# On minimal Gorenstein Hilbert functions

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Lefschetz Properties in Algebra, Geometry and Combinatorics
KRAKÓW 24/29 June 2024
(joint work with L. Bezerra, R. Gondim, G. Zappalà)

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For instance, real orientable manifolds, projective varieties, Kahler manifolds, convex polytopes, matroids, Coxeter groups and tropical varieties are examples of categories for which the ring of cohomology is an Artinian Gorenstein $\mathbb{K}$-algebra.
The fundamental point is that these algebras can be characterized as algebras satisfying Poincaré duality.

## Problem

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We deal with standard graded Artinian Gorenstein $\mathbb{K}$-algebras over a field of characteristic zero. A natural and classical problem consists in understanding their possible Hilbert function, sometimes also called Hilbert vector. When the codimension of the algebra is less than or equal to 3, all possible Hilbert vectors were characterized in [22]; in particular, they are unimodal, i.e. they never strictly increase after a strict decrease.

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By Poncaré duality, the Hilbert function of $A \in \mathcal{AG}_{\mathbb{K}}(r,d)$ is a symmetric vector $\mathrm{Hilb}(A) = (1, r, h_2, \dots, h_{d-2}, r, 1)$, that is $h_k = h_{d-k}$. There is a natural partial order in this family given by:

$$(1, r, h_2, \dots, h_{d-2}, r, 1) \preceq (1, r, \tilde{h}_2, \dots, \tilde{h}_{d-2}, r, 1),$$

if $h_i \leq \tilde{h}_i$, for all $i \in \{2, \dots, d-2\}$.

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if $h_i \leq \tilde{h}_i$, for all $i \in \{2, \ldots, d-2\}$.
The maximal Hilbert functions are associated to compressed algebras and completely described in [13]. In fact the Hilbert vector of a compressed Gorenstein algebra is a maximum in $\mathcal{AG}_{\mathbb{K}}(r,d)$.

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Moreover, given two comparable Gorenstein Hilbert functions, it is not true that any symmetric vector between them is Gorenstein. Some partial results in this direction were obtained in [25] and called the interval conjecture.

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Moreover, given two comparable Gorenstein Hilbert functions, it is not true that any symmetric vector between them is Gorenstein. Some partial results in this direction were obtained in [25] and called the interval conjecture.
We **conjecture**:
A class of Artinian Gorenstein Hilbert algebras called full Perazzo algebras always have minimal Hilbert function, fixing codimension and length.

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As a consequence of our methods we give a new proof of part of a known result about the asymptotic behavior of the minimum entry of a Gorenstein Hilbert function.

While it is known that non unimodal Gorenstein $h$-vectors exist in every codimension greater than or equal to 5 (see [4]), it is open whether non unimodal Gorenstein $h$-vectors of codimension 4 exist.

Historically, the first such example of a non unimodal Gorenstein $h$-vector was given by Stanley, that showed that the $h$-vector $(1, 13, 12, 13, 1)$ is indeed a Gorenstein $h$-vector and the non unimodality occurs here in degree 2 (see [22][Example 4.3]).

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We say that a vector is totally non unimodal if

$$h_1 > h_2 > \ldots > h_k \text{ for } k = \lfloor d/2 \rfloor.$$

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From Macaulay-Matlis duality, every standard graded Artinian Gorenstein $\mathbb{K}$-algebra can be presented by a quotient of a ring of differential operators by a homogeneous ideal that is the annihilator of a single form in the dual ring of polynomials. Full Perazzo algebras are associated with full Perazzo polynomials, they are the family that we will study in detail.

Perazzo polynomials are related to Gordan and Noether theory of forms with vanishing Hessian (see [10, Chapter 7] and [20]).
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In [8] we study full Perazzo algebras focusing on socle degree 4, showing that they have minimal Hilbert vector in some cases. Now we deal with codimension greater than 13. In the case of socle degree 4 we recall the known results:

## Ideas

- We have constructed a family of Gorenstein algebras called **Full Perazzo algebras** and our main result is that, for small $m$, precisely $m = 3, 4, 5$, the Hilbert vectors of Full Perazzo algebras of type $m$, are always minimal. Moreover, we have given a simple proof of Stanley's conjecture and we have pointed out that the $h$-vector of the Stanley's example is a special case of a Full Perazzo algebra.

## Ideas

- We have constructed a family of Gorenstein algebras called **Full Perazzo algebras** and our main result is that, for small $m$, precisely $m = 3, 4, 5$, the Hilbert vectors of Full Perazzo algebras of type $m$, are always minimal. Moreover, we have given a simple proof of Stanley's conjecture and we have pointed out that the $h$-vector of the Stanley's example is a special case of a Full Perazzo algebra.
- We have introduced another family of Artinian Gorenstein algebras having non unimodal Gorenstein $h$-vectors: the **Turan algebras** that are Artinian Gorestein algebra presented by quadrics. We gave a conjecture about the asymptotic behaviour of Artinian Gorenstein algebra presented by quadrics.

# Preliminaries

We deal with standard bigraded Artinian Gorenstein algebras, i.e. Artinian Gorenstein algebras, $A = \oplus_{k=0}^{d} A_k$, such that

$$\begin{cases} A_d \neq 0 \\ A_k = \displaystyle\bigoplus_{i=0}^{k} A_{(i,k-i)} \text{ for } k < d. \end{cases}$$

The pair $(d_1, d_2)$, such that $A_{(d_1,d_2)} \neq 0$ and $d_1 + d_2 = d$, is said the socle bidegree of $A$.

## Remark 1

Since $A_k^* \simeq A_{d-k}$ and since duality is compatible with direct sum, we get $A_{(i,j)}^* \simeq A_{(d_1-i,d_2-j)}$.

# Full Perazzo Algebras

## Definition

Let $\mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ be the polynomial ring in the $n$ variables $x_1, \ldots, x_n$ and in the $m$ variables $u_1, \ldots, u_m$. A **Perazzo polynomial** is a reduced bihomogeneous polynomial $f \in \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]_{(1,d-1)}$, of degree $d$, of type

$$f = \sum_{i=1}^{n} x_i g_i \qquad (1)$$

with $g_i \in \mathbb{K}[u_1, \ldots, u_m]_{d-1}$, for $i = 1, \ldots, n$, linearly independent and algebraically dependent polynomials in the variables $u_1, \ldots, u_m$.

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## Remark 2

A Perazzo polynomial $f \in \mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_m]_{(1,d-1)}$ of degree $d$ is a Nagata polynomial, hence the algebra $A = Q/\operatorname{Ann}(f)$, associated to $f$, where $Q = \mathbb{K}[X_1, \dots, X_n, U_1, \dots, U_m]$ is the ring of the differential operators, can be realized as a Nagata idealization of order 1, socle degree $d$ and codimension $n + m$.

By above Remark (2), we can give the following definition:

**Definition**

Let $f \in \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]_{(1,d-1)}$ be a Perazzo polynomial of degree $d$. The algebra $A = Q/\operatorname{Ann}(f)$ associated to $f$ is called **Perazzo algebra** and it is a bigraded algebra of socle degree $d$ and codimension $n+m$.

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Now we fix $m \geq 2$ and we consider the $m$ variables $u_1, \ldots, u_m$. Let us consider $M_j = u_{j_1}^{\alpha_{j_1}} \cdots u_{j_m}^{\alpha_{j_m}}$ for $j = 1, \ldots, \tau_m$ where $\tau_m := \binom{m+d-2}{d-1}$ and $\alpha_{j_1} + \cdots + \alpha_{j_m} = d-1$.

## Definition

A bihomogeneous polynomial $f \in \mathbb{K}[x_1, \ldots, x_{\tau_m}, u_1, \ldots, u_m]_{(1,d-1)}$ of degree $d$ of type:

$$f = \sum_{j=1}^{\tau_m} x_j M_j \tag{2}$$

is called **Full Perazzo polynomial** of type $m$.

## Remark 3

As in Remark (2), let $f \in \mathbb{K}[x_1, \dots, x_{\tau_m}, u_1, \dots, u_m]_{(1,d-1)}$ be a Full Perazzo polynomial of type $m$ and of degree $d$, the algebra $A = Q/\operatorname{Ann}(f)$, associated to $f$, where $Q = \mathbb{K}[X_1, \dots, X_{\tau_m}, U_1, \dots, U_m]$ is the ring of the differential operators, can be realized as a Nagata idealization of order 1, socle degree $d$ and codimension $m + \tau_m$.

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## Proposition 1

*Let $A$ be a Full Perazzo algebra of type $m \geq 2$ and socle degree $d$. Then, for $k = 0, \dots, \lfloor \frac{d}{2} \rfloor$, $h_k = \dim A_k = \binom{m+k-1}{k} + \binom{m+d-k-1}{d-k}$.*
*In particular, its Hilbert function is totally non-unimodal for $r >> 0$.*

To verify that the Hilbert vector is asymptotically totally non unimodal it is enough to see that as a function of $m$, $h_k(m) \simeq \frac{1}{(d-k)!} m^{d-k}$ for $k \leq d/2$.

## Corollary 2

*For every $d \geq 4$ there is a positive integer $r_0$ such that for all $r \geq r_0$ there is an Artinian Gorenstein algebra with socle degree $d$ and codimension $r$ having a totally non unimodal Hilbert vector.*

## Definition

Let $r, d$ be fixed positive integers and let $H$ be a length $d+1$ symmetric vector $(1, r, h_2, \ldots, h_{d-2}, r, 1)$. We say that $H$ is a **minimal Artinian Gorenstein Hilbert function** of socle degree $d$ and codimension $r$ if:

1. there is an Artinian Gorenstein algebra such that $\text{Hilb}(A) = H$;
2. $H$ is minimal in $\mathcal{AG}(r, d)$ with respect to $\preceq$. To be precise, if $\hat{H}$ is a comparable Artinian Gorenstein Hilbert vector such that $\hat{H} \preceq H$, then $\hat{H} = H$.

We now present the Weak full Perazzo Conjecture.

## Conjecture

Let $H$ be the Hilbert vector of a full Perazzo algebra of type $m$ and socle degree $d$. Then $H$ is minimal in $\mathcal{AG}(r, d)$.

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## Definition

Let $0 < k < d$ be an integer, we can define the following functions:
$\mu_k(r,d) = \min_{A \in \mathcal{AG}} \{\dim A_k\}\,, \quad \delta_k(r,d) = r - \mu_k(r,d).$

Fixed $d$, we can consider the above functions without dependence by $d$, hence $\mu_k(r,d) = \mu_k(r)$ and $\delta_k(r,d) = \delta_k(r)$. We have the following lemmas:

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**Lemma 1**

$$\mu_k(r+1) \leq \mu_k(r) + 1 \quad \forall k = 0, 1, \ldots, d$$

As a consequence of Lemma (1), we have the following:

**Lemma 2**

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**Lemma 3**

The function $\mu_k(r)$ is non-decreasing in $r$.

## Remark 4

Notice that from the previous lemmas it is easy to see that both functions $\mu_k$ and $\delta_k$ satisfy the following monotonic behavior for each $r$.

$$\mu_k(r+1) = \mu_k(r) \quad \text{or} \quad \mu_k(r+1) = \mu_k(r) + 1.$$

$$\delta_k(r+1) = \delta_k(r) \quad \text{or} \quad \delta_k(r+1) = \delta_k(r) + 1.$$

We now present a stronger conjecture.

## Definition

Let $r, d$ be fixed positive integers and let $H$ be a length $d+1$ symmetric vector $(1, r, h_2, \dots, h_{d-2}, r, 1)$. We say that $H$ is a **strongly minimal Gorenstein Hilbert function** of socle degree $d$ and codimension $r$ if:

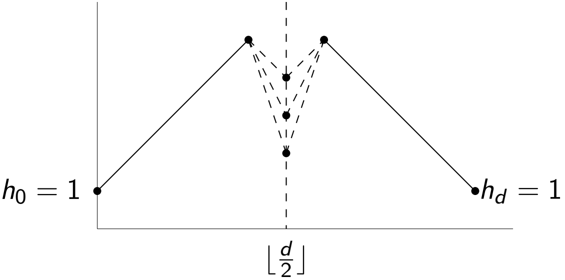
1. there is an Artinian Gorenstein algebra such that $\mathrm{Hilb}(A) = H$;
2. $H$ is minimal in $\mathcal{AG}(r, d)$ with respect to $\preceq$;
3. $h_k = \mu_k(r)$ for every $k \in \{2, 3, \dots, \lfloor d/2 \rfloor\}$;
4. For every $k \in \{2, 3, \dots, \lfloor d/2 \rfloor\}$, we have $\delta_k(r-1) < \delta_k(r)$.

We now have the conditions to propose the Strong full Perazzo Conjecture.

## Conjecture

Let $H$ be the Hilbert vector of a full Perazzo algebra of type $m$ and socle degree $d$. Then $H$ is strongly minimal in $\mathcal{AG}(r, d)$.

Fixed $k = \lfloor \frac{d}{2} \rfloor$, for every $A \in \mathcal{AG}_{\mathbb{K}}(r, d)$, we can have different Hilbert vectors that are non unimodal, and the study of the asymptotic behavior of the function $\mu_k(r)$ gives us information about the valley of the Hilbert function in the middle:



# A conjecture of Stanley

Now we analyze the case $d = 4$. Consider $\mathcal{AG}_{\mathbb{K}}(r, 4)$ the family of all Gorenstein algebras in codimension $r$ and socle degree 4. For every $A \in \mathcal{AG}_{\mathbb{K}}(r, 4)$, the Hilbert vector will always be of type

$$(1, r, \mu(r), r, 1)$$

denoting $\mu_2(r)$ by $\mu(r)$.

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denoting $\mu_2(r)$ by $\mu(r)$.
We gave a new short proof of the Theorem in [16] solving Stanley's conjecture.

**Theorem 3**

*Let $A \in \mathcal{AG}_{\mathbb{K}}(r,4)$ be a Gorenstein algebra of codimension $r$. Then*

$$\lim_{r\to\infty} \frac{[\mu(r)]}{r^{2/3}} = 6^{2/3}.$$

Then we study Gorenstein Hilbert functions of algebras with socle degree 4 and 5. Part of the results in socle degree 4 can be found in [8].
In socle degree 4, a Gorenstein sequence is of the form

$$(1, r, h, r, 1).$$

Let $\mu(r)$ be the integer such that $(1, r, \mu(r), r, 1)$ is a Gorenstein sequence, but $(1, r, \mu(r) - 1, r, 1)$ is not a Gorenstein sequence.
It is well known that $(1, r, h, r, 1)$ is a Gorenstein sequence if and only if $\mu(r) \leq h \leq \binom{r+1}{2}$ (see [2]). We set $\delta(r) = r - \mu(r)$.

## Definition

We say that the Gorenstein sequence $(1, r, \mu(r), r, 1)$ is *minimal*. Moreover we say that the Gorenstein sequence $(1, r, \mu(r), r, 1)$ is *strongly minimal* if $\delta(r - 1) < \delta(r)$.

By Remark 5.4 in [8], if $(1, r, \mu(r), r, 1)$ is strongly minimal, then $\delta(r) = \delta(r-1) + 1$.
The minimal $r$ such that $(1, r, \mu(r), r, 1)$ is not unimodal is $r = 13$ ([19]). So $\delta(r) = 0$ for $r \leq 12$.

**Proposition 4**

*$\delta(r) = 1$ iff $13 \leq r \leq 19$.*
*Consequently the sequence $(1, 13, 12, 13, 1)$ is strongly minimal.*

## Corollary 5

$\delta(20) = 2$. *Consequently the sequence* $(1, 20, 18, 20, 1)$ *is strongly minimal.*

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## Proposition 6

*Let* $m \geq 3$. *We have that*

$$\delta\left(m + \binom{m+2}{3}\right) \geq \binom{m}{3}.$$

By consequences of the theorems of Green and Macaulay we have:

## Theorem 7

$\delta(24) = 4$ *and* $\delta(40) = 10$.

## Corollary 8

$\delta(25) = 4$.

## Corollary 9

$\delta(23) = 3$. *Consequently the sequence* $(1, 24, 20, 24, 1)$ *is strongly minimal.*

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**Proposition 10**

$2 \leq \delta(21) \leq \delta(22) \leq \delta(23) \leq 4$.

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## Proposition 10

$2 \leq \delta(21) \leq \delta(22) \leq \delta(23) \leq 4$.

## Proposition 11

$20 \leq \delta(62) \leq 21$.

## Proposition 12

$\delta(26) = 4 = \delta(27)$.

In socle degree 5, a Gorenstein sequence is of the form

$$(1, r, h, h, r, 1).$$

Let $\mu(r)$ be the integer such that $(1, r, \mu(r), \mu(r), r, 1)$ is a Gorenstein sequence, but $(1, r, \mu(r) - 1, \mu(r) - 1, r, 1)$ is not a Gorenstein sequence.
It is well known that $(1, r, h, h, r, 1)$ is a Gorenstein sequence iff $\mu(r) \leq h \leq \binom{r+1}{2}$ (see [2]).
We set $\delta(r) = r - \mu(r)$.

## Definition

We say that the Gorenstein sequence $(1, r, \mu(r), \mu(r), r, 1)$ is *minimal*. Moreover we say that the Gorenstein sequence $(1, r, \mu(r), \mu(r), r, 1)$ is *strongly minimal* if $\delta(r-1) < \delta(r)$.

## Proposition 13

*In socle degree* 5 *we have*
$\delta(r) = 0$ *iff* $r \leq 16$.
$\delta(r) = 1$ *iff* $r = 17$.
*For* $18 \leq r \leq 25$, $\delta(r) = 2$

## Proposition 14

*Let $m \geq 3$. We have that*

$$\delta\left(m + \binom{m+3}{4}\right) \geq \frac{m+5}{4}\binom{m}{3}.$$

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$$\delta\left(m + \binom{m+3}{4}\right) \geq \frac{m+5}{4}\binom{m}{3}.$$

By the theorem of Green and Macaulay we can prove:

## Theorem 15

*For $m \in \{3,4,5,6,7,8,9,10\}$, we have $\delta\left(m + \binom{m+3}{4}\right) = \frac{m+5}{4}\binom{m}{3}$. That is, the full Perazzo conjecture is true in these cases.*

## Corollary 16

*The Gorenstein vector*

$$(1, 18, 16, 16, 18, 1)$$

*is strongly minimal.*

Consider the family of full Perazzo algebras of type $m = 3$ and socle degree $d \geq 4$. Its Hilbert function is given by $h_k = \binom{k+2}{k} + \binom{2+2q-k}{2q-k}$, for $k \leq \lfloor d/2 \rfloor$ and by symmetry we get $h_{d-k} = h_k$.

## Theorem 17

*Every full Perazzo algebra with socle degree $d \geq 4$ of type $m = 3$ has minimal Hilbert function.*

In this section we give a simple proof of Theorem 3.6 in [2].
Let $P_m = m + \binom{m+d-2}{d-1}$ be the codimension of a Full Perazzo algebra of type $m$.
Denote by $\mu_{d,k}(r)$ the minimal entry in degree $k$ of a Gorenstein $h$-vector with codimension $r$ and socle degree $d$.

### Lemma 4

$$\mu_{d,k}(P_m) \geq \binom{m+d-k-1}{d-k}$$

.

## Theorem 18

*Let $A$ be a Gorenstein algebra of codimension $r$ and socle degree $d$. Then, for all $k < [\lfloor d/2 \rfloor]$*

$$\lim_{r\to\infty} \frac{\mu_{d,k}(r)}{r^{\frac{d-k}{d-1}}} = \frac{((d-1)!)^{\frac{d-k}{d-1}}}{(d-k)!}.$$

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# THANK YOU