_i \rightarrow S_{i+d}$  (transposed):

			$x^{i+d-2}y^2$					
x <sup>i</sup>	$\begin{bmatrix} d \\ 0 \end{bmatrix}$	$\binom{d}{1}$	$\binom{d}{2}$		$\binom{d}{d}$	0		ړ ٥
$x^{i-1}y$	0	$\binom{d}{0}$	$\binom{d}{1}$		$\binom{d}{d-1}$	$\binom{d}{d}$		0
÷	:	•	$\begin{pmatrix} d \\ 2 \end{pmatrix} \begin{pmatrix} d \\ 1 \end{pmatrix}$	÷	÷	•	÷	:
y <sup>i</sup>	0	0	0		$\binom{d}{d-i}$	$\binom{d}{d-i+1}$		$\binom{d}{d}$

Deleting columns corresponding to monomials in J and rows corresponding to monomials not in I gives a matrix representing  $\times (x + y)^d : [I/J]_i \rightarrow [I/J]_{i+d}$  where the *n*th row is of the form

$$\begin{bmatrix} \begin{pmatrix} d \\ k_n \end{pmatrix} & \begin{pmatrix} d \\ k_n+1 \end{pmatrix} & \dots & \begin{pmatrix} d \\ k_n+c-1 \end{pmatrix} \end{bmatrix}.$$

Then doing a series of iterative column operations relying on the recursive relation of binomial coefficients, we finally get a matrix of the form

$$\begin{bmatrix} \binom{d+c-1}{k_1+c-1} & \binom{d+c-2}{k_1+c-1} & \cdots & \binom{d}{k_1+c-1} \\ \binom{d+c-1}{k_2+c-2} & \binom{d+c-2}{k_2+c-2} & \cdots & \binom{d}{k_2+c-2} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{d+c-1}{k_s+c-1} & \binom{d+c-2}{k_s+c-1} & \cdots & \binom{d}{k_s+c-1} \end{bmatrix}$$

Based on our hypotheses on I and J, the fact that S and S/J have the SLP and that S/J is Gorenstein, we can prove that at least the binomial coefficients on the main diagonal of a particular maximal minor of are nonzero.

Applying the corollary of the LGV Lemma, this implies maximal rank of the original matrix for I/J, hence I/J has the strong Lefschetz property.

# Modules over k[x,y,z]

It is not necessarily true that similar modules in three variables have the SLP. For example, the k[x, y, z]-module I/J where  $I = (x^2, y^2, z^2)$  and  $J = (x^3, y^3, z^3)$  fails the WLP and SLP.

However, we can consider tensor product extensions using the following:

### Theorem (Lindsey, 2011)

For  $M = \bigoplus_{j=p}^{q} M_i$  a graded Artinian S-module with the SLP,  $M \otimes_k k[z]/(z^c)$  has the SLP for all  $c \ge 0$  if and only if the Hilbert series of M is almost centered, i.e.,  $h_{p+i-1} \le h_{q-i} \le h_{p+i}$  for all  $1 \le i \le \lfloor \frac{q-p}{2} \rfloor$  or  $h_{q-i+1} \le h_{p+i} \le h_{q-i}$  for all  $1 \le i \le \lfloor \frac{q-p}{2} \rfloor$ .

#### Theorem

Let  $M = (x^{\alpha}, y^{\beta})/(x^{a}, y^{b}, z^{c})$  be a k[x, y, z]-module where  $0 \le \alpha \le a$  and  $0 \le \beta \le b$ . Then M has the SLP for all  $c \ge 0$  if  $\alpha \ne \beta$  and  $max\{\alpha, \beta\} \le 2$  or  $max\{\alpha, \beta\} = min\{\alpha + \beta, a, b\}$ .

## **Central Simple Modules**

• Let A be a standard graded Artinian k-algebra,  $\ell \in A$  a linear form with r the smallest positive integer for which  $\ell^p = 0$ . Harima and Watanabe defined the **ith central simple module** of A with respect to  $\ell$  as the nonzero quotients of the form

$$V_{i,\ell} = rac{(0:\ell^{f_i}) + (\ell)}{(0:\ell^{f_i-1}) + (\ell)}.$$

where  $r \ge f_i > f_{i+1} \ge 1$  for all *i*.

• Harima and Watanabe (2007) gave an equivalent condition for a graded Artinian *k*-algebra with a symmetric Hilbert series having the SLP in terms of central simple modules. A weaker version is true even for non-symmetric Hilbert series:

### Theorem

Let A be a graded Artinian k-algebra with a Hilbert series that is not necessarily symmetric. Then A has the SLP if there exists a linear form  $\ell \in A_1$  such that  $\widetilde{V}_{\ell} = \oplus(V_{i,\ell} \otimes k[t]/(t^{f_i}))$  has the SLP.

## Codimension 3 k-algebras of type 2

 Boij, Migliore, Miró-Roig, Nagel, and Zanello proved that level monomial algebras of type two in codimension 3 have the WLP. In the non-level case, Cook and Nagel completely characterized when such algebras as

$$A = k[x, y, z]/(x^{a}, y^{b}, z^{c}, x^{\alpha}z^{\gamma}, y^{\beta}z^{\gamma})$$

have the WLP. Their combinatorial method of proof involved lattice paths, perfect matchings, and lozenge tilings of triangular regions. But this method cannot also be used to study the SLP.

• However, the central simple modules of A (with respect to z) are of the form

$$V_{1,z} = \frac{(0:z^{c}) + (z)}{(0:z^{c-1}) + (z)} \cong \frac{k[x,y]}{(x^{\alpha},y^{\beta})}$$
$$V_{2,z} = \frac{(0:z^{\gamma}) + (z)}{(0:z^{\gamma-1}) + (z)} \cong \frac{(x^{\alpha},y^{\beta})}{(x^{a},y^{b})}.$$

Notice that  $V_{2,z}$  has the SLP by the previous corollary (and  $V_{1,z}$  has the SLP since it is a monomial complete intersection)!

## Codimension 3 k-algebras of type 2

Tensoring  $V_{1,z}$  with  $k[z]/(z^c)$  gives another monomial complete intersection with the SLP, and

$$\mathcal{W}_{2,z}\otimes k[z]/(z^{\delta})\cong rac{(x^{lpha},y^{eta})}{(x^{a},y^{b},z^{\gamma})}$$

which we saw (using Lindsey's result) also has the SLP in some cases. By looking at the Hilbert series of the above modules, we can study when the direct sum  $\tilde{V}_{\ell}$ , and hence A, has the SLP:

### Theorem

Let  $I = (x^a, y^b, z^c, x^{\alpha}z^{\gamma}, y^{\beta}z^{\gamma})$  be an Artinian monomial ideal in S = k[x, y, z]where  $0 < \alpha < a$ ,  $0 < \beta < b$ , and  $0 < \gamma < c$ . Then S/I has the strong Lefschetz property if any of the following conditions hold:

**1.** 
$$\alpha + \beta - 1 \le a + b - c \le \alpha + \beta + 1;$$

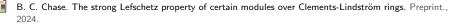
2. 
$$min\{\alpha, \beta\} \neq max\{\alpha, \beta\} = min\{\alpha + \beta, a, b\}$$
 and  
 $max\{\alpha, \beta\} - \gamma - 1 \le a + b - c \le max\{\alpha, \beta\} - \gamma + 1;$ 

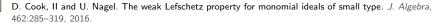
**3.**  $min\{\alpha, \beta\} < max\{\alpha, \beta\} \le 2$  and  $a + b + \gamma \le c + 2$ .

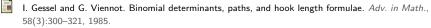
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