Trends in Mathematics Education Research

EDITED BY

MIROSŁAWA SAJKA

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Wydawnictwo Naukowe UKEN
30-084 Kraków, ul. Podchorążych 2
tel./faks 12 662-63-83, tel. 12 662-67-56
e-mail: wydawnictwo@uken.krakow.pl

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List of Contributors

Agnieszka Bojarska-Sokołowska, University of Warmia and Mazury in Olsztyn, Faculty of Social Sciences, POLAND, bojarska@matman.uwm.edu.pl

Ali Bozkurt, University of Gaziantep, Department of Mathematics and Science Education, TURKEY, alibozkurt@gantep.edu.tr

Zuzana Gönciová, Pavol Jozef Šafárik University in Košice, Faculty of Science, Institute of Mathematics, SLOVAKIA, zuzana.gonciova@student.upjs.sk

Monika Grigaliūnienė, Vytautas Magnus University in Kaunas, Education Academy, LITHUANIA, monika.grigaliuniene@vdu.lt

Veronika Hubeňáková, Pavol Jozef Šafárik University in Košice, Faculty of Science, Institute of Mathematics, SLOVAKIA, veronika.hubenakova@upjs.sk

Antonín Jančařík, Charles University in Prague, Faculty of Education, CZECH REPUBLIC, antonin.jancarik@pedf.cuni.cz

Edyta Juskowiak, Adam Mickiewicz University in Poznań, Faculty of Mathematics and Computer Science, POLAND, edyta@amu.edu.pl

Marlene Kafui Amusuglo, Charles University in Prague, Faculty of Education, CZECH REPUBLIC, mkamusuglo@outlook.com

Elena Kok, Osnabrück University, Department of Mathematics and Computer Science, GERMANY, elekok@uos.de

Monika Krišáková, Pavol Jozef Šafárik University in Košice, Faculty of Science, Institute of Mathematics, SLOVAKIA, monika.krisakova@student.upjs.sk

Esperanza López Centella, University of Granada, Department of Mathematics Education, SPAIN, esperanza@ugr.es

Bożena Maj-Tatsis, University of Rzeszów, Institute of Pedagogy, POLAND, bmaj@ur.edu.pl

Barbara Nawolska, University of the National Education Commission in Krakow, Department of Preschool and Early Childhood Pedagogy, POLAND, barbara.nawolska@uken.krakow.pl

Jarmila Novotná, Charles University in Prague, Faculty of Education, CZECH REPUBLIC and CeDS, Université de Bordeaux, FRANCE, jarmila.novotna@pedf.cuni.cz

Edyta Nowińska, Osnabrück University, Department of Mathematics and Computer Science, GERMANY, enowinska@uos.de

Begüm Özmusul, University of Gaziantep, Department of Mathematics and Science Education, TURKEY, ozmusul@gantep.edu.tr

Sławomir Przybyło, University of the National Education Commission in Krakow, Department of Mathematics, POLAND, slawomir.przybylo@uken.krakow.pl

Marta Pytlak, University of Rzeszów, Institute of Pedagogy, POLAND, mpytlak@ur.edu.pl

Roman Rosiek, University of the National Education Commission in Krakow, Institute of Technology, POLAND, roman.rosiek@uken.krakow.pl

Mirosława Sajka, University of the National Education Commission in Krakow, Department of Mathematics, POLAND, miroslawa.sajka@uken.krakow.pl

Ingrid Semanišinová, Pavol Jozef Šafárik University in Košice, Faculty of Science, Institute of Mathematics, SLOVAKIA, ingrid.semanisinova@upjs.sk

Matej Slabý, Pavol Jozef Šafárik University in Košice, Faculty of Science, Institute of Mathematics, SLOVAKIA, matej.slaby@upjs.sk

Alena Šturcová, Charles University in Prague, Faculty of Education, CZECH REPUBLIC, alena.steflickova@seznam.cz

Estelle Szafran-Florian, European High School in Krakow, POLAND, estelle@szafran.me

Konstantinos Tatsis, University of Ioannina, Department of Primary Education, GREECE, ktatsis@uoi.gr

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Preface

This book emerged from the project *Trends in Mathematics Education Research* (TiMER), an initiative supported by the *European Society for Research in Mathematics Education* (ERME). The primary aim of the TiMER project was to foster and advance research in mathematics education.

The intended audience for this book includes teacher educators of pre-service and in-service mathematics teachers at all educational levels, as well as the teachers themselves, mathematics education researchers, policymakers, and all those with an interest in the field of mathematics education.

By presenting a wide range of topics, this book addresses various facets of mathematics education, spanning multiple educational stages. These range from the development of mathematical thinking in early childhood education through the different grades of primary and secondary education and extend into issues of mathematics teacher training for mathematics educators.

Introduction

Three key components are distinguished in the classical theoretical framework of interactions within the mathematics teaching-learning process: the student, the teacher, and the subject of instruction – mathematics. These elements are interconnected through bilateral relationships, emphasising their mutual influence on one another.

This book begins with a focus on students' perspectives before transitioning to an analysis of the teacher's role and training in mathematics education.

The first part focuses on students' understanding of selected mathematical concepts, examining their intuitive and formal knowledge. The research presented aims to describe how these concepts are comprehended, identify difficulties and misconceptions, and provide insights into addressing these challenges. The findings can inform the design of learning environments and educational activities, guide education policy, and contribute to effectively structuring the core curriculum.

Research in this domain necessitates diverse methodologies, tools, and techniques. These range from developing fundamental instruments designed to elicit specific cognitive activities in young children – as exemplified in the chapter by Marta Pytlak, *Which Road Is Longer? – Intuitions of 5–6-Year-Old Children Related to the Concept of Measure and Taking Measurements* – to the application of methodological triangulation and advanced apparatus in studies on formal knowledge usage. An example of the latter is the research on mathematical modelling skills conducted by Mirosława Sajka and Roman Rosiek in *Walking up the Stairs: An Excerpt from Research Involving Eye Tracking on Understanding Function as a Tool for Describing Movement*.

When researchers employ tests as investigative tools, the process entails rigorous validation and interpretation, as illustrated in the study by Jarmila Novotná and Alena Šturcová, *Tool for Diagnostics of Students' Difficulties in CILL (Content and Language Integrated Learning)*. Estelle Szafran-Florian's chapter, *Mathematisation and Modelling – Comparing the Performances of IB DP (International Baccalaureate Diploma Programme) and Polish Programme Students*, explores the notion of function in practical contexts and contributes to discussions on curriculum design and its impact on student achievement.

The book's second part continues to focus on learners but shifts towards analysing their metacognitive skills. Examining them necessitates solving carefully designed problems that elicit specific modes of reasoning. This section begins with a chapter by Bożena Maj-Tatsis and Konstantinos Tatsis, *Perspectives on Young Students' Mathematical Reasoning*, which provides a literature review on mathematical reasoning in early primary education. The authors highlight the varied definitions of the term and underscore the role of problem selection and teacher facilitation in the reasoning process. Following this, Esperanza López Centella presents an *Analysis of Tasks from a Hejný Method Mathematics Textbook for the Sixth Grade*, employing qualitative content analysis to examine the selection of tasks designed to foster critical thinking. Building on this theme, Edyta Juskowiak's chapter, *Ways of Solving Mathematical Tasks by Students Aged 14–15 as Manifestations of Critical Thinking*, offers an in-depth analysis of student responses to a non-standard task, identifying elements of critical thinking in their reasoning and attempts of argumentation.

The chapter by Edyta Nowińska and Elena Kok, *Challenging Aspects of Metacognitive Support in the Classroom and How to Prepare Teachers for Them*, explores the complexity of fostering metacognitive support and presents a theoretical framework that enhances teachers' ability to foster students' metacognitive development and critical thinking. This discussion serves as a transition to the subsequent parts of the book, which are dedicated to the role of teachers and teacher training. While the teacher's influence is an underlying theme throughout the book, the following two sections focus more explicitly on this aspect.

The book's third and most comprehensive part focuses on mathematics teachers, particularly their professional knowledge and the training of pre-service teachers, examining how their subject-specific and pedagogical knowledge evolves throughout their preparation. Research on various aspects of mathematical knowledge for teaching plays a critical role in this discussion. Monika Grigaliūnienė's chapter, *Knowledge of Mathematics Teachers from the*

Perspective of Their Students, offers a qualitative exploration of teachers' knowledge – or its gaps – from students' viewpoint, utilising a focus group methodology.

The subsequent three chapters investigate mathematics teachers' specialised knowledge, particularly in the context of courses designed to foster knowledge of functional and algebraic thinking. Different methodological perspectives are employed: Matej Slabý and Ingrid Semanišinová analyse *Pre-service Teachers' Knowledge of Students' Misconceptions About and Difficulties With Functions*, with a focus on pedagogical content knowledge; Veronika Hubeňáková, Monika Krišáková, and Zuzana Gönciová examine *Slovak Pre-service Mathematics Teachers' Knowledge About Linear Function Definition and Their Beliefs About Mathematics*; and Begüm Özmusul and Ali Bozkurt implement the other methodological approach, *An Experimental Study on Middle School Pre-Service Mathematics Teachers' Algebraic Knowledge for Teaching.*

Barbara Nawolska's chapter, *Reductive Reasoning of Pedagogy Students in the Process of Solving a Text Task Entitled: How Many Pearls Were in the Casket?*, presents a detailed analysis of reasoning when solving a non-standard task by pre-service teachers' of early childhood education, highlighting the need to shape reductive reasoning skills in this group.

A thorough assessment of pre-service teachers' knowledge at the outset of their university education is essential for diagnosing their educational needs and designing effective training programs. In this regard, Mirosława Sajka and Sławomir Przybyło provide the chapter *Diagnosis of School Mathematics Knowledge and Skills of Students Entering University to Become Mathematics Teachers*.

The book's final section is devoted to proposals for enhancing pre-service mathematics teacher education at different educational levels. A critical aspect of this discussion is the need to thoroughly analyse mathematics curricula in early education. Comparative studies, such as the one presented by Marlene Kafui Amusuglo and Antonín Jančařík in *Colleges of Education Early Grade Mathematics Curriculum and National Kindergarten Mathematics Curriculum in Ghana: A Comparative Analysis*, offer valuable insights in this regard.

Moreover, all innovative approaches to pre-service teacher education merit further dissemination. Agnieszka Bojarska's chapter, *The Use of an Interactive Form of Classes to Motivate Pre-service Teachers of Early Childhood Education to Solve Mathematical Problems*, exemplifies an unconventional yet effective teaching method designed to engage and inspire future educators.

The organisation of the chapters presented here represents one of many possible arrangements, reflecting the inherently complex nature of the teaching and learning of mathematics. A comprehensive perspective that acknowledges the interplay of these various elements is crucial to enhancing the effectiveness of mathematics education. We hope that the findings from the presented research, along with the theoretical insights, will underscore the importance of understanding students' and teachers' mathematical knowledge. The book also introduces teaching and diagnostic tools, such as problem sets, which may serve as valuable resources for educators and researchers in enhancing mathematics education.

Acknowledgement

This book was co-funded by the *European Society for Research in Mathematics Education* (ERME) as part of its *Initiative for Supporting Emerging Communities for Mathematics Education Research in Eastern Europe* under the project entitled *Trends in Mathematics Education Research* (TiMER). The publication of this book would not have been possible without the financial support of the *University of the National Education Commission* in Krakow (formerly the *Pedagogical University of Krakow*).

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Lara Alcock (Loughborough University, UK)
Matthew Inglis (Loughborough University, UK)
Jarmila Novotná (Charles University in Prague, Czech Republic)
Edyta Nowinska (Koblenz University, Germany)
Benjamin Rott (University of Cologne, Germany)
Naďa Vondrová (Charles University in Prague, Czech Republic)

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I extend my gratitude to all those who contributed to the publication of this book, including the editorial and production teams at *Scientific Publishing House of the University of the National Education Commission in Krakow*, as well as the professionals responsible for proofreading, editing and typesetting with formatting. Their dedication and expertise have played a vital role in bringing this volume to completion.

Finally, I would like to express my deep gratitude to the authors for their insightful contributions and all study participants whose engagement made this book possible. Their collective efforts have enriched our understanding of students' and teachers' mathematical knowledge, offering theoretical perspectives and practical tools to enhance mathematics education. I hope that the findings and resources presented in this book will serve researchers, educators, teacher trainers, and teachers in their ongoing work.

Mirosława Sajka

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PART I

The Understanding of Chosen Mathematical Concepts in Students

Marta Pytlak
University of Rzeszów

CHAPTER 1

WHICH ROAD IS LONGER? – INTUITIONS OF 5–6-YEAR-OLD CHILDREN RELATED TO THE CONCEPT OF MEASURE AND TAKING MEASUREMENTS

Summary: Mathematics education in kindergarten is the subject of many studies. Children aged 5-6 are especially open to new educational experiences. They have a natural ability to explore and learn. Moreover, many activities they undertake in everyday life have great potential to develop their intuition regarding specific mathematical concepts. One such issue is the concept of measure and taking measurements. A study conducted among preschool children revealed their intuitions about these concepts – preschoolers are able to distinguish basic phenomena related to measure and measurements. They understand the distance between objects as straight-line distance between them, and such "straightness" is equated with perpendicularity. The obtained results are a good starting point for broader research in the field of understanding the concept of measure and taking measurements in preschool children.

Keywords: measure understanding, taking measurements, distance, preschool education.

1. Introduction

There is increasing talk about the importance of preschool education in a child's development. Its importance is emphasised especially in relation to preparing children for early school education (Gruszczyk-Kolczyńska, 2009; 2012). Children's mathematics on the preschool level is based on two main pillars: arithmetic and geometry. In relation to arithmetic, competences are primarily developed in the scope of counting, converting, and comparing the size of sets. In geometry, preschool education focuses primarily on recognising the shapes of basic figures (triangle, square, rectangle, circle).

The issues of measure and taking measurements are essentially omitted, but it is not a new and unknown concept for a child. Without even knowing it, he/she encounters this concept everyday. Many studies emphasise the importance of knowing measure and taking measurements (NTCM, 2000). However, there is a lack of extensive research in this area, especially at the preschool stage. Meanwhile, geometry and measurement give us a lot of opportunities to develop children's competences and talents. As Marija van den Heuvel-Panhuizen and Kees Buys (2004, p. 10) write:

> Both measurement and geometry enable children to make connections with their daily environment. Both domains offer mathematical tools, each in their own way, to structure the physical world and to get a grasp on it. Moreover, they both lead to wonderment. And thus to the development of mathematical disposition with is characterized by an exploring attitude, a certain perseverance in solving problems, and a sensitivity to the beauty of mathematical structures and solutions.

The very issue of measure and taking measurements in formal terms is often difficult. It requires the understanding of operations at a specific level (Gruszczyk-Kolczyńska, 1992). However, children aged 5-6 years function at the preoperational level. Therefore, it is important to use the child's own experiences and introduce them to the world of mathematics in the most natural and friendly way for them. The development of skills and competences should take place through activities and the child's active participation – such an approach, called Natural Mathematics, is described by Dwyer and Elliget (1970). Among the important principles of this approach are:

1. Skills are not separated and learned first and applied later to a problem. They are learned in the process of solving many problems.
2. Neither are concepts separated and learned first and then applied later to a problem. They are also learned in the process of solving many problems. This is the opposition to the current procedure of using many exercises to illustrate a concept (...) (Dwyer & Elliget, 1970, p. 35).

The following questions remain open: What experiences do preschool children have with the concept of measure and taking measurements? What intuitions do they have and how can they be used to develop a correct and complete understanding of the concept of measure and measurements? An attempt to answer these questions is the subject of this paper.

2. Theoretical Framework

Intuition plays an important role in children's learning. In the teaching process, the teacher should refer to the child's intuitive approach to mathematical issues. Using natural beliefs can aid learning.

> Intuitive ability is the essential source for learning at all stages, including the conscious choice of skill or concept itself. Intuition is the initial platform for the beginning learner, the vehicle for progress of intermediate learner, and the means of breakthrough to new knowledge by the advanced scholar. At all steps along the way, intuition is used as the ultimate appeal when systematized comprehension breaks down, whether its purpose is "remedial" or creative (Dwyer & Elliget, 1970, p. 37).

The issue of measure and measurements is not completely alien to the child. Children's first experiences with these concepts are in the form of comparisons. Children compare who is taller, who has a longer foot, who built a bigger tower. Their own idea of measure and measurements is slowly being built. The use of these natural tendencies of comparing can be a starting point for targeted activities in the development of "measuring" competences (MacDonald & Lowrie, 2011).

Taking measurements itself is usually associated with the use of a more or less standard measuring tool. There are a lot of studies that present differing approaches to this issue (see Gómezescobar, Guerrero, & Fernandez-Cezar, 2020). Some authors believe that using a measuring tape requires the child to understand the very essence of measurement. In a way, this aligns with Piaget's approach: before a child can use a ruler properly, he or she should first become familiar with the concept of length. Vygotsky opposes this approach, by pointing out in his approach that the ruler is a type of tool that children can adapt and use even before they are formally introduced to the concept of measure. This is noted by Clemens and Stephan (2004).

The process of measuring is linked to a particular activity. Interestingly, when measuring lengths, children naturally make use of one-dimensional geometry, even though they function in three-dimensional space. The measurement process involves several aspects, as mentioned by Buys and de Moor (2008). They distinguish between measuring by comparison, by using a unit (standard or non-standard), and by using appropriate measuring tools. To fully understand the concept of measure, a child should be able to go through all three of these stages. Here, it is very important to strongly relate to the child's everyday experiences, as is often emphasised by van den Heuvel-Panhuizen (2008). In developing a proper understanding of the concept of measure and taking measurements, the use of both standard and non-standard units of measurement is important (Haylock & Cockburn, 1989; Boulton-Lewis, Wilss, & Mutch, 1996; MacDonald & Lowrie, 2011). Measuring itself should not be treated and taught as a separate activity, but rather as a specific combination of concepts and skills that develop over time (Clements & Stephan, 2004). In order to nurture the concept of measurement and the ability to take measurements in children, it is necessary to give them appropriately prepared "measurement tasks" (Mac Donalds & Lowri, 2011).

As mentioned in the introduction, in Polish preschool education, a lot of time is devoted during mathematics classes to issues related to counting. Many games and tasks are related to counting and converting. Geometric problems appear as an "afterthought", usually in the form of recognition of figures and shapes. Measurement-related content is often overlooked. Comparing objects is done more in the form of comparing sizes of sets, than in the context of measuring, length, or size. Hence, the idea of examining children's competences in the scope of measure and taking measurements was born.

3. Research Methodology

This research is part of a larger project on the understanding of the concept of measure and measuring by preschool children (5-6 years old).

The aims of the whole project are:

1. Recognising children's intuitions regarding various aspects of measure and taking measurements,
2. Examining the ability to take measurements and use different units of measurement,
3. Developing competences of measuring and the understanding of the concept of measure (in relation to taking measurements),
4. Developing certain tools and didactic proposals supporting the understanding of the concept of measure.

The research was carried out in two stages: The first stage consisted of a pilot study, followed by the research stage. At each stage, the procedure was very similar: first, the intuitions and beliefs of 5-6-year-old children were examined in the scope of measure and taking measurements, followed by studying how children use their intuitions to solve tasks related to measuring lengths and comparing them. After completing the pilot studies, evaluation of the research tool and its further testing are planned.

The pilot studies aimed to answer the following questions:

1. What intuitions and beliefs do 5-6-year-old children have about taking measurements at the pre-introduction stage?
2. How do children measure different objects, and what units do they use?
3. Are children able to compare different objects considering their specific dimensions (can they develop a universal unit, rules for taking measurements)?
4. Are children able to apply the learned and developed principles of taking measurements when solving related tasks?

The research group during the pilot study consisted of children aged 5-6 attending a public kindergarten (18 subjects in step 1 and 20 in the following steps).

The pilot studies were carried out as follows:

Step 1. First, a diagnosis was made of all children participating in the study in the context of their intuitions and beliefs related to taking measurements. For this purpose, an individual diagnostic interview was conducted with each child, during which he or she solved three tasks. The researcher conversed with the child, but did not evaluate his or her answers. The aim was to simply assess what the child knows about taking measurements.

Step 2. During group activities, the children solved tasks together related to the concept of measure and taking measurements. The aim of these activities was to stimulate interaction between the children. The influence of the differences between children in their thought processes when approaching a given task, as well as the final solution to the problem were observed. The aim of the children's joint discussion with the researcher was to draw attention to important aspects related to measure and taking measurements and to systematise the knowledge on this subject (what it means to measure length, how to take measurements, what to pay attention to when measuring).

Step 3. Conducting classes, during which children solved tasks related to taking measurements on their own or in small groups (3-4 people), including analysis of the children's work and observation of whether there has been a change in their approach to the issue of measure and taking measurements.

Step 4. Analysis of the collected research material. Redesigning of activities supporting the development of competences related to taking measurements and shaping the understanding of the concept of measurement, preparation for research on a wider research group.

After the pilot studies, proper research is planned, the course of which will be similar to the first stage of this research. The studies will begin with individual diagnostic interviews. Then, the children will go through a series of group meetings during which they will develop their skills in taking measurements through play. The next step will involve individual tests to check their understanding of measure and taking measurements, ending with classes to consolidate the knowledge learned.

As of now, the pilot studies have been completed. Diagnostic studies were also carried out among children participating in the research. The next stages of the study are in preparation.

In this paper, I will focus on presenting the course of the pilot studies (especially from steps 2 and 3) and discussing the results obtained therein. The collected data were analysed qualitatively. Detailed results of the pilot studies from step 1 were presented during the SEMT conference (Pytlak & Maj-Tatsis, 2021).

4. Step 1

4.1. Organisation and Course of Pilot Studies – Step 1

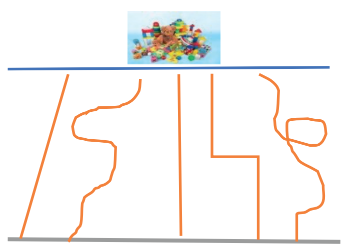
Step 1 in the pilot study was based on one-on-one meetings with every child. The aim of the interview was to identify the child's intuitions in terms of measurements. The child was asked to attempt to define what it means to "measure". We were also interested in how the children would compare the lengths of two objects (here it was two trains made of blocks) and how they would check which object was longer.

The conversations took place on the premises of the kindergarten, in a secluded place, so that nothing would distract the children while they worked. During this conversation, the researcher presented the child with three tasks to solve. There was no time limit to solve the problems, although usually these conversations lasted up to 10 minutes. While working on solving them, the researcher conducted a conversation with the child, the purpose of which was to obtain information on how to solve the tasks. The questions were informative rather than evaluative. It was primarily about acquiring information from the child without valuing or judging their answers. All of the child's answers were considered correct.

The research material consisted of videos recorded during each of the meetings and protocols prepared on their basis. The children's working methods, as captured on film, and their conversation with the researcher were analysed.

The tasks given to the children were as follows:

Task 1. Instruction for the children:
Ania loves to play with toys. She's standing on the green line and would like to get to her toys [the researcher physically places the figure on the green line]. She wonders which way to go. She wants to take the shortest route. Help Ania make the right choice. Why is this path the shortest? Is there any way to check this?



Aim of the task: To investigate the children's intuitions about measuring the distance between two objects (their understanding of the distance and the "shortest path" between two objects).
Expected results: The child will indicate the third path as the correct one, intuitively referring to its perpendicularity in relation to the start and end lines.

Task 2. Instruction for the children:
Kuba and Jaś are arranging trains from colorful blocks. Here are Kuba's blocks, and here are Jaś's. Which of the boys do you think will make a longer train out of their blocks? Why? Try assembling these two trains. Was your prediction right? How can you tell which of the stacked trains is longer?

Note: The blocks the children received were the same length.
Aim of the task: To investigate the children comparing the length of objects with their intuitions in this area (e.g. arranging them on a common line).
Expected results: The child will point to the pile with more blocks, can refer to the size of the set; when assembling the two trains, he will place them one below the other on the same line.

Task 3. Instruction for the children:
Ala and Kasia are arranging trains. Here are Ala's bricks, and here are Kasia's. Which girl do you think will build a longer train? Why? Try to assemble these two trains. Was your prediction right? And how can you tell which of the assembled trains is longer?

Note: Each of the children had blocks of a different length (one had 8 longer blocks, the other 6 longer blocks, and after arranging the trains, the longer one was the one made of longer blocks).
Aim of the task: To investigate whether the children will pay attention to the "unit" of measurement and how they will compare the length of objects (whether they make use of their experience from working on task 2).
Expected results:

1. The child will point to a set of longer blocks as the one from which a longer train can be built, and the assembly of the blocks will confirm this hypothesis

or

2. The child will point to a set of shorter blocks as the one from which a longer train can be built, and the assembly of the blocks and comparison of the objects will lead to a change of mind and the discovery that not only the number of components, but also their size matters.

4.2. Analysis of Results from Step 1

In task 1, each of the proposed roads (except No. 2) was selected. The justifications for the choice were quite surprising. This can be seen in the table below:

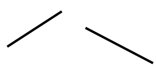
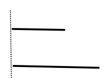
Table 1. The results from the task 1

Path no.	**1**	**2**	**3**	**4**	**5**
Choice made (in %)	30	0	55	5	10
Justification	Because it's straight	-	Because it's straighter Because it's not crooked Because it's the shortest/fastest	Because they are so straight here (pointing to individual sections)	Because it's so twisted Because it has loops

The analysis of the research material showed that children have considerable intuitions in the field of measure and taking measurements. The results obtained from task 1 showed that the vast majority identified the distance between the objects as a straight line connecting the two objects. At the same time, their understanding of the "shortest way" was twofold. For some of the children, it was the "straightforward" way (here, it was a straight line without loops or bends, which could be verified visually). A large group understood the shortest path as a section perpendicular to the starting line (which is consistent with the mathematical understanding of the concept of distance).

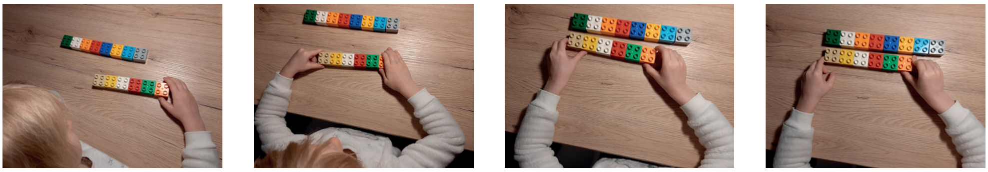
When attempting to solve the task, two of the children immediately provided the answer, without arranging trains from the prepared blocks. Except for one case, all the answers were correct. When asked by the researcher to justify their answer, the children usually counted both sets of blocks, referring to arithmetic justification (comparing the size of sets). Some of them started to build trains only after a suggestion from the researcher. Here, the following methods of arrangement can be distinguished here:

Table 2. Ways of arranging trains by children

Description of arrangement method	Arranging two trains in one line	Arranging two trains diagonally to each other	Arranging two trains parallel to each other, with offset	Arranging two trains parallel to each other, on a common starting line
Visualisation				
number	6	5	4	3

The chosen method of arrangement determined, in a sense, the justification for the answer to the question regarding the longer train. When the researcher asked: "Can you show me that this train is longer?" some of the children recalculated the components of each train. The children who arranged the trains on one starting line immediately pointed to the longer train and specified how much longer it was than the other. In a situation where the trains were parallel to each other, but offset, the children moved one of the trains so that they started on the same line. Some of the children performed similarly, arranging the trains in a single line or diagonally.

Figure 1. Comparing trains made of blocks by a child



However, there were subjects (6 children) who, despite providing the correct answer about the longer train, could not justify it in relation to the trains they had arranged. These were the children who arranged the trains diagonally or in a single line. As justification, they provided only the size of the two sets and referred to the comparison of numbers. For them, it was the numbers themselves that mattered, not their representation (here: in the form of two trains). Three children, when asked by the teacher to show which train was longer, combined the two into one very long train. One of the children's comments on the situation was as follows: "This train is the longest". It seems that for these children, the fact that they were building with identical blocks meant that they did not have to compare them with each other (they were the same objects). The number of elements used was important. Therefore, what they were comparing was not the physical sets, but their size.

The children transferred their experience from task 2 to task 3. This can be seen in the summary of the results of their work:

Table 3. Results for tasks 2 and 3 from the first part of the pilot studies

		Task 2	Task 3
First answer: Indication of expected answer/indication of wrong answer		17/1	9/9
Justification by reference to size of sets		15	7
Aranging method	In line side by side	6	3
	Obliquely	5	0
	In parallel with offset	4	0
	One below another, from common starting line	3	14

Some children did not initially pay attention to the different lengths of the blocks used to build the trains, hence the wrong answers and wrong justifications. Seeing the different lengths of the blocks, the children started to assemble the trains themselves, which made them reassess their answer. This time, their method of assembly was significantly different. Even if a child started assembling the trains in parallel with an offset, they would very quickly move them to the same starting line.

The results obtained from tasks 2 and 3 showed that when "measuring," children paid attention to both the unit and the method of making measurements. At the same time, their experience from task 2 was transferred into their way of solving task 3. By comparing two objects with each other (here in the form of two trains of different lengths), they placed them one under the other, on the same starting line. However, this method of comparison was especially important in a situation where the trains were made of blocks of different lengths (which can be equated to a different unit of measurement). When the two objects were made of identical blocks, the children did not feel the need to make physical comparisons. In this case, it was enough for them to simply count the elements in each of the trains.

Thus, in this situation, "measuring" was equated to the size of individual sets, while "arranging trains" was treated as different arrangements of a given set.

5. Step 2

5.1. Organisation and Course of Pilot Studies – Step 2

In step 2 of the pilot study, group activities with children were conducted. The preschoolers were divided into two groups of 10 children. Each group worked separately under the supervision of a teacher, performing the same tasks. Each of the meetings was recorded. The children worked in groups, as I wanted them to interact with each other. I also wanted to see if the influence of the group would cause a change in their approach to the task. When talking with the group, the teacher did not evaluate the answers of individual students, subjecting their correctness to the judgment of their classmates. Each of the children had the opportunity to speak freely.

During the meeting, the children were presented with two tasks. These were presented in the form of stories, illustrated with physical props. The aim of the tasks was to find out what intuitions and beliefs the children have about measuring, primarily about understanding distance as the shortest segment between objects, perpendicular to the end line.

The tasks presented to the students were as follows:

Task 1.
In the land of toys, a doll, a ball, some blocks, and a teddy bear decided to pay each other a visit. Who is the teddy bear furthest away from? And who is closest to the ball? How can we check this?

Figure 2. Initial situation for task 1 in the second step of the pilot studies

When presenting the task to the children, the teacher placed the aforementioned toys on the floor. They were arranged to ensure differences in distance between them (Fig. 2) – it was not always possible to visually assess which distance was the greatest, which was meant to steer the children towards discussion and action.

This task did not impose any particular method of solution, nor did it suggest that the distances between the toys should be measured. The idea was to provoke the students into discussing the problem posed: How do we compare distances between objects? A string prepared by the teacher was used as a means of measurement in this task. In addition, the children used their own feet or hands to measure distances (measuring distances with feet and hands).

Task 2.
a) The animals organised a ball-throwing competition. A fox and a bunny were the judges, and an owl, a beaver, and a deer stood on the starting line. Each player made one throw. Which animal won the competition? Why?

Figure 3. Initial situation for task 2 in the second step of the pilot studies

b) The first judge was the fox. He used feet to measure the distances. The measurement, according to the fox, looked like this:

Figure 4. How the fox took measurements

The fox announced his verdict: The beaver won. Do you agree with the verdict of the fox judge? Justify your opinion.

c) The bunny started his judgment. With his feet, he measured each of the distances between the player and his ball. It looked like this:

Figure 5. How the bunny took measurements

The bunny announced his verdict: there is a draw, everyone threw the same distance. Do you agree with the bunny's verdict? Justify your opinion.

As in the case of task 1, the teacher introduced the above three tasks, demonstrating their content. First, he placed all three players on a common starting line, and imitating the throw of balls, he placed them on the carpet at a certain distance from the players. Showing how the individual animals judged, he placed measuring instruments on the carpet (here they were paper feet of the same length).

At the start of the task, the students were able to propose their solutions. Then, these solutions were compared with two erroneous solutions.

5.2. Analysis of Results from Step 2

Task 1

When starting to solve the situation from task 1, the children unanimously agreed that individual distances should be measured. There were various proposals regarding taking measurements, usually by determining the distance using arm width. The teacher's proposal of using a string to measure distances was met with approval. The children correctly marked the distances between the individual toys with a string. They also proposed a method of finding out which segment is the longest – by comparing the strings with each other. The shortest of them was the reference point by which the length of the others was determined. One of the boys took the first of the strings and put it to the second and third in turn. At the same time, he commented on the results obtained: "This one is longer", "This one is shorter".

Ex*: What toy is the teddy bear farthest away from?
A: The doll,
F: No, the blocks.
Ex: Ala says doll, and Franek says blocks. Which one of them is right? How do we check this?
F: You have to measure the strings.
Ex: Then measure the strings.
F: [takes the strings and places them down one by one, with the centers of the strings at the same level] This one is longer and this one is shorter.
A: No. [adjusts the arranged strings so that they all start at a common point, the teddy bear]
Ex: Oh, Ala gathered all of the strings together. Is that a good thing? Do you agree?
F: You can see here that this one is longer and these are shorter.
Ex: You can see it very well.

* Ex – Experimenter; Ch – Children. Other capital letters represent the statements of the individual children.

In the quoted fragment of the conversation, we can notice that when comparing objects (here: strings illustrating the distance between toys), the children do not always pay attention to proper placement. The visual aspect is important. If something is clearly visible and certain properties can be visually evaluated, then additional aspects do not have to be taken into account. However, some children already have a need to organise the rules and put them into practice at this stage. If the child's proposal does not negate the generally accepted solution, and instead presents it more clearly, then the other children accept the proposal with approval.

Figure 6. Measuring and comparing distances with strings

In addition to measuring distances with strings, the children also suggested the use of measuring tape, which could be a reference to everyday life experiences. In addition, attempts were made to use arms (spreading arms to the measured distance), legs (measuring the distance by spreading legs) or oneself (one's own body). However, the children themselves noticed that it is not always possible to measure something in this way. It is also difficult to compare two lengths accurately with such measurements. The teacher suggested measuring the distance with feet. First, he measured the distance between the teddy bear and the ball with his feet, and then asked one of the girls to measure the same distance. Two different results were obtained: 3 feet for the teacher and 6 feet for the girl. In children, this situation did not cause any conflict, and it was noted that the results differ due to different foot lengths. It also seems that the children did not see the need to use the same measuring instrument for all measurements. The only thing that mattered was how it was measured.

Ex: Laura took a measurement and came up with a 6. I'll also take a measurement [he measures the distance between the teddy bear and the ball with his feet, counting loudly] one, two, three. So, who is right? Is it a distance of 6 or 3?
F: Well, it's because Laura has smaller feet.
Ex: Oh, so if the feet are smaller, the result will come out a certain way, and if they are bigger, it will end up different, right? [children nod]
F: Because bigger feet take up more space and smaller feet take up less.
Ex: If I wanted to measure the distance between all of the toys, do I have to measure with the same feet, or can I measure some and Laura some?
Ch: I don't know [nodding their heads]
Ex: Well, listen, Laura measured the distance from the teddy bear to the ball and it came out as 6 feet. I measured the distance from the teddy bear to the blocks and I also came up with 6 feet. You said that the teddy bear is further away from the blocks than the ball, right?
Ch: [nodding their heads]
Ex: But both measurements resulted in 6. So, can you take measurements with different feet to compare distances? Do they have to be the same rates?
Ch: [silence, no answer]

The children unanimously approved of the teacher's method of measuring distances with their own feet and began to use it themselves. They also very quickly noticed that the result of the measurement depends on the measuring instrument used. Hence, obtaining different values for the same distance did not cause any cognitive conflict in the children. They were a bit surprised when the teacher pointed out two identical results obtained when measuring two different distances (from the teddy bear to the blocks and from the teddy bear to the ball). They knew that the distances differed greatly, but they could not explain why the number was the same in both cases. What was surprising here was that, despite previous experiences with various "measurement units", the children were unable to make use to this knowledge. Perhaps the measurement process itself was more important to them, and the unit used only mattered if it referred to the same measurement.

Task 2

The second task consisted of two parts. The starting situation was always the same: There were three players on the starting line who threw balls a certain distance. First, the children evaluated the throws themselves and decided which player won. They agreed on who placed first and last. They justified their choice of winner by pointing out that the distance from the starting line to the ball is the longest. In addition, they used their hands to indicate the routes of sections of particular distances. Here, it was clear that the children made use of their intuitive understanding of the distance between objects as the shortest distance between them.

In the second part of the task, the children were asked to assess the correctness of distances measured by two judges. In both cases the measurement was performed incorrectly and the results were different from those obtained by the children in the first part of the task. The first referee, a fox, measured the distance from himself to the balls thrown by individual players. The children did not negate the fox's measuring method. It followed the accepted rules for measuring the distance between objects (between the fox and the balls): from one object to another, in a straight line. Objections were raised only due to the result, which differed from the one obtained by the children – this is when they paid attention to which sections were supposed to be compared. They quickly checked the solution and made changes to the arrangement of the elements. Measuring the distances with their feet, the children tried to keep the parallelism of individual sections and start them from the same line. The recalculation of the elements (feet acting as units of measurement) in individual segments referred rather to size comparisons of appropriately ordered sets. When the children provided the numerical value of the measured distance, the cardinal aspect became apparent.

In the next part of the task, it turned out that the animal judge measured the distances in such a way that each competitor achieved the same result. The children spotted the mistakes right away. First of all, they disagreed with the result itself, clearly indicating that the distances achieved by the ball throwers are different. They referred primarily to the visual representation of the task. In addition, the preschoolers paid attention to the method of measuring itself: The feet were not placed in a straight line, but in an arc.

6. Step 3

6.1. Organisation and Course of Pilot Studies – Step 3

In step 3 of the pilot study, classes were conducted with the entire group of preschoolers. The children were divided into teams of 3-4 subjects, whose task was to measure the length of various objects in the room (e.g. the length of the window sill, the dimensions of the table). The selection of teams was fully arbitrary. The children were given measuring instruments – the paper feet used during previous tasks. They also had pieces of paper on which they could write down the results of their measurements. At the end of the task, everyone shared the results and compared them with each other.

6.2. Analysis of Results from Step 3

All teams actively participated in the task. Initially, the children quickly divided the roles: One person was responsible for measuring, another for recording the results. At some point, however, everyone decided to measure and compare their results with each other. As soon as a different result was found, they checked if the measurement was made correctly. They paid a lot of attention to the correct use of measuring instruments: The feet were to be placed in a straight line, one behind the other, without gaps. They were very careful to apply the rules they had developed in their previous classes. The children wrote down the results of their measurements on pieces of paper. The results contained the number indicating the length or width of the tables, sometimes accompanied by an illustration.

One of the groups made as many as three measurements. Each of the children measured the length of the bench on their own, and wrote down each of the results on a piece of paper. However, three different values appeared: 10, 8, 6. Each of these results was treated as correct by the children on the team.

Figure 7. The girls measuring the length of the bench and recording the results of their work

Ex: Did you get three different results?
Ch: Yes.
Ex: And they're all of the same bench? Are they all correct?
Ch: Yes.
Ex: Can you explain this to me? Because I don't understand [he points to the drawing and the three numbers written in circles].
A: Well, at first, we measured like this [she points to the long side of the bench] and we came up with 10 feet.

M: And then I measured it again, but I ran out of feet and I got 8.
A: And here [points to the short side of the bench] there were 6 feet.

The children were aware of how to take a measurement. They correctly applied the principles of measurement and recorded the results obtained. The task was so interesting for them that each of them wanted to make their own measurements. In addition, while measuring the length of the bench, both girls took measurements at the same time. Unfortunately, they lacked "measuring material", so one girl arranged 10 feet along the entire length, and the other placed only 8. Although they knew that only one of these values was correct, they wrote them both down on a piece of paper. Perhaps they wanted to show the results of their own measurements. Since they got different results, both appeared in the record. Measuring the width of the bench, both girls got the same result (6 feet), hence they wrote only the one result on a piece of paper.

The boys took measurements in a completely different way. First, they placed their feet around the bench, creating a kind of border. However, when writing down the dimensions of the bench on a piece of paper, they only provided the length and width (6 and 10, Fig. 8).

Figure 8. The boys measuring the length of the bench and recording the results of their work

After all of the feet were placed, the boys began to count how many of them could fit on each side of the bench. However, they obtained different results because the feet were not positioned correctly.

K: Here it will be 6 [points to the shorter side], and here [begins to count quietly] one, two, three...
W: Here's 10, and here's 5 [points to the other side of the bench]
K: [finishes counting] Eight, nine. No, it's supposed to be 6 and 9.
Ex: And how did you guys do?

W: Something doesn't add up.
Ex: Something doesn't add up? What is it?
W: Well, because it turned out differently.
Ex: Then check again if everything is correct.
K: [looks at the feet] It's because it's too loose here, they should be closer [he moves the feet closer together]. And now it's 6
W: Ah, yes [corrects position on the long side]
Ex: And how is it now?
K: Now there's 6 here [pointing to the shorter side] and 10 there [pointing to the longer side].

The boys were definitely aware of how to take a measurement. Initially, they arranged the measuring tapes around the perimeter of the bench, and by providing its dimensions, they counted the elements on its shorter and longer side. Each of the boys counted the lengths on his side. Obtaining different results provoked them to analyse the situation and verify the correctness of the measurement. They realised their mistakes very quickly. After fixing them, each side had the same measurement. Analysing the way they worked, it can be seen that they used the principles of taking measurements developed during the previous meetings: measuring units (here, paper feet) should be arranged in a straight line, one behind the other, without breaks.

7. Summary and Conclusion

The children were very eager to take part in the research and actively participated in solving individual tasks. They also tried to justify their answers, strongly supporting each other with gestures.

The results obtained during the first step of the pilot study suggest that children aged 5.5-6 already have some intuitions about the concept of measure and taking measurements. They typically interpret the distance between objects as a "segment perpendicular to both". This coincides with the formal mathematical understanding of distance. Moreover, the word "straightness" is used by them to denote both a straight line and a perpendicular one.

The second step of the pilot study confirmed these assumptions. The children had their own intuitions about measuring. They understood the distance between objects as the shortest distance connecting them, the end of which is perpendicular to the final object. This is in line with the notion of distance that students face during math education in primary school. The comparison of such segments was carried out through visual verification. They had to be placed parallel to each other and observed to see which one protruded from the others. The arrangement of these sections parallel to each other and from a common starting line facilitated such visual verification, but was not considered a necessary condition.

The introduction of measuring with feet can be equated with the use of a unit of measurement. Measuring with different feet (teacher and student) means using different units of measurement. However, when it came to taking measurements, the unit as such was not a priority. More important here was the process of measuring: Keeping a straight line, accurately measuring units (here: moving foot by foot). The children were aware that by measuring the same distance with different feet (i.e., different units) they would get different results. However, they were not fully aware that in order to compare different sections, they had to use the same unit. Perhaps, at this stage of education, children are not yet fully ready for this type of reasoning. It is possible that if they had more experience with measuring length with different measuring instruments (i.e., using different units) they would understand the point of using the same measuring instrument.

For the children taking part in the study, the key word was "to measure". They understood taking measurements as an activity, a certain procedure, and not as providing a numerical value. The important parts concerned carrying out the measuring process, whether the right direction and straightness of the line were maintained, whether the measuring instrument (unit) was applied evenly. The numerical result being a result of this process was of secondary importance. This was especially evident when working on the second task. When assessing the fox and the bunny, the main consideration was whether the section was straight and connected the starting point with the end point. Hence, the arcs determined by the bunny were immediately treated as an erroneous way of measuring. On the other hand, the children agreed that the straight lines drawn by the fox were the correct way to measure the distance at which the balls were thrown, even though they ran to the individual balls from the fox and not from the players who threw them. The children were able to develop common, universal rules for taking measurements, and they were able to apply the adopted principles in their further work, as shown in the third step of the pilot study. The rules adopted by children for measuring and comparing lengths are in line with the generally accepted principles in this area. Similar results regarding the determination of measurement principles can be observed in the research of Tzekaki (2018).

The attitude of the children participating in the pilot studies and the results obtained are encouraging. The presented series of meetings with preschoolers and the set of tasks provided to them seems to be a good starting point for developing children's competences in the field of measure and taking measurements. Detailed research on a much broader research group can verify this hypothesis and help to develop a didactic proposal for kindergartens on how to develop competences in the scope of the concept of measure and taking measurements in children.

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Mirosława Sajka & Roman Rosiek
University of the National Education Commission

Chapter 2

WALKING UP THE STAIRS: AN EXCERPT FROM RESEARCH INVOLVING EYE-TRACKING ON UNDERSTANDING FUNCTION AS A TOOL FOR DESCRIBING MOVEMENT

Summary: The ability to interpret graphs is crucial not only in learning mathematics and science in general, but also in everyday life. This also concerns the ability to present, in such graphical form, activities that every student experiences in their everyday life, such as walking up the stairs. Our chapter shows an introductory analysis of the difficulties students experience with describing such an activity in graphical form. We discuss an excerpt from a broader study on understanding the function as a tool for describing movement which involved 64 secondary school students and was carried out in methodological triangulation using a questionnaire, eye-tracking technology, and an open-ended in-depth interview. The analysis of the results provides us with information on the students' interpretation of the graphs, showing that, despite understanding the graphs at the level of "reading the data" and "reading between the data", the majority of respondents did not correctly identify the graph of the movement. Different interpretations were revealed, and most of the wrong answers have their root in the "picture" misconception, consisting of identifying the graph of the movement with the trajectory of the movement or the design of the stairs. A lack of covariational thinking in this context was evidenced in some students. In the light of the research results, the task proved to be difficult (success rate below 0.5) despite the different methodological approaches used, while the respondents considered it rather easy in their self-assessment. Both this fact as well as the oculographic data recorded during the eye-tracking part of the study show a high inference of the intuitive System 1 (Kahneman, 2011) in the solution of this task. However, for 20% of the respondents, the use of another research method was sufficient to improve their answers, because it led them to self-reflect on the task, which triggered analytical thinking and activated System 2.

Keywords: understanding function, covariational reasoning, functional thinking, function graphs, misconceptions, eye-tracking.

1. Introduction

The ability to interpret graphs is crucial in the process of learning mathematics, physics, and science in general. It is also an important part of everyday life for any adult, as it allows to analyse the content of graphs appearing in daily news or in popular science.

The ability to mathematise and describe phenomena in the real world around us using the language of mathematics, is crucial in any mathematics curriculum, and is at the heart of teaching physics.

We have chosen to analyse the movement of walking up the stairs and its description in the form of a graph as the topic for consideration in the presented chapter, as such movement is an elementary and common experience of every student.

2. Theoretical Background

2.1. Understanding the Notion of Function in the Covariational Aspect

The notion of function is one of the most fundamental concepts in mathematics and its teaching, and is used for modelling in mathematics and science, as well as everyday life.

The notion of function can be understood through several aspects. In this chapter, we accept four aspects of understanding the concept of function as described, among others, by Pittalis et al. (2020) and approved as a theoretical background for the European FunThink project, in which the authors of this work were involved. This project aims at enhancing functional thinking at different stages of mathematics education and implementing in five European countries (FunThink Team, 2021; Frey et al., 2022). The function aspects are as follows:

1. Function **as an input-output assignment** stresses the computational aspect of function, perceived as a request to do a calculation (e.g., Sfard, 1991);
2. Function **as a dynamic process of covariation** between the independent and the dependent variable which includes quantitative reasoning and multiplicative objects, coordination of changes in quantities or values, and ways in which an individual conceives the variation of quantities (e.g., Thompson & Carlson, 2017);
3. Function **as a correspondence relation** focuses on the particular relation between the independent and the dependent variable, including mapping (e.g. Skemp, 2012);
4. Function **as a mathematical object**, that can be examined, compared with, or connected to other mathematical objects (e.g., Sfard, 1991; Sajka, 2003; Lichti & Roth, 2019).

In this chapter, we focus mainly on the understanding of function as a dynamic process of covariation (2) as well as an object (4).

Carlson et al. (2002) identified five mental actions that students undertake when manipulating the magnitudes or numerical values of covarying quantities in the context of functions. Each level encompasses the mental actions and behaviours associated with all the preceding levels.

Thompson and Carlson (2017) revised the framework proposed by Carlson et al. (2002) by providing an in-depth description of the students' progression in learning, incorporating aspects of the covariational reasoning developmental models of Saldanha and Thompson (1994) and Castillo-Garsow et al. (2013).

The major levels of covariational reasoning are distinguished and described by Thompson and Carlson:

> Smooth continuous covariation: The person envisions increases or decreases (hereafter, changes) in the value of one quantity or variable (hereafter, variable) as happening simultaneously with the changes in another variable's value, and the person envisions both variables varying smoothly and continuously.
>
> Chunky continuous covariation: The person envisions changes in one variable's value as happening simultaneously with the changes in another variable's value, and they envision both variables varying with chunky continuous variation.
>
> Coordination of values: The person coordinates the values of one variable (x) with the values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
>
> Gross coordination of values: The person forms a gross image of the values of quantities varying together, e.g., "This quantity increases while that quantity decreases". The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in the two quantities' values.
>
> Precoordination of values: The person envisions the variation of two variables' values, but asynchronously – one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.
>
> No coordination: The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values (Thompson & Carlson, 2017, Table 13.4, p. 441).

2.2. Understanding Function Graphs

A crucial aspect of developing the understanding of the notion of function is understanding function representations and providing students with diverse representations (Sierpinska, 1992). Using representations and mastering representational change by students is also a complex research problem in itself (e.g., Even, 1998; Ronda, 2015). In this chapter we focus on understanding the graphs of functions.

2.2.1. Graph of a Function

Semadeni (2002) stresses the epistemological difficulties concerning the concepts "ordered pair" and "function" and its consequences. In order to understand the graph of a function, an understanding of many other concepts is needed. It requires an understanding of the coordinate system, i.e. also the number axis, and therefore an intuitive understanding of a function being a bijection. It also requires understanding of the coordinates of a point, and therefore the notion of an ordered pair or a two-term sequence, i.e., also a function. This results in a looping of the concepts (Semadeni, 2002).

Moreover, the term 'graph of a function' in school can be understood in two ways: *as a set of points of the form (x, f(x))* or as *a drawing*, a sketch representing a function, of varying accuracy (e.g., Turnau, 1990).

A graph of a function is one of these representations of functions which favour a rather structural understanding (Sfard, 1991), because it allows to grasp a function in one glance.

What is more, a graph is also an example of a *multiplied object* – as describing "two measurements at the same time" (Thompson & Carlson, 2017) and as a *multiplied object* it allows for the reconstruction of co-variation – can be unpacked when it is needed.

2.2.2. Graph Misconceptions

An important branch of research in mathematics and physics education is the analysis of misconceptions, also related to the understanding of the concept of a function and the interpretation of representations. Our chapter specifically addresses misconceptions related to understanding and interpreting graphs. Leinhardt et al. (1990) made an insightful study of misconceptions, also in the context of interpreting graphs, which is still relevant today. Among other things, they wrote:

> Misconceptions are features of a student's knowledge about a specific piece of mathematics knowledge that may or may not have been instructed. A misconception may develop as a result of overgeneralizing an essentially correct conception, or may be due to interference from everyday knowledge. To qualify, a misconception must have a reasonably well-formulated system of ideas, not simply a justification for an error. So although misconception does not need to be an entire theory, it should be repeatable and/or explicit rather than random and tacit. Some misconceptions can be traced logically to intuitions. For example, students' tendency to interpret graphs iconically may be related to their intuitions regarding picture reading (Leinhardt et al., 1990, p. 5).

Leinhardt et al. (1990) distinguished misconceptions in the following eight categories: *What is and is not a function* (including ideas about what graphs of functions should look like); *Correspondence; Linearity; Continuous versus discrete graphs; Representations of func-*

tions; Relative reading and interpretation; Concept of variable; Notation. The *relative reading and interpretation* misconception included three other types of misconceptions: 1. Interval/point confusion, 2. Slope/height confusion, and 3. Iconic representation. In the following decades, a number of studies have found that the misconception diagnosed by Leinhardt et al. (1990) of a tendency to interpret graphs iconically, as a picture of a trajectory of movement, is common among students. We refer to it as the "picture" misconception later in this chapter.

2.2.3. Levels of Graph Comprehension

Two main categories, *interpretation* and *construction,* can be distinguished in the context of general actions which can be undertaken by learners while working with graphs. Leinhardt et al. (1990) stress that in most cases, construction requires interpretation of specific values, a part, or the whole representation, as well as of the properties of the concept it is representing.

Several authors considered the kinds of questions that graphs can be used to answer (e.g., Bertin, 1983; Carswell, 1992; Curcio, 1987; McKnight, 1990; Wainer, 1992). Friel et al. (2001) have characterised these approaches and consolidated them into three levels of graph comprehension:

> (...) an elementary level focused on extracting data from a graph (i.e., locating, translating); an intermediate level characterized by interpolating and finding relationships in the data shown on a graph (i.e., integrating, interpreting), and an advanced level that requires extrapolating from the data and analyzing the relationships implicit in a graph (i.e. generating, predicting). At the third level, questions provoke students' understanding of the deep structure of the data presented (Friel et al., 2001, p. 130).

We use Curcio's (1987) terminology when referring to these three levels, that is, *read the data*, *read between the data*, and *read beyond the data*, described below.

1. **Read the data:** Students answer questions about specific data represented in a graph. Reading them requires students to understand what a graph is, what the axes represent, where the data are located and what they represent.
2. **Read between the data:** Students are able to identify and explain relationships in the data presented in the graph. This requires students to understand not only the basic structure of the graph (described in 1), but also to understand the differences in the values or types of data presented.
3. **Read beyond the data:** Students make predictions based on the data and relationships presented in the graph. In order to do this, students not only need to understand the structure of the graph and the relationships in it, but also the context in which the data are presented. In other words, students need to be able to use the information presented in the graph to answer questions beyond the data in the graph.

2.3. Dual-Process Theory in Cognitive Psychology and Mathematics Education

When analysing the results of the research, it was also important to refer to general psychological thought processes, as they were relevant to the purposes of this research. In our case, these were the models presented by Kahneman (2011) as well as Fishbein (1987) and Vinner (1997).

Kahneman (2011) presented a model of human cognition, known also as the Dual-Process Theory, based on two modes or 'systems' of thinking.

System 1 "operates automatically and quickly, with little or no effort and no sense of voluntary control". Kahneman (2011) provides a list of examples of the automatic activities that are attributed to System 1, among which are, for instance, the answer to 2 + 2 = ?, detecting that one object is more distant than another, understanding simple sentences, or locating the source of a sudden sound.

> Several of the mental actions in the list are completely involuntary. You cannot refrain from understanding simple sentences in your own language or from orienting to a loud unexpected sound, nor can you prevent yourself from knowing that 2 + 2 = 4 or from thinking of Paris when the capital of France is mentioned. Other activities, such as chewing, are susceptible to voluntary control but normally run on automatic pilot (Kahneman, 2011, p. 23).

System 2 conversely "allocates attention to the effortful mental activities that demand it, including complex computations. The operations of System 2 are often associated with the subjective experience of agency, choice, and concentration" (Kahneman, 2011, p. 22). Kahneman provides examples such as: monitoring the appropriateness of one's behaviour in a social situation, counting the occurrences of the letter "a" in a page of text, telling someone your phone number, parking in a narrow space (for most people, except garage attendants), comparing two washing machines for overall value, filling out a tax form, checking the validity of a complex logical argument.

Leron and Hazzan (2006) stress that these two modes operate in different ways, are activated by different parts of the brain, and have different evolutionary origins, because System 2 is evolutionarily more recent than System 1, and it reflects cultural evolution.

There are many researchers who explore concepts developed in cognitive psychology and interrelations between them in the field of mathematical education. Leron and Hazzan (2006), for example, described the Dual-Process Theory in the context of the field of mathematics education and made a comparative summary on intuition vs. analytical thinking in mathematics education, based, among others, on the works of Fischbein (1987) and Vinner (1997). Their analysis can be summarised in Table 1.

Mathematics education literature shows that the definition and function of intuition are similar to those of System 1 (Table 1). Both are characterised by immediacy, high accessibility, automaticity and effortlessness, and both are considered to be mostly useful and reliable under normal everyday conditions but are prone to errors under more complex and abstract conditions (Leron & Hazzan, 2006).

> *Intuitive knowledge is immediate knowledge; that is, a form of cognition which seems to present itself to a person as being self-evident.* [. . .] In all these instances, one deals with apparently immediate forms of cognition (Fischbein, 1987, p. 6; italics in the original).

Table 1. A comparison of terminology between mathematics education and dual-process theory (interpretation based on Leron & Hazzan, 2006)

<table>
<tr><td rowspan="3">Dual-Process Theory perspective</td><td colspan="3">Cognition</td><td></td></tr>
<tr><td rowspan="2">SYSTEM 1</td><td colspan="2">SYSTEM 2</td><td></td></tr>
<tr><td>Rule-governed, serial thinking</td><td>Self-monitoring</td><td></td></tr>
<tr><td rowspan="2">Mathematics education perspective</td><td>Intuition</td><td>Analytical thinking</td><td>Self-monitoring</td><td>Beliefs, resource management, etc.</td></tr>
<tr><td colspan="2">Cognition</td><td colspan="2">Metacognition</td></tr>
</table>

Moreover, it is worth mentioning that when relying on intuitive knowledge in the context of mathematics education, the student does not know how to explain his/her reasoning, does not know where the conclusions he/she draws come from, such knowledge may also consist in using mere, not always well formed, *concept images* (Vinner, 1983).

Comparing System 2 and a part of metacognition, Leron and Hazzan (2006) also find similarities in the self-monitoring function of System 2 and the same part of metacognition, but other parts differ in the two theoretical approaches.

Finally, the term “cognition” is used in a partially similar way. However, from a mathematics education perspective, self-monitoring belongs to metacognition, not to cognition, as the Dual-Process Theory states. For example, Vinner (1997) seemed to reserve cognition for analytical thinking only:

> [...] much effort is devoted to find cognitive interpretations for many types of behavior for which, perhaps, a different type of interpretation is more suitable. Furthermore, much didactic effort is invested in ‘cognitive corrections’ where perhaps a different type of correction would be more effective. By saying this, I am not denying the importance of cognitive research. I am asserting, however, that not every event in a mathematics learning can be explained in cognitive terms, and that it is a fallacy to assume that the cognitive approach is adequate for almost every situation in mathematics learning (Vinner, 1997, pp. 97–98).

Vinner (1997) defined another term: *pseudo-analytic processes,* in which students superficially select elements in the problem and apply procedures used with typical questions due to their superficial similarity with previous problems. The pseudo-processes are "simpler, easier, and shorter than the true conceptual processes" (Vinner, 1997, p. 101), thus many students unconsciously apply them.

Leron & Hazzan (2009) stress the monitoring role of System 2 in relation to System 1. However, they also note that many of the misconceptions come from the combined failure of both S1 and S2. Therefore, they propose that the most important educational implication is the need to train people to be aware of the way S1 and S2 operate, and to include this awareness in their problem-solving toolbox.

> This suggestion has an interesting (almost paradoxical) recursive nature: It in effect implies that S2 should monitor not only the operation of S1 (its standard role), but the S1-S2 interaction as well; thus, S2 has to monitor its own functioning in monitoring S1.In a way, we might say that an operation of a "System 3" is needed here (to monitor S2),but in practice, this function is recursively assumed by S2 itself. (Stanovich (2008) attempted to formulate a tri-process theory). While monitoring and critiquing S1 is one of the reasons S2 has evolved in the first place, monitoring the S1-S2 interaction seems to be what Geary (2002) has called biologically secondary skills, one which will not normally develop without explicit instruction (Leron & Hazzan, 2009, p. 270).

In our study, we use psychological terminology (System 1 and System 2) because we are unable to resolve whether cognition (analytical thinking) or metacognition (self-monitoring) will be responsible for overcoming the intuition that activates System 1, as resulting from the chosen methodology of our research.

3. Methodology

3.1. Purpose of the Study

The aim of the study was to answer the question: *How do secondary school students, using a selected example in a real-life situation, analyse distance-time movement?*

3.2. Methodological Triangulation

In designing the research, methodological triangulation was applied. Thus, the research was designed to be composed of the following three parts:

1. The eye-tracking part of research (ET),
2. A written *Questionnaire* (Q) including, inter alia, a self-assessment of skills in mathematics and physics, a short written test, as well as a reflection on the solutions to the tasks from the first part, and a subjective assessment of their difficulty,
3. An individual, open, in-depth *Interview* (I).

3.2.1. Eye-Tracking

The methodology was intended to allow us to record and examine the way in which the content of the task was analysed, to distinguish the strategies and distractors that were taken into account when solving the task, to precisely record the duration of the analysis of each visual scene, mainly by analysing the scan-path records, i.e. the order and number of fixations on the areas of each AOI. By using this methodology, we seek answers to the following research questions:

- Do students only analyse mathematical models?
- Do students use graphics and drawing to illustrate the staircase?
- Can we distinguish the criterion used to provide an answer?

3.2.2. Written Questionnaire

The questionnaire was carried out in paper-pencil format and contained three sections, as described below.

a) Self-assessment of skills in mathematics and physics
The students completed a short self-assessment questionnaire of their mathematics and physics skills, along with a metric in which they provided their name, gender, and the type of the class they attended.

b) A short test
Then, they answered questions in the context of a task designed to test the students' first two levels of graph comprehension, i.e. their ability to "read the data" and "read between the data" in a graph of a function and interpret it in a situation involving drone flight, among other things. We were keen to ensure, before analysing the students' skills of research context, that none of the students had elementary difficulties in reading motion graphs, which is clearly a prerequisite for undertaking the mathematisation process.

c) Reflection on tasks

The purpose of the final section of the questionnaire was to reflect on each task from the eye-tracking section, provide answers for these tasks once again, and assess their difficulty. This section of the questionnaire also served as a scaffolding for the in-depth interview on each task.

3.2.3. Open In-Depth Interview

This methodology aimed to provide a more meaningful insight into the students' reasoning by asking students to describe their own reasoning, to describe the reasoning behind their answer, and to share the difficulties they experienced in analysing the content of the task and attempting to solve it.

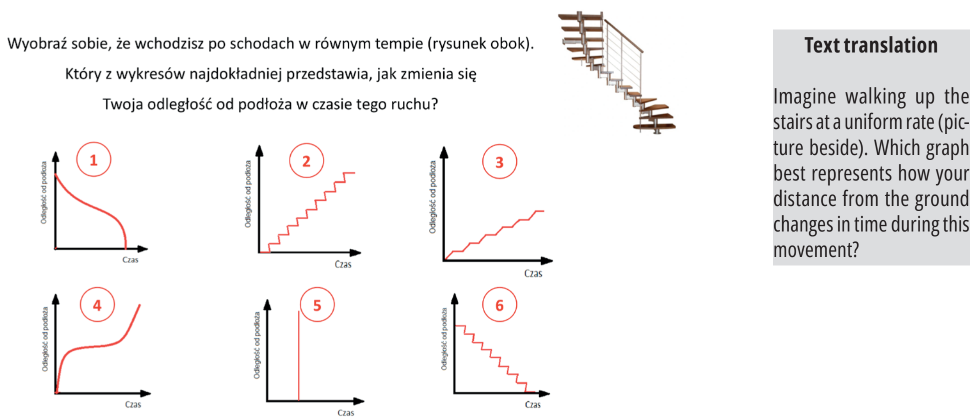
4. Research Tools

4.1. Eye-Tracking – Task Regarding Walking up the Stairs

The research concerns a verbally described real-life situation regarding movement modelled by means of a distance-time graph – which therefore directly concerns covariational reasoning, the second aspect of function understanding.

In the current chapter, we limit the analysis to one task, designed by authors, which is presented in Figure 1.

Figure 1. Original slide used for eye-tracking part: Walking up the stairs



Walking up the stairs is an activity that everyone experiences and observes in everyday life from an early age, so it is both easy to recall from one's own experience and from observing others. On the other hand, such a movement provides opportunities for multifaceted interpretation on a macro and micro scale as well as for considering functional relationships in terms of covariational reasoning. This is why we have chosen such a context for the task.

The students were asked to choose one of the six graphs presented that best describes the movement. In the proposed graphs, the vertical axis is described as distance from the ground and the horizontal axis is time. Figure 1 shows the slide as used in the research.

Due to the eye-tracking methodology, the task was formulated so that its solution did not require any calculations or drawings and was designed as a multiple-choice task. Students provided their best answer orally – they could, e.g., state 'number two' or 'I don't know – next'.

In the task, an illustration of a staircase with a complex spiral shape was intentionally included alongside the description of the task in order to extend the possibilities for interpreting the graph, to diagnose the amount of people (and their approach) who analysed the illustration of the staircase and whether this had a significant influence on the choice of answer. In addition, such a staircase addressed the possible misconception of equating the distance-time graph with the trajectory of the movement. Additionally, concerning the spiral shape, it is harder to abstract horizontal movement, which has no effect on height.

In the formulation of the task, six graphs are provided as distractors. We will refer to them as G1–G6. Among them, four purposely refer to the shape of the staircase: G2 and G6 (the shape of the successive steps of the staircase), and G1 and G4 (the spiral design of the staircase from the illustration). In addition, graph G5 illustrates the variation of height while ignoring the variation of time. Graphs G5, G2, and G6 purposely depict non-functional relationships, and in graphs G2 and G6, the sections that are not horizontal have been sloped so that it is impossible to use them in the context of a height-time graph (going back in time).

The axes of the coordinate system intentionally have no units, so there are two possibilities for the correct answer depending on whether we are analysing movement on a micro scale (G4 is then possible – as climbing one step of stairs, standing on it, then climbing another) and on a macro scale, analysing a larger number of stairs (G3). The proposed graphs do not exhaust all possible interpretations. Due to the eye-tracking methodology, it was necessary to provide several distractors to choose from and provide elimination options. A graph in the form of a fragment of an increasing linear function, which could be the correct answer under appropriate assumptions, was intentionally not included. Such an option would certainly have been the most attractive distractor due to the curriculum's emphasis on this type of function in mathematics and the association with uniform motion in physics. However, we wanted the students to think about this functional relationship by analysing it, and not to give an answer solely on the basis of school experience, quick associations, and being reminded of the shape of a graph they had seen during their lessons.

4.2. Written Questionnaire

4.2.1. Self-Assessment of Skills in Mathematics and Physics

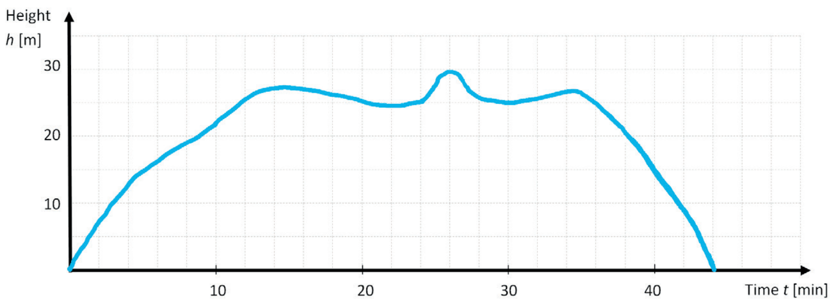
The questionnaire asked: *How would you rate, on a scale of 1 to 10, your level of knowledge in mathematics? How would you rate, on a scale of 1 to 10, your level of knowledge in physics?* The students were asked to mark their answers on a scale of 1-10, where 1 meant 'I have great difficulty in learning mathematics / physics', and 10 meant 'I am definitely doing well with it'.

4.2.2. Short Written Test

There were two additional tasks in the questionnaire. In this chapter, we present only one of them with its first two sub-tasks, designed to test the ability to analyse graphs:

The graph shows the change in the height of a drone during its flight. Answer the following questions.

Figure 2. Task about a drone inspired by Janowicz and Wesołowski (2019, p. 173)



a) How long did the flight last?
b) What was the maximum height reached by the drone?

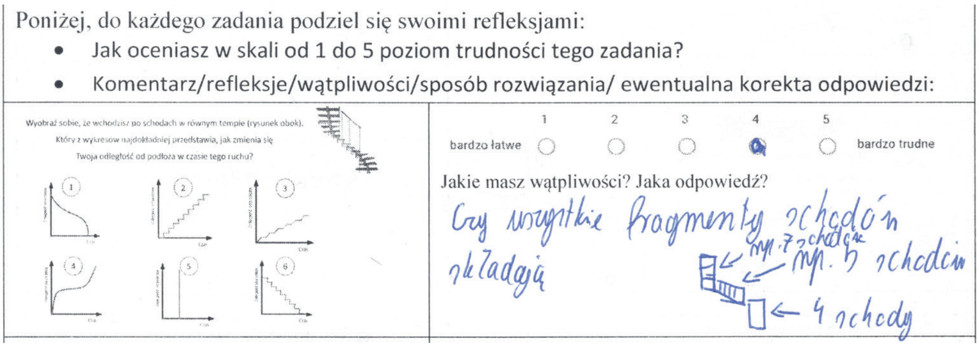
The task had been used in previous research. Both questions relate to levels of graph comprehension (Curcio, 1987). The question formulated in sub-item (a) diagnoses level 1, the ability to "read the data", as students are expected to read the specific drone flight time data presented in the graph and therefore understand what a graph is, what the axes represent, where the data are located, and what they represent. In contrast, the question formulated in sub-question (b) diagnoses level 2, "reading between data", as students identify and explain relationships in the data presented in the graph, in this case to find the maximum value in the given data for the height to which the drone ascended, which requires students to understand the differences in the values.

4.2.3. Reflection on Tasks

As the research was composed of more tasks, the students were given a worksheet to fill in during the interview, which at the same time provided a set of open-ended questions for the interview (Figure 3). The instructions were as follows:

Below, for each task, share your thoughts:
- On a scale of 1 to 5, how would you rate the difficulty level of this task?
- Comments/reflections/doubts/solution method/corrections to your answer:

Figure 3. Worksheet fragment of the first participant [P01]



Each task was then restated in the form of a printout of the slide (Figure 3, down left column), and space was provided next to it to mark, on a scale of 1-5, the difficulty of the task, where 1 meant very easy and 5 meant very difficult. The following questions were presented below: *What doubts do you have? What is your answer?* Figure 3 provides the response to the questionnaire for this task as given by the first participant in the study (we coded participants sequentially in P01 format), who marked the ease of the task as 4 and shared his/her doubts about whether all parts of the staircase in each part of the curve consist of the same number of steps – 'e.g., with 7 steps of stairs, e.g., 5 steps of stairs, 4 steps of stairs'.

4.3. Interview

The interview was conducted in the form of the students' free responses to these inquiries: *Comment on the solution to the task; Do you have any doubts?; Do you want to share a reflection?; Why did you choose that answer?; Justify.* The researcher often asked 'Why?' and did not provide any feedback to the participants. The respondents signed a confidentiality clause that they would not discuss these tasks with other students.

To stress the students with the prospect of being recorded, an indirect form was adopted – the use of a speech-to-text transcription application (Google Docs Voice Typing). The student was able to correct and authorise the recorded speech and was asked to speak loudly and slowly, pausing between sentences. Sometimes a participant was asked to wait a while so that necessary corrections could be made to the transcribed text.

5. Participants and Research Procedure

The research was carried out in one of Kraków's secondary schools with a good reputation and a high level of teaching, as measured by the results of the *matura* exit exam. A total of 64 students took part in the study. These were students from classes of different profiles and from different grades: 9, 10, 11. Participation in the study was voluntary, but it was necessary to provide written consent for participation signed by parents or legal guardians of the student, or personally by adult students (over 18 years of age).

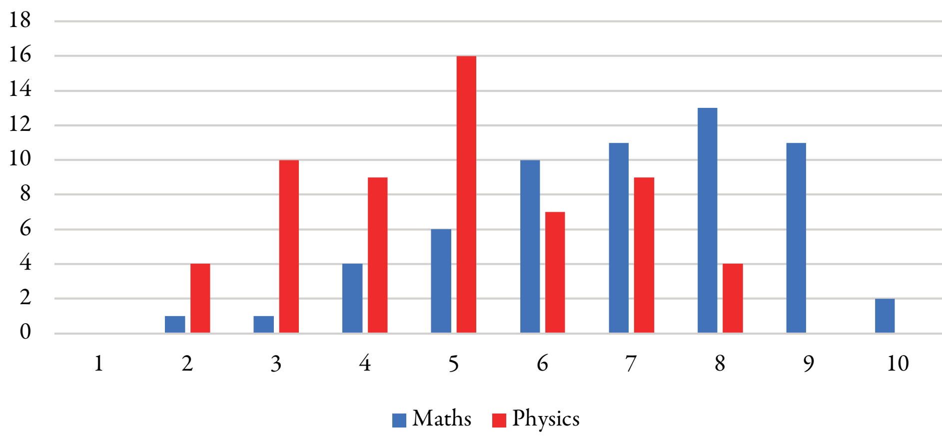
The study's stages were carried out chronologically according to the research methodology, under laboratory conditions, identical for all study participants. Thus, the eye-tracking study began in a separate room. Eye movements were recorded using the Tobii Pro X3-120 eye-tracking system with a sampling rate of 120 Hz. Prior to data collection, the system was calibrated individually for each participant using a 9-point calibration algorithm. The questions were displayed on a 24" monitor positioned 70 cm from the participant's eyes. Participants proceeded to the next question by selecting an answer. After the eye-tracking test, the participant moved to the next room, where he/she first completed the questionnaire independently for parts a) and b). Then, after completing part b), an open-ended interview began, in which the participant reflected on the tasks in the eye-tracking part of the questionnaire and once again noted down his or her response to the task, assessed its ease, and commented on the task.

6. Presentation of General Results

6.1. Self-Assessment of Skills in Mathematics and Physics

On average, participants in the research rated their skills in mathematics two marks higher (mean 6.97) than in physics (mean 4.93), while the modal value equalled 5 for physics marks and 8 for mathematics, 3 marks higher. The distribution of self-assessments is shown in Figure 4.

Figure 4. Participants' self-assessment of skills in mathematics and physics



6.2. Introductory Task Regarding Drone

For question (a), all but one person correctly answered that the flight lasted 44 min, and only one person gave a duration of 42 min, which was due to a mistake in reading the unit on the correct axis – this person assumed that one grid corresponded to 1 minute. It can therefore be concluded that the respondents have reached level one of graph comprehension: 'reading the data' (Curcio, 1987) – they can directly read data from a graph, i.e. they understand what a graph is, the convention of reading data from a graph in a coordinate system, the role of the ordinate and abscissa axes, where the data are located, and what they represent.

In turn, for question (b), 100% of survey participants gave the correct answer: 30 m.

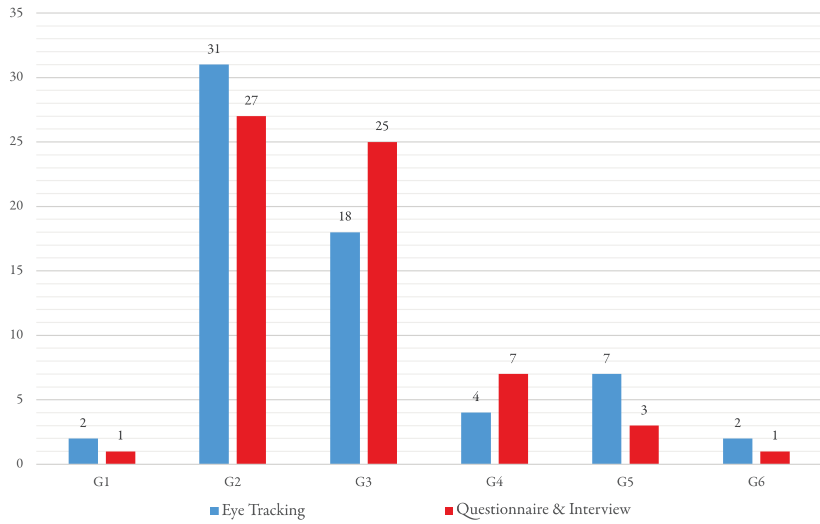
This part of the survey was designed to ensure that these basic levels of graph comprehension were achieved by students.

6.3. Overall Results – Walking up the Stairs

6.3.1. Results in ET and Q&I Parts of the Research

The overall results for responses to the task about walking up the stairs in two parts of the research, i.e., Eye-Tracking (ET) and Questionnaire & Interview (Q&I), are summarised in Figure 5.

Figure 5. Walking up the stairs: answers from Eye-Tracking and Questionnaire & Interview parts

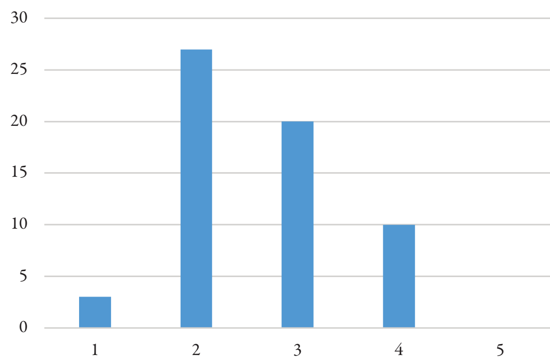


The responses that were considered correct, because they represented a non-decreasing function, were shown in graphs G3 and G4. However, no one in the interview argued the choice of answer G4 in the context of the micro-scale (climbing one step of the staircase), and the students' choice of G4 was mainly due to other factors such as misconceptions or other additional assumptions in interpreting the shape of the staircase and even randomness, as we discuss in the next paragraph. Assuming, therefore, that the fully correct answer is only G3 in our sample, the success rate of this task was 0.28 in the eye-tracking (ET) part and 0.39 in the questionnaire part combined with the interview (Q&I). In both approaches, therefore, the task can be considered **difficult**, as it falls with its success rate within the range of 0.20–0.49 (Niemierko, 1999). When taking into account those G4 answers which can be considered correct with an additional assumption (e.g. P58), and those where precise reasoning was not provided because the student did not want to or could not describe them, therefore eliminating from the G4 answers only those which were selected on the basis of faulty reasoning – the task can still be considered difficult.

6.3.2. The Participants' Assessment of the Ease of the Task

Despite the above, the respondents considered this task as **rather easy** in their self-assessment (Figure 6).

Figure 6. Ease of the task on walking up the stairs according to the participants



The average ease score was 2.63, and answer 2 was the most frequent, indicated by 27 survey participants. In addition, no research participant found this task very difficult, and three people found this task very easy.

6.3.3. Changes in Answers

The changes in responses between the eye-tracking part of the research (ET) and the questionnaire and interview reflection part (Q&I) are interesting. Table 2 shows the numbers of individual responses in both parts of the research.

The correct answer G3 was given by 18 people in the ET part (Table 2, row ET G3), while the number of correct answers increased from 18 to 25 in the Q&I part (Table 2, column Q&I G3). The number of incorrect answers G5 decreased from 7 in the ET part to 3 in the Q&I part.

The expected changes from a G2 answer in ET to a G3 answer in Q&I occurred in 6 people, and an additional 2 people changed their answer from G2 to a functional answer G4. Moreover, 2 people changed their G5 answer to G3, and one from G5 to G4, one from G6 to G3 and one from G2 to G1 (and therefore to a functional answer). Thus, almost 20% of the survey participants changed their non-functional answer to an answer representing a function graph.

Table 2. Numbers of individual responses G1-G6 in each part of the research

	Q&I G1	Q&I G2	Q&I G3	Q&I G4	Q&I G5	Q&I G6
ET G1			1			1
ET G2	1	22	6	2		
ET G3		3	15			
ET G4				4		
ET G5		1	2	1	3	
ET G6		1	1			

ET – Eye-Tracking part; Q&I – Questionnaire & Interview

7. Chosen Oculometric Results and Analysis

7.1. Methodology for the Analysis of Oculographic Data

Scan-path records were chosen as the primary qualitative criterion for analysis, as they directly reflect the order and manner in which the individual respondents analysed the task content.

The area of the task slide was then subdivided into Areas of Interest (AOIs), allowing for analysis in terms of gaze-dwelling on particular AOIs. AOIs were defined in the wording of the task, in the individual distractors to be selected (graphs G1–G6), and in the area of the staircase illustration. The area and shape of every AOI containing graphs were the same. The oculographic data on the AOIs were then statistically analysed, with the number of recorded fixations by individual subjects on the selected areas of the AOI task chosen as the relevant criterion for eye-tracking analysis.

In order to visualise the distribution of fixations obtained in the sample group, the number of fixations, from smallest to largest, was divided into 10 intervals of equal length, for which a histogram was made, expressing the number of subjects for whom the number of fixations falling within the interval was recorded.

The mean value of the number of fixations for all test subjects for this task was calculated, resulting in 194 fixations.

We further identified groups of participants for whom the number of fixations during the entire task was less than 75% of the group mean value, which in the case of our data is also equivalent to 25% of all participants – those for whom the lowest number of fixations was recorded during the solution of the entire task. We refer to this group as 'fast' participants because, according to Kehneman's (2011) theory, we can assume that they activated System 1 while analysing the visual scene of the task by analysing it briefly and making a quick decision. It is worth noting that, according to Kahneman (2011), if the fast system is not activated, System 2 (called slow) is assumed to be activated. We refer to all subjects outside of the distinguished 'fast' group as 'non-fast'. In our analysis, we further distinguish another group, which we refer to as 'very slow', for whom more than 125% of the average number of fixations in the group of all participants was registered during their work on the task.

Another methodological approach was to analyse the AOI with a staircase illustration. In this context, three groups were also distinguished: the first one being the 'no illustration' group of subjects, whose number of fixations on this AOI was less than or equal to 5, the second was the 'with illustration' group, where the subjects showed more than 5 fixations, and the third was the 'prolonged illustration' group of subjects, characterised by intensive support of reasoning through visual analysis of the illustration, with at least 30 fixations on this AOI.

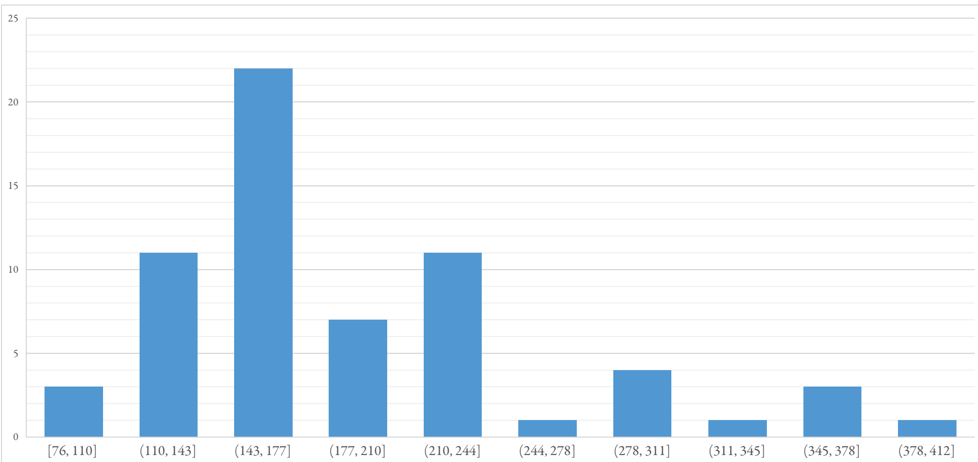
7.2. Correctness of Answers in Relation to the Number of Fixations

7.2.1. Number of Fixations in the Study Sample

The recorded fixations in the entire task for all subjects ranging from 76 to 412 are shown in the histogram in Figure 7.

This distribution shows, among other things, that only in 10 subjects out of 64, fixation numbers greater than the median were registered (245-412). There are 54 subjects in the first five ranges (fixation number from 76-244). Thus, the visual scene analysis performed by the 10 individuals with the highest number of fixations significantly deviates from the rest of the subjects due to their higher fixation numbers.

Figure 7. Histogram of the distribution of the number of individuals whose number of fixations falls within each interval of length 34



7.2.2. Results Obtained in the 'Fast' Group

Table 3 shows the distribution of responses in the 'fast' group, which consists of 25% of all respondents – equivalent to the group that had less than 75% of the average number of fixations.

Table 3. Distribution of answers given in the 'fast' group (n=16)

	G1	G2	G3	G4	G5	G6	total
'fast'	0	8	2	1	4	1	16

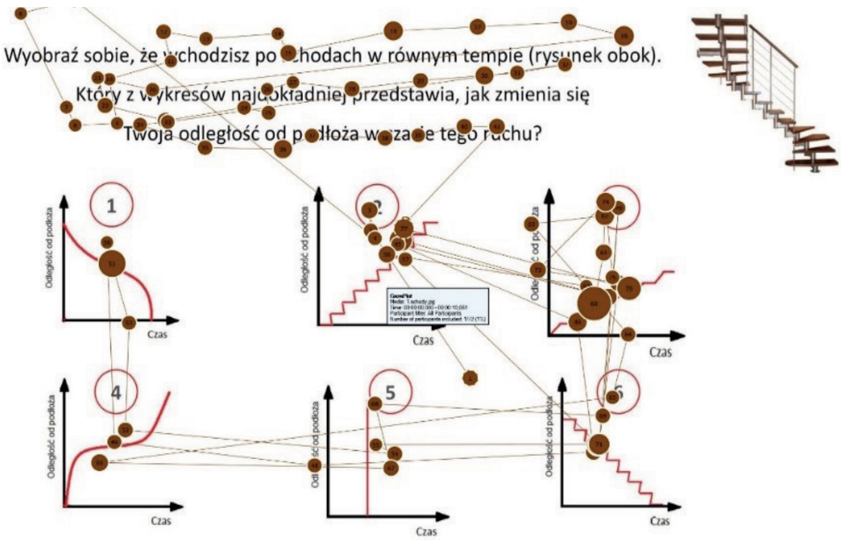
The ratio of correct G3 answers to incorrect G2 answers is 1:4, with a similar ratio of selected functional to non-functional relations of 3:13 (0.23) in this group.

This group is likely to be predominantly made up of those who attempted to provide answers using System 1 or those who guessed the answer.

However, eye-tracking technology makes it possible to identify the participants in this group who were guided by their expertise, completed the tasks very quickly, and indicated the correct answer.

Below is an example of a scan path for the participant who achieved the lowest number of fixations in the entire sample group (76) and who, after analysing the task text, without analysing the staircase illustration, indicated the correct answer (Figure 8).

Figure 8. Fastest participant P51 (least number of fixations, expert knowledge)



It shows that the respondent analysed the visual scene of the task in a very effective way, did not make use of the illustration of the stairs at all, read the content of the task once, went back to the relevant phrases from the content of the task: 'which of the graphs' and 'your distance', and looked at each of the answers. The respondent immediately paid particular attention to graphs G2 and G3, the saccades between these graphs are evidence of comparative analysis – the respondent chose one of them as correct. Graph G3 was analysed by the participant the most – the description of the ordinate axis was verified and found to be correct. The low number of fixations (3-5) on graphs G1, G2, G3, and G4 indicates that it was eliminated as an answer, and the arrangement of the saccades indicates that they were analysed in one group, after the correct answer had already been selected, in order to ascertain and eliminate them.

7.2.3. Results Obtained in the 'Non-Fast' Group

If we were to remove the 'fast' group (i.e. 16 people) from the pool of all participants, the remaining students can then be tentatively identified with the group that could hypothetically trigger analytical thinking, or System 2 according to Kahneman (2011). It is clear from Table 4 that there is a greater number of correct G3 responses in this group relative to the number of incorrect G2 responses, and the ratio is approximately 0.7, the highest among the distinguished groups by this total fixation count criterion (Table 4). The ratio of functional to non-functional responses is even higher, at approximately 0.78.

Table 4. Distribution of answers given in the 'non-fast' group (n=48)

	G1	G2	G3	G4	G5	G6	total
'non-fast'	2	23	16	3	3	1	48

7.2.4. Results Obtained in the 'Very Slow' Group

An additional criterion identified another group of 'very slow' participants, i.e. participants whose number of fixations during the whole task equalled a value greater than 125% of the average number of fixations. This group is included in the previous one, eleven people were thus singled out. Table 5 shows the distribution of their answers.

Table 5. Distribution of answers given in the 'very slow' group (n=11)

	G1	G2	G3	G4	G5	G6	total
'very slow'	0	5	3	2	1	0	11

The ratio of correct G3 responses to incorrect G2 responses is 0.6 in this group, and the ratio of functional to non-functional relationships is higher than in previous groups, at around 0.83.

7.2.5. Results Obtained in the 'No Illustration' Group

Additional analysis was performed on the participants who did not analyse the staircase illustration. To isolate such participants, we assumed that these participants recorded, at most, 5 fixations on the AOI with the staircase illustration. This gives 17 participants.

Table 6 illustrates the distribution of responses given by these participants.

Table 6. Distribution of answers given in the group 'no illustration' (n=17)

	G1	G2	G3	G4	G5	G6	total
'no illustration'	0	10	5	1	1	0	17

In this group, the ratio of correct G3 answers to incorrect G2 answers is 0.5. The ratio of functional to non-functional relationships is approximately 0.83.

7.2.6. Results Obtained in the 'With Illustration' Group

Participants who had more than 5 fixations on the AOI with the staircase illustration are included in the 'with illustration' group. Table 7 shows the distribution of responses in this group, which shows that the ratio of correct G3 responses to incorrect G2 responses is approximately 0.62.

Table 7. Distribution of answers given in the group 'with illustration' (n=47)

	G1	G2	G3	G4	G5	G6	total
'with illustration'	2	21	13	3	6	2	47

7.2.7. Results Obtained in the 'Prolonged Illustration' Group

There were 16 subjects who analysed the staircase illustration for the longest time, achieving at least 30 fixations on the AOI containing the staircase illustration. The record number of fixations on the AOI of the image was as high as 197.

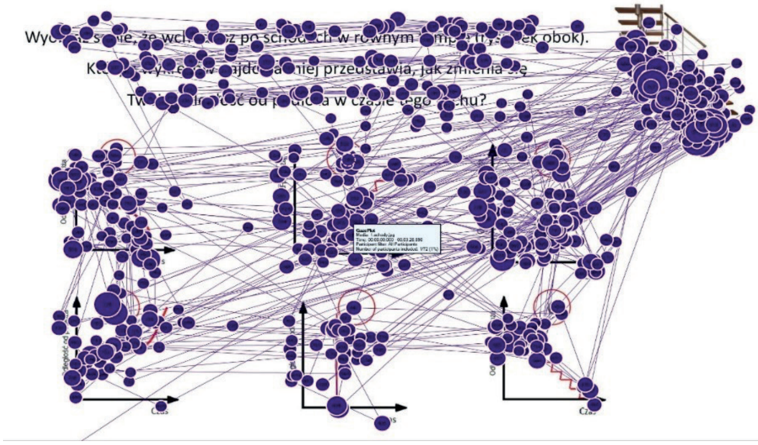
Table 8. Distribution of answers given in the 'prolonged illustration' group (n=16)

	G1	G2	G3	G4	G5	G6	total
'prolonged illustration'	0	6	5	1	3	1	16

In this group, the ratio of correct G3 responses to incorrect G2 responses is 0.83 – interestingly, the same as in the 'no illustration' group – and the ratio of functional to non-functional relations is about 0.6.

Below, we provide an example of a scan path of a P13 participant with a very high fixation count (total fixation count of 165), an in-depth analysis of the wording of the task and particular distractors and, above all, of the staircase illustration (56 fixations on the illustration alone), followed by an identification of the wrong answer G6.

Figure 9. Scan path P13, high number of fixations in illustration area (56), wrong answer G6



8. Chosen Qualitative Analysis

Due to the vastness of the different ideas observed in this study and the amount of possible approaches to analysing and interpreting the results, in this paragraph we focus only on the work of those who chose answer G5 in the eye-tracking part, as we consider this wrong answer as the most contradictory in terms of understanding the concept of function, revealing the lowest level: no coordination in the context of covariational reasoning (Thompson & Carlson, 2017).

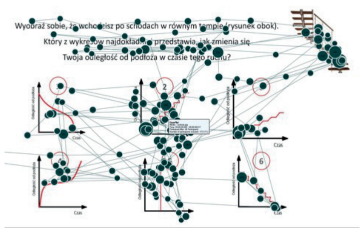
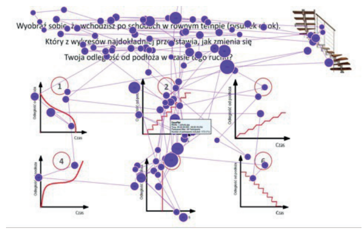
We present information about these participants sequentially: participant number and group they belong to, scan path while working on this task, statement during the interview about the task, their self-assessment in maths and physics, and self-assessment of the difficulty of the task.

8.1. Changing the Answer from G5 to G3 (n=2)

Two respondents changed their answer from G5 to the correct answer. An analysis of their scan path allows us to conclude that both subjects assessed the entire visual scene associated with the task (see scan paths, Table 9). It is interesting to note the differences in speed between the subjects in the context of eye-tracking when working on the task. One of them

(P43) was classified as a 'fast' response (total fixation count 115), the other belonged to those with a moderate total fixation count (184). Both used drawing analysis, while P43 analysed the drawing briefly (fixation count: 7), and P41 analysed it overlong (fixation count: 21).

Table 9. Results of those who changed their answer from G5 to G3 during the interview (n=2)

No	P41	P43
Scan path		
Interview	*The first time I chose answer five, but now I think it was the wrong answer. Perhaps this mistake was due to the fact that I was, in a sense, not yet fully warmed up and now I would like to think about this question again. This time I would choose answer three. I think this graph best illustrates what is being asked in this task.*	*I answered 5. I suggested that it was uniform. But I thought about the fact that after all, here's the time... and there's no time progression, so it would take one moment to move, yet there are no instruments. And it has to be a function, right? [researcher does not respond, student continues] ...It should be.... it assigns a value to a particular argument and this here argument was one, and there were multiple values on the straight line, this would mean that at one time I would be in all possible positions on the staircase i.e. something comparable to 'warp 10 – superluminal speed', multilocation [laughs].*
Self-assessment	in Maths: 8	in Maths: 6
	in Physics: 7	in Physics: 7
	Difficulty: 4 – difficult	Difficulty: 2 – easy

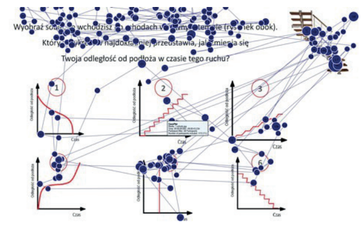
Person P41 was parsimonious in explaining his/her reasoning, he explained his/her change of mind was due to 'not being warming up' – while it cannot be excluded that the subsequent tasks in the whole series of the research, which also dealt with the analysis of movement and the interpretation of graphs in this context, may have influenced his/her choice of answer the second time around, the didactic role of the examples provided during the research cannot be excluded, which we consider a positive phenomenon. The student rated this task as difficult, which may also suggest that the actual process of changing his/her mind required effort and the student took into account his/her earlier wrong answer in the evaluation.

Participant P43 did not need the support of the illustration during the eye-tracking part and did not use it. On the other hand, in the interview with person P43, it can be seen how he/she, on his/her own, with an internal dialogue, makes reflections about the functionality of the graph. He analyses the fact that multiple values cannot be achieved in one moment of time. He compares this with metaphor and humour by referring to science-fiction works. Independently arrives at the correct answer. Assesses the task as easy, is convinced of the correctness of his/her new answer.

8.2. Changing the Answer from G5 to G4 (n=1)

Only one participant (P58) changed the answer from G5 to G4. At the same time, this person was placed in the 'prolonged illustration' analysis group (fixation count 30). When analysing the visual scene, his/her hesitation with answer G4 is visible.

Table 10. Results of the person who changed their answer from G5 to G4 during the interview (n=1)

No	P58
Scan path	
Interview	*The only graph that works for me in this task is graph four, because all of the graphs that have some sort of staircase look are not going to make sense to me, because at no point in the movement up these stairs are we at the same distance from the ground, we are just going up all the time. Diagram 4 makes the most sense. It also seems that halfway up those stairs they are probably a little less steep* [shows illustration] *so the distance might change less rapidly.*
Self-assessment	in Maths: 10
	in Physics: 5
	Difficulty: 3 – moderate

The influence of the illustration on his/her interpretation shines through in the student's statement, and the intensity of the analysis of the illustration is confirmed by the oculographic data. The participant chose answer 4 because of the shape of the staircase, assuming that the staircase is almost flat in the 'bend' part. It can be seen in his/her scan path that there are a lot of fixations on the 'bend' part of the staircase. This is an additional assumption added to the task, at which the G4 graph becomes possible and correct, however this shows the misconception of the picture. The student was swayed too strongly by the illustration and failed to consider that if a part of the bend was flat while being at the same distance from the ground as suggested by diagram G4, then that part of the bend would not be made up of stairs. Meanwhile, the illustration shows a regular staircase, just in perspective. Therefore, the student's answer was influenced by a misconception regarding the picture and an incomplete interpretation of the situation depicted in the drawing, requiring spatial imagination.

In addition, the student likely has a habit of climbing stairs quickly and smoothly, as he did not take into account the fact that when standing on a step the height does not change for a while, which he clearly emphasised with the words: 'at no point in the movement up these stairs are we at the same distance from the ground, we are just going up all the time'.

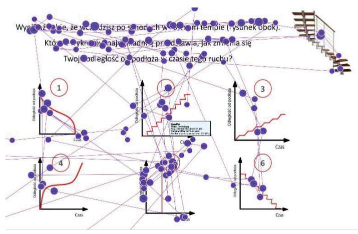
It is undoubtedly gratifying to see this reflection, followed by the student's selection of the G4 functional relation as the answer. Re-reflection on the task resulted in the activation of covariational reasoning.

This respondent is an example of someone who rightly rated his/her mathematics achievement at the top of his/her class. This participant is in a class with an extended level of mathematics (Grade 9), which shows that the task is challenging even in a group of such able students. He was able to overcome the fundamental difficulty in the task and rated the task's difficulty as moderate.

8.3. Changing the Answer from G5 to G2 (n=1)

Only one person (P70) changed his/her answer from G5 to G2. Table 11 shows his/her data.

Table 11. Results of the person who changed his/her answer from G5 to G2 during the interview (n=1)

No	P70 ('fast')
Scan path	
Interview	*Answer 2 because the graph of the relationship of the distance from the ground to time resembles the construction of a staircase.* *EXP: 'You chose 5 first, why?'* *Because this answer seemed the most logical to me.* *EXP: 'What made you change your mind?'* [longer reflection, smile, no answer].
Self-assessment	in Maths: 9
	in Physics: 4
	Difficulty: 2 – easy

Person P70 made a progression, as from a G5 response, indicating a lack of consideration of two changes varying simultaneously, he/she selected a G2 graph, which, although still non-functional and wrong, does roughly account for both changes – height and time.

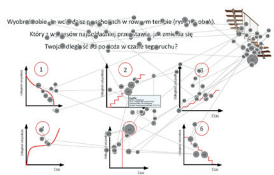
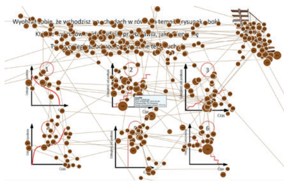
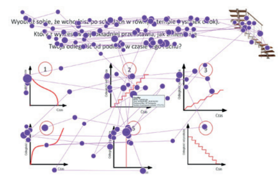
This individual went from *No coordination* to the penultimate *Chunky continuous covariation* level of covariational reasoning in the second approach (Thompson & Carlson, 2017).

8.4. Those Who Stayed with Their Choice G5 (n=3)

It is interesting to note their affiliation to the previously listed groups, as two of them are from the ‘fast’ group, and one is from the ‘non-fast’ group. In addition, two of them analysed the illustration for an ‘prolonged’ time, and one of them did it quickly.

Table 12 shows the work on the task of the three students who remained stable in their answers by choosing G5 in both parts of the task.

Table 12. Results of participants who stayed with their choice G5 (n=3)

No	P08 – ‘fast’ and ‘prolonged illustration’	P10 – ‘slow’ and ‘prolonged illustration’	P12 – ‘fast’ and ‘with illustration’
Scan path			
Interview	The participant in the interview did not want to comment on this task. He limited himself to a written statement in the questionnaire: *I had my doubts about whether it was not possible to move sideways.*	*The request was not very clear. I think it would have been better to add relevant details so that there was no doubt. Change the wording of the tasks to make them clearer, e.g. Stairs, was perhaps not quite clear – it is not clear where the stairs are.*	*I think the answer is number 5. I'm generally weak at maths, so I make broad guesses. I almost guessed in all of them. I don't know how to do this, I don't have any comments. I don't want to talk about it.*
Self-assessment	in Maths: 8	in Maths: 6	in Maths: 5
	in Physics: 8	in Physics: 5	in Physics: 5
	Difficulty: 2	Difficulty: 2	Difficulty: 2

Two individuals P08 and P12, however, are among the fastest responders (total fixation count: 134 and 143), while P10 is from the ‘very slow’ group (total fixation count: 366). The three individuals did not change their minds. However, it is difficult to assign P10 to a literally fast system; his/her way of looking at the slide shows a systematic and very detailed analysis of the visual scene. This respondent, however, lacked interpretive skills and covariational reasoning skills. In spite of this, he was confident in the correctness of his/her answer, as he rated the task as easy, which indicates that he based his/her answer on intuitive knowledge, i.e. the experience of walking up the stairs, which proved to be an insurmountable obstacle.

9. Discussion and Conclusions

9.1. Self-Assessment of Skills in Mathematics and Physics

It can be concluded that the self-assessment of knowledge in mathematics among the participants of the study was high, as on a scale of 1 to 10 the mean score was 6.97 and the modal score was 8; while in regard to physics it was average, as the mean score was 4.93 and the modal score was 5. These results show that the students do not underestimate their self-assessment in this area, therefore their attitude towards mathematics and physics tasks in general is not affected by anxiety or stress caused by the subject. Additionally, in the case of our respondents, this positive attitude towards mathematics and physics is due to the specific nature of the school, which is attended by ambitious students who maintain high grades in these subjects.

9.2. Graph Comprehension

In the written part of the research, all participants in the research achieved basic levels of graph comprehension (Friel et al., 2001). All can be considered to have achieved the first level – 'reading the data' (Curcio, 1987) – they can directly read data from a graph, i.e. they understand what a graph is, they understand the convention of reading data from a graph in a coordinate system, the role of the ordinate and abscissa axes, where the data are located, and what they represent. In addition, all students demonstrated the level of 'reading between data' (Curcio, 1987), as by correctly answering question (b) they demonstrated an understanding not only of the basic structure of a graph (explored in subsection (a)), but also an understanding of the differences in the values of the data represented by identifying and explaining the relationships in the data involving the selection of the largest value in a given set.

Thus, the students possessed the necessary knowledge and understood the tools they were meant to use when describing a given situation from everyday life in the form of a graph.

9.3. Overall Results versus Self-Assessment of Task Difficulty – System 1

The answers that were considered correct were G3 and G4, but no one in the interview argued the choice of answer G4 in the context of the micro-scale (climbing one step of the staircase), and the students' choice of G4 was mainly due to other factors, i.e. either misconceptions or other additional assumptions when interpreting of the shape of the staircase. Assuming that only G3 was the correct answer, in both methodological approaches the task **was difficult** for the test participants, as its success rate fell within the range: 0.20–0.49 (Niemierko, 1999), with a success rate of 0.28 in the eye-tracking part and 0.39 in the questionnaire part. In the case of assuming as a correct result those G4 answers which are possible with an additional assumption (e.g. P58, see description 7.3.2) and those for which the exact reasoning is unknown because the student did not want to or was unable to provide it, and therefore eliminating from the G4 answers only those which were selected on the basis of faulty reasoning – the task is still interpreted in both methodological approaches as difficult.

At the same time, the respondents found the task rather easy in their self-assessment – the average self-assessment on a scale of 1-5 was 2.63 with response G2 being the most frequent, 3 people found the task very easy, and no one found the task very difficult.

This discrepancy between the self-assessment and the test results indicates that System 1 was triggered in the majority of the respondents, resulting in a quick, intuitive, and, in most cases, incorrect answer. The respondents were not aware that they had solved the task incorrectly, they did not have significant doubts about the answer, and therefore considered the task easy. This belief was certainly also reinforced by the context of the task describing a basic activity from everyday life.

9.4. Change of Answer in the Second Type of Approach – System 2

The distribution of responses in both parts of the research was reported in detail in Table 2. The improvement in some of the results may have been due to various factors. First of all, the subjects approached the solution of the same task again under different conditions, so the task was already familiar to them and the thought process involved in solving it was likely to continue. In addition, the respondents felt less stressed than in the eye-tracking part due to knowing all the tasks this time as well as not being recorded. They were able to reflect on their own first and then recount their interpretations in an open-ended interview. Sometimes, by speaking their thoughts out loud, they noticed mistakes. There was no rush in this part of the research. This methodology promoted and indirectly triggered analytical thinking, System 2, as it required reflection. For those who corrected their answer, it was therefore sufficient to calmly reflect on the task again, triggering System 2, in order to give the correct answer. This indirectly indicates that the first attempt at solving the task lacked this reflection, i.e. it was based on System 1.

However, the vast majority of the respondents, 44 out of 64, did not change their mind in the Q&I part compared to the ET part (see the diagonal in Table 2). The choice of incorrect answer G2 in the ET part appeared to be more stable than the choice of the aforementioned incorrect answer G5, whose incorrectness was noticed by 4 people. In contrast, answer G2 remained unchanged by 22 people, despite their reflection on the task. This answer remained the most frequently chosen answer (by 27 people in total), gaining an additional 5 responses in the Q&I section from people who had chosen a different answer in the ET section. Unfortunately, some of these changes were in an undesirable direction, as 3 people changed their answer G3 in the ET part to G2 in the Q&I part. Undesirable changes are indicative of uncertainty and perhaps randomness in making the first choice, lack of knowledge regarding functional relationships, and, possibly, insufficient activation of System 2.

The selection of responses G2 and G5 indicates the activation of the 'picture' misconception, as G2 simply depicts a diagram of the staircase and G5 depicts the direction of upward movement when climbing the stairs. On several occasions, answer G4 was chosen by participants on the basis of the same misconception (e.g., P58, see 7.3.2), as revealed in the interviews. Answer G1 depicts a diagram of the shape of the staircase with a curve as in the picture, and G6 again depicts a diagram of the staircase, only leading 'down' (going from left to right). Thus, all answers except G3 may be indicative of a 'picture' misconception triggered while working on this task.

In addition, the choice of answers representing non-functional relationships (G2, G5, G6) demonstrates an inability to carry out covariational reasoning in the context of this task. It can be concluded that as many as 40 individuals selecting these answers, or 63% of the respondents in the ET part, and 36 individuals after reflection in the Q&I part, still more than half of the respondents (56%), did not reach the level of *smooth continuous covariation* (Thompson & Carlson, 2017). These individuals were focused on the shape of the trajectory (the 'picture' misconception). The drone task reassures us that the subjects were aware of the importance of the horizontal axis, which describes time, and were able to read the data from a graph. Thus, the 'picture' misconception, reinforced by everyday experience, was such a strong factor and obstacle that even the activation of System 2 did not overcome it.

Moreover, the choice of G5 was indicative of the respondents' inability of covariational reasoning in this task – they stopped their interpretation at the change of only one variable – height, ignoring the simultaneous change of the other variable – time, i.e. they reasoned only variationally, which represents a level of *no coordination* (Thompson & Carlson, 2017).

9.5. Conclusions Based on Oculographic Analysis

In the context of fast thinking, it appeared that among the 16 participants classified as 'fast', two demonstrated expert knowledge, while the rest of the respondents succumbed to erroneous associations. 'Speeding' through the task tended to result in incorrect answers, but the research documented two instances of rapid expert thinking.

The 'very slow' group in the criterion of fixation numbers obtained the best ratio of functional to non-functional responses. However, this result is even better in the 'non-fast' group, after excluding those from the 'very slow' group. The number of fixations on the task and its long and detailed analysis did not lead to significantly better results for the subjects.

However, in the 'non-fast' group, the ratio of those who answered correctly to those who did not is higher than in the groups of participants whose answers are characterised by a significantly lower number of fixations on the task. The increase in time spent working on the task did not correlate directly with an increase in correct answers, while the activation of System 2 in the 'non-fast' group, excluding the 'very slow' group, had such an effect.

The analysis, or lack thereof, of the illustrations in the task did not prove to be a significant factor affecting performance. In both the 'no illustration' group and the 'prolonged illustration' group, the ratio of correct G3 answers to incorrect G2 answers was exactly the same: 0.83.

10. Summary

The participants can be considered to represent a relatively high level of knowledge and skills in mathematics and physics and a high motivation to learn, as evidenced by their attendance to a prestigious and selective high school, their willingness to participate in this maths and physics study, their very serious approach to solving the tasks, their self-assessment of their skills in mathematics and physics, and the correctness of their work on the drone task. This reinforces all the more the conclusions reached in our study. The task proved to be difficult for the respondents, even though it was set in the context of an elementary activity from everyday life and was rated by them as easy.

In the context of this task, we found that the success rate was only 0.28 on the first attempt, so the level of *smooth continuous covariation* (Thompson & Carlson, 2017) in the context described would be very difficult to achieve even for gifted students. The conclusions are obvious if we want to predict the solvability of this task in a group of less able teenagers.

The first conclusion is therefore that covariational reasoning should be taught in school and that more time should be devoted to it.

In the first eye-tracking approach to solving the task, 72% of the participants succumbed to the 'picture' misconception. Thus, school teaching practice should take into account and design measures to eliminate this obstacle through targeted didactic interventions in this direction, which is another conclusion of our study.

One possible way to do this is to trigger System 2 activation, which can be achieved in different ways. In our study, reflection on the task was a methodologically planned procedure. It is noteworthy that almost 20 percent of the participants in the study changed their non-functional answer to an answer representing a function graph only as a result of their own spontaneous reflection; in this way, the respondents improved their answers by activating System 2. The implementation of critical thinking and self-monitoring is, and should continue to be, a concern for all teachers, especially mathematics and physics teachers, and science teachers in general.

In the context of fast thinking, a tendency to succumb to associations with the trajectory of movement and to identify the shape of the staircase with the diagram was revealed. Taking into account our research methodology, the way in which we have defined 'fast', only 2 people in the group were fast thinkers.

The increase in time spent working on the task did not correlate in the longest cases with an increase in correct answers. The best results were obtained in the group of people who belonged to neither the 'fast' nor the 'very slow' group; in this group we assume that System 2 of slow, analytical thinking was spontaneously activated. In contrast, the poorer performance in the 'very slow' group most likely has its origins in other knowledge deficits and uncertainties, which the subjects were unable to overcome despite many attempts. On the other hand, they were certainly highly motivated to give the correct answer.

The analysis or lack of analysis of the illustrations in the task did not prove to be a significant factor affecting performance on the task. In both the 'no illustration' group and the 'prolonged illustration' group, the ratio of correct G3 to incorrect G2 answers did not differ significantly. Climbing stairs is a well-known activity. However, the analysis of the illustration contributed to the discovery of further misconceptions related to the non-standard and complex shape of the stairs in the illustration.

11. Follow-up

The research is part of a broader investigation, and the partial analysis cited here concerns only a part of the results of the responses to one task among those used in the investigation. Analyses of the results of this and subsequent tasks will be continued and explored further from different perspectives.

The paper presents a preliminary methodological approach to categorising participants as 'fast' based on the overall fixation count analysis. This approach represents an initial step and should be further refined in future research. The task is relatively complex compared to those typically employed in Dual-Process Theory studies in psychological literature, as it requires multiple decisions to reject response options. It is plausible that some participants rely on System 1 thinking for certain parts of the task, while engaging System 2 thinking for others. The next methodological approach will focus on the number of fixations on each rejected response option, combined with a more detailed and sophisticated analysis.

Moreover, other alternative theoretical backgrounds will be implemented to interpret the results of the study, for example, such as the two types of thinking: predicative and functional, or analysis from the point of view of representation translation or modelling approaches.

In addition, activities have been undertaken as part of the *Embodying Math & Physics Education* (EMPE) project to design and implement learning environments for students that address the difficulties described.

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Alena Šturcová
Charles University, Faculty of Education, Prague

Jarmila Novotná
Charles University, Faculty of Education, Prague; CeDS, Université de Bordeaux, France

Chapter 3

Tool for Diagnostics of Students' Difficulties in CLIL (Content and Language Integrated Learning)

Summary: The article is focused on CLIL (Content and Language Integrated Learning), integrating a non-linguistic subject (here represented by mathematics) and a foreign language (here represented by English) for students. The aim is to introduce a type of didactic test which involves testing both language and content, and also simultaneously helps the teacher to diagnose whether the students' difficulties lie in the content or language portion. The test is analysed with statistical indicators to determine whether it can be used for other classes. Students' results from different testing rounds are compared.

Keywords: CLIL, Content and Language Integrated Learning, integrated learning, assessment in CLIL, diagnostics, testing, tests.

1. Introduction

CLIL is an abbreviation of the English term "Content and Language Integrated Learning". It is a method that integrates the teaching of a foreign language and a school subject. The concept of CLIL is rapidly spreading in Europe. It is a method that allows students to learn a foreign language in a natural environment. Students do not learn the language for the language itself, but they learn it while using it directly to gain knowledge in other areas.

Language education in Czech schools is compulsory from the third grade (the pupils are usually 9 years old). The pupils should be at A2[1] level when they leave elemen-

[1] The level of English is described according to CEFR, which is an international standard for describing the level of foreign language (CEFR, 2023).

tary school at 15 and at B1 level if they pass the secondary school exit exam (Maturitní zkouška, Anglický jazyk, 2023). The CLIL method is used in Czech schools and is officially supported, but its prevalence is not extensive. There are official learning programs for teachers, and several projects aimed at CLIL have been undertaken in the last twenty years. However, the method is not enshrined in Czech law, and its adoption depends on individual schools and teachers.

The dual focus of CLIL teaching is naturally reflected in the assessment of students taught through the use of this method. The teacher can either assess only the content subject, only the language portion, both subjects separately, or both subjects at the same time. The relevant topics are how to assess both subjects at the same time, how a test assessing both subjects can be designed, and how to identify whether a student struggles in the content subject, language, or both, and how the teacher can diagnose this. The diagnostics provide important feedback for the teacher, who can gain a better overview of the student's results and progress, as well as feedback from the results and the class.

This article focuses on CLIL integrating mathematics and English, and its purpose is to introduce a type of test for diagnosing whether students' difficulties lie in mathematics or English though didactic testing. The test was designed from the position of a teacher who uses diagnostics to analyse the difficulties of individual students as well as the whole class. Our aim is to inspire other CLIL teachers to extend their assessment tools. It is possible to use the test in its given form or it can be modified to suit different topics. The additional purpose of our research was to determine if the test we created is suitable for use in different classes.

We carried out studies where we administered the test to different groups of students, not only in English (L2) but also in Czech (L1 – first language), then analysed and compared the results, and subsequently analysed the test. This research method allowed us to conduct several types of analyses. We analysed the results of selected individual students in each version of the test (as a teacher in a class would do), the results of the students in a given group, we compared the results of students in individual rounds of tests, and we also analysed the test itself using statistical indicators.

In the theoretical part of the article, we introduce CLIL and describe its advantages and problematic aspects. We also focus on the language used and the difficulties students encounter in learning through the CLIL method, as the analyses of the students' solutions are partly based on the terminology and language used. We also describe the assessment in CLIL and alternative tests that preceded the test used in this research.

2. Theoretical Background

2.1. CLIL (Content and Language Integrated Learning)

CLIL is a method that combines the teaching of a foreign language and a content subject in such a way that equal emphasis is placed on both the language and the subject. The lesson, or teaching in general, has two objectives – one related to the content and the other to the language. Language is taught through content, and the content is taught through language. Mehisto, Marsh, and Frigols (2008) point out a third objective that CLIL should fulfil, which is the development of students' learning skills and strategies.

CLIL is a method of teaching that makes use of a foreign language, and there are several perspectives on how to define CLIL. A broader definition considers CLIL as an umbrella term for any teaching where content and language are integrated, making it superior to bilingual education (e.g., Ball, 2012). In teaching, especially with advanced students, the distinction between bilingual education and CLIL is often blurred. A narrower understanding sees CLIL as a methodology with specific characteristics that differ from bilingual education (e.g., Coyle, Hood, & Marsh, 2010). For the purposes of this article, CLIL is viewed in its narrow understanding.

CLIL encompasses teaching through short language activities (called language showers), individual lessons or longer periods, and can also be part of various projects. Ball (2009) distinguishes between "hard CLIL" and "soft CLIL". Stannard (2017) elaborates on this, describing "hard CLIL" as teaching content in a language other than the native language. This type of teaching is often conducted by teachers of the content subject, not language teachers, which might limit them due to their potential lack of language skills as well as language teaching expertise. "Soft CLIL" focuses more on the language aspect of teaching, where interesting topics are taught in a language which is non-native for students, and teachers are more often qualified language teachers.

When planning CLIL lessons and creating materials for teaching and assessing, teachers must consider various aspects of teaching, including the language students require in order to understand the content and use it in the classroom. Šmídová (2012) describes three types of language and refers to them as: the language of specialised terms, academic language, and peripheral language. In the realm of specialised terms, she emphasises that some terms are already familiar to students but in a different context – for instance, "square" might be known in Czech language as "náměstí" (public square) and "čtverec" (geometric square), but students might not be acquainted with the usage of "square" as "na druhou" (to the power of two). Concerning academic language, it encompasses not only individual words but also knowledge of entire language structures and grammar.

For example, if a student is unfamiliar with modal verbs, they will not be able to express themselves even if they are acquainted with the specialised terminology. Šmídová categorises peripheral language as encompassing the basic and commonly used phrases in teaching, such as greetings and basic instructions. In CLIL teaching, teachers should be aware that most subjects have their specific terminology, which students need to know in both languages, not only in L2 but also in L1.

The specificities of CLIL teaching are experienced by the students and the teacher during the teaching process, and by the teacher also when creating or adapting CLIL materials. Mathematics is characterised mainly by specific terminology and procedures. Mathematics has its unique language, for instance, Novotná and Hofmannová (2000) introduce L3 – the language of mathematics, in addition to L1 and L2 (second language, the one integrated with mathematics). They point out that mathematical language has specific grammatical structures and a vast amount of terminology used exclusively for mathematics, often involving non-verbal communication and the use of visual and graphic materials.

Mathematical language has been studied by a number of authors. In this article we consider mathematical language as described by Pimm and Keynes (1994), who state that mathematical language covers several areas:

- Spoken language used during mathematics teaching (from both teacher's and student's perspectives),
- Specialised language – the use of mathematical terms and phrases ("mathematics register"),
- Language used for writing texts, e.g., in textbooks and word problems, including graphical representations,
- Language of written symbols,
- Language used by students to think about mathematics,
- Language used by students to communicate with each other.

The mathematical language used depends on the students' age, level, and the topic they are studying. Older and more mathematically educated students should be able to express themselves with more precision and complexity. At that level, mathematics becomes more general and complex, and the specific topic being taught also affects the language used, for instance, when teaching conic sections or word problems.

Students face difficulties in learning mathematics through CLIL on several levels. Again, it depends on the students' age and level as well as the topic. Numerical tasks (e.g., equations, expressions) can be solved without knowledge of the language, but understanding the process often requires comprehension of specialised terminology and notation not encountered in regular language classes. Moreover, these are not just individual technical terms, but phrases encompassing entire sentences, which often differ greatly from English – e.g., expressions with powers and square roots, etc.

These challenges can be categorised into five groups, as outlined by Kubínová (2018) (based on her research which was, in turn, based on a study by Novotná and Moraová (2005)) (supplemented with own examples):

- Common expressions – these are expressions from general English that might be too challenging for students, either due to their level or because they describe a socio-cultural context unfamiliar to students,
- Cultural specifics – primarily different ways of expressing units, time, and dates,
- Grammar – the most significant challenge here is the distinct structure of English sentences. Czech has a much freer word order, whereas English uses a fixed word order and specific sentence structures. Differences also exist in inflection, articles, tense usage, etc.
- Mathematical notation – this involves different symbols and notations,
- Mathematical terminology – mathematics has a rich and specific terminology, and difficulties for students can arise from terms that do not have equivalents in the other language or where one term in the first language encompasses multiple meanings in the second language.

Teachers and students also have to contend with differences resulting from the different curricula and didactics of mathematics in different countries.

2.2. Assessment in CLIL – Mathematics and English

Assessment in CLIL occurs at multiple levels, similar to school assessment in general, and can be viewed from various perspectives. We can assess the teaching unit (a lesson, part of a lesson, or several lessons forming a logical unit), including its planning, implementation, and outcomes. We can also assess the students and their results, performances, or progress. Additionally, we can assess the effectiveness and efficiency of the CLIL method. These areas overlap in some aspects. Besides these areas, various other aspects can be assessed, such as CLIL materials, etc.

Our research deals with the assessment of CLIL students and their results from the point of view of the relationship between content and language. A CLIL teacher might primarily assess the content, focus only on language, or assess both content and language independently (for example, conducting one test for terminology and another for content in L1), or attempt a form of assessment that considers both language and content simultaneously. The research studies and our experiences confirm that CLIL teachers primarily prioritise assessing the content. For example, Reierstam (2015), analysing the assessment and perception of biology and history teachers in Sweden, states that these CLIL teachers tended to assess content rather than language skills. Regarding language, they only focused on terminology. Hönig (2010) carried out a similar study involving history teachers in Austria. She states that they concentrated on content, but the language accuracy was also important, so the language was assessed indirectly.

Our experience shows that the teacher's assessment of content or language also depends on the students' proficiency level in the language. If their language proficiency is very good, the teacher is more likely to assess specialised terminology from the language as the primary focus. For instance, if the teacher knows that the students are proficient in modal verbs, he or she can safely use them without testing their understanding. Based on our experience, language testing in CLIL can be approached in two ways. Either the teacher wants to test whether the students have mastered language material that is independent of the mathematics content being covered, or the teacher wants to test language material, whose understanding directly affects the students' comprehension of the mathematics content. In the latter case, the testing primarily ensures that students can comprehend both content and language together.

Students can be assessed on the basis on their language skills (speaking, listening, writing, and reading), content knowledge, critical and logical thinking, communication and interaction skills, ability to work independently, teamwork, practical skills, etc.

Assessment of students in CLIL classes occurs at several levels. It involves ascertaining how well the students have mastered the learning material – this can be done through various types of tests, graded oral testing, etc. The teacher influences which aspect of the learning material will be tested by their choice of testing methods. For example, during oral exams, the teacher can assess pronunciation in L2 (which is typically not assessed in written exams) or assign independent or group work and assess the progress or output of this work. The teacher also influences the assessment through the terminology used and can utilise various graphical aids such as pictures or graphs. If a mathematical problem is given numerically, the students can solve it without understanding the language. The teacher can also influence the testing process by specifying the expected outputs during the examination, for example whether they require responses or comments in L2.

Lo, Lui, and Wung (2019) conducted a study on how science teachers provide instructions during assessment and the scaffolding methods they use to help students acquire language skills. Among their findings, they discovered that the majority of teachers focus on content. However, the authors also provide examples of tasks where the teachers assessed language skills, such as assigning students to write coherent paragraphs. They also state that teachers may be compelled to solely evaluate content if it involves hard CLIL.

2.3. Alternative Tests – Research by Hofmannová, Novotná, Pípalová, Šteflíčková (Šturcová)

The issue of assessing integrated mathematics and English teaching has been studied by Hofmannová, Novotná, and Pípalová (2004, 2011). In a 2004 study, they presented a test whose results illustrate that a student's ability to understand the language influences their ability to solve mathematical problems. They created a test focusing on comparisons. Students are asked to translate verbal expressions (e.g., "There are roughly four times as many people in Bristol as in York" (2004)) into mathematical language. From a mathematical perspective, the test deals with transforming verbal language into equations and inequalities using symbols such as =, <, ≈, +. From a language perspective, it primarily focuses on specific vocabulary related to comparisons, specific adverbs and verbs, comparative and prepositional phrases, etc. By comparing the results for ten different expressions, it is possible to identify the areas where students face difficulties.

In another study, Novotná (2011) introduced another type of test, structured on the basis of grading individual tasks, wherein by comparing answers to these tasks it is possible to distinguish what the student does not understand. Some tasks test the use of language tools, but the mathematical difficulty remains the same, while others maintain the use of the same language tools but differ in mathematical difficulty. For some tasks, neither component changes.

Both of these tests deal with different mathematical concepts, but in both cases, students are asked to convert verbal expressions into mathematical language.

Šteflíčková (Šturcová) (2014) also explored alternative tests, building on previous research, and created three different tests to determine the areas which are challenging to students. The first test was entirely verbal, requiring students to identify a mathematical operation from its description and then do it. One of the tasks was, for instance, "Add 10/7 to 3/14 and divide by 6". If a student was able to rewrite the sentence into mathematical notation, they likely understood the language; if they translated it correctly but could not solve it, the difficulty was probably rooted in mathematics. If a student was unable to rewrite the sentence and therefore could not solve the task, they faced language difficulties, but it was impossible to determine whether they would have been able to solve the task if it had been presented differently. Each operation appeared multiple times in the test, allowing the teacher to assess whether the student genuinely understood the language and was capable of solving the task thanks to comparing solutions in different parts of the test. The second test (inspired by Novotná's test in 2011) focused on geometric plane figures – polygons. The test was verbally presented and the students were supposed to analyse figures they were familiar with and search for those that met specific conditions related to sides, angles, congruent sides and angles, perimeters, and areas. From a linguistic perspective, the test consisted of specific terminology associated with geometric figures and the grammar of modal verbs. One of the tasks, for example, was: A polygon. It must have equal angles. It can't have all sides equal". To solve this task correctly, the student must understand mathematical terminology i.e. "polygon," "equal," "angles," and "sides"; be familiar with modal verbs like "must" and "can't"; and have a grasp of geometric shapes and their properties. The tasks were designed to build upon each other, repeating terminologies and concepts, providing the teacher with deeper insights into the student's understanding. The third test consisted of a set of word problems that intertwine various tenses and different types of percentage problems. These problems intentionally included information irrelevant to solving the word problems for the students to eliminate through proper understanding of time sequences.

All of these tests can be modified in various ways.

3. Research

3.1. Research Questions

1. Which didactic tests are suitable tools for identifying difficulties in CLIL teaching when integrating mathematics and English?
2. Is the test we have designed suitable for determining whether a student has difficulties in English or mathematics? How can the teacher diagnose this?
3. According to statistical indicators, is the test we have designed suitable for use in teaching?

3.2. Analysis of Didactic Test Designed for CLIL Teaching

Based on the research conducted by Hofmannová, Novotná, Pípalová (2004; 2011) as well as Šteflíčková (Šturcová) (2014), alternative tests were created (Šturcová, 2024) that can be used to determine whether a student has difficulties in mathematics or English. One of these tests was used for the research portion of the article. We decided to carry out this type of research with our test to be able to analyse more data and, despite the fact that our research sample was small, the research could be a base for some other – more extensive – research studies. In the following text, we will conduct an analysis from several perspectives. We will perform an analysis of the statistical indicators of the test to assess whether the test is suitable for repeated use in teaching. We will also analyse and discuss the students' results in terms of identifying areas of difficulty regarding content and language. We will demonstrate how the diagnostics are conducted on individual tasks. We will also discuss and compare the results in both rounds in order to see how the results of individual students change.

3.3. Methodology and Use of Research Tools

For the experiment, we used a test focusing on geometric plane figures in mathematics and basic terminology related to plane geometry in English. More details are provided later.

The experiment was conducted in such a way that the test was given to a class where the students were divided into groups of four based on their proficiency levels in English and mathematics. In each group, there were students with approximately the same proficiency levels in both subjects. This proficiency level was assessed based on the students' grades from their latest report card and their performance in English and mathematics classes, as judged by their teachers. Two students in each group took the test in English, and the other two in Czech. Subsequently, a CLIL lesson was conducted during the class, focusing mainly on vocabulary, but also revisiting basic mathematical concepts during vocabulary practice. Due to time constraints, a class that already had knowledge of the mathematical content was selected, so the test aimed to assess what the students remembered, although it was primarily designed as a test to assess the outcomes of learning. This lesson took part one week after the first test. Approximately a week after this lesson, the second round of testing took place, with tests administered in such a way so that each group included: one student who took both tests in English, one who solved the first test in English and the second one in Czech, one who solved the first test in Czech and the second one in English, and finally, one who took both tests in Czech. As a result, four groups of students were formed. The first group solved both tests in English, the second group solved both tests in Czech, the third group solved the first test in English and the second one in Czech, and the fourth group solved the first test in Czech and the second one in English. An analysis of the tests and an assessment of the student' solutions were conducted.

3.4. Research Sample

The experiment was conducted in one class of students aged 16–17 consisting of 5 boys and 15 girls in the 2^{nd} year of a secondary vocational school[2]. The experiment occurred in two

2 Secondary vocational schools are four-year institutions that educate students preparing for a specific profession while also providing the option to continue their studies at a university. Following a nine-year elementary school, students can choose to attend a secondary vocational school or a grammar school, which is a four-year general secondary school where they prepare for university studies. Al-

rounds: in the first round, a total of 20 students participated (10 students solved the test in English, 10 in Czech), and in the second round of testing, 15 students participated (7 students solved the test in English, 8 in Czech). Mathematics is taught three times a week in the second year and the content involves topics related to the exit exam – they are the same for all secondary schools.

3.5. Test

The created test is a non-standardised cognitive test of proficiency, objectively scorable (Chráska, 2007). It is a multiple-choice test where one out of four answers is correct. Four options were chosen to reduce the likelihood of guessing, eliminating the need for having to work with a guessing correction (Chráska, 2007) (although this would be possible and, if there were many tasks, appropriate). The test's simplicity in evaluation was the reason for having only one correct answer.

The test aimed to assess whether and how students mastered the content, and if they faced difficulties, whether they were in English or in mathematics. The test content focused on planimetry – geometric figures on a plane. Mathematically, the content covered basic plane figures in five subcategories: lines and their parts, angles, relative position of lines, triangles, and polygons. From English, basic mathematical terminology related to planimetry was included in the test, primarily testing memory and understanding of specified terms.

The test was constructed on the basis of a specification table, where each task was categorised according to Bloom's taxonomy of learning objectives (Čábalová, 2011). The test comprised 20 tasks, with 4 tasks in each subcategory. Each task specified what the student should have mastered in mathematics and in English, and what additional English terms they must know. We did not include the most common terms used in English, such as "to be", "can" etc. In some tasks, it was necessary for students to be familiar with mathematical symbols, which, in this case, were nevertheless the same in both languages, and students should know them from regular mathematics classes.

Tables 1, 2, 3, 4, and 5 show specification tables for different subcategories.

ternatively, they can enroll in a vocational school, typically lasting three years, where they receive vocational training and, upon completion, enter the workforce.

Subcategory "Lines and their parts"

Table 1. Specification table for subcategory "Lines and their parts"

Task number:	Bloom's taxonomy	Mathematics content	English (specialised terms)	Additional English expressions
1	Remembering	To match basic mathematical concepts (straight line, line segment)	line segment straight line	choose in order
2	Understanding	To estimate the length of a line segment		determine long length
3	Applying	To apply the definition: A straight line passes through exactly two different points	straight line point	determine how many pass through different exactly
4	Applying	To determine how many line segments are in the picture	line segment	determine how many

Tasks in subcategory "Lines and their parts"
Figures 1, 2, 3, and 4 present the tasks in the subcategory "Lines and their parts".

Figure 1. 1st task in test

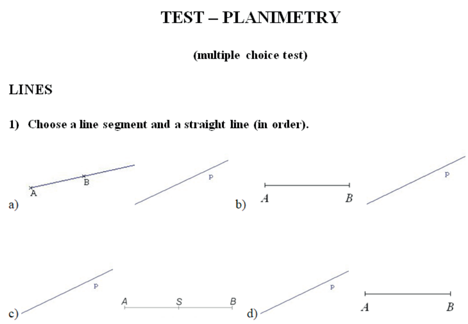


Figure 2. 2nd task in test

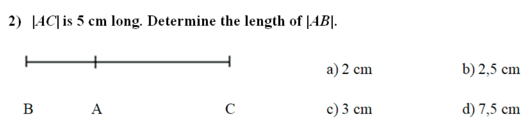


Figure 3. 3rd task in test

3) There are four different points. Determine how many different straight lines can we draw if one line passes through exactly two points.

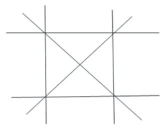
a) 2 b) 4 c) 6 d) 8

Figure 4. 4th task in test

4) Determine how many line segments are in the picture.

a) 6 b) 20

c) 8 d) 10



The test is constructed so that if a student answers task 1 incorrectly, it is likely that they will also answer task 3 and 4 incorrectly, indicating a problem with English. It will not be possible to determine from the English version of the test whether they also have a problem with mathematics. If a student answers task 1 correctly but answers task 3 incorrectly, they likely have a problem with mathematics. If a student answers task 1 correctly but answers task 4 incorrectly, this also likely signifies a problem with mathematics. Task 2 primarily tests mathematical knowledge.

Subcategory "Angles"

Table 2. Specification table for subcategory "Angles"

Task number:	Bloom's taxonomy	Mathematics content	English (specialised terms)	Additional English expressions
5	Remembering	To know the basic types of angles (right, obtuse, acute, straight)	right angle obtuse angle acute angle straight angle	determine
6	Applying	To determine the size of vertically opposite angles	angle	determine size
7	Understanding	To estimate angle sizes using knowledge about angles	angle	guess size
8	Analysing	To perform an analysis when adding angles To know basic mathematical concepts (right, obtuse, acute, straight angle)	right angle obtuse angle acute angle straight angle	

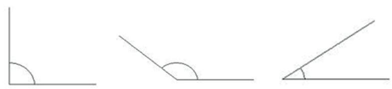
Tasks in subcategory "Angles"

Figures 5, 6, 7, and 8 present the tasks in the subcategory "Angles".

Figure 5. 5th task in test

ANGLES

5) Determine the names of these angles (in order).



a) acute, straight, obtuse b) right, acute, obtuse

c) right, obtuse, acute d) straight, obtuse, supplementary

Figure 6. 6th task in test

6) Determine the size of angle α.

a) 45° b) 80°

c) 100° d) 135°

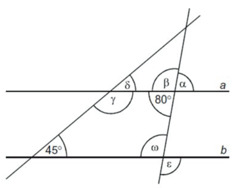


Figure 7. 7th task in test

7) Guess the size of the angle in the following picture.

a) 100° b) 120°

c) 135° d) 160°

Figure 8. 8th task in test

8) Determine which angles could give a straight angle.

a) three obtuse angles b) three acute angles

c) one right and one obtuse angle d) one right and one acute angle

If a student answers task 5 incorrectly and does not know English terminology, nothing can be determined from a mathematical perspective. If a student answers task 8 incorrectly but task 5 correctly, they probably have a problem with mathematics. Tasks 6 and 7 primarily test mathematical knowledge.

Subcategory "Relative position of two lines"

Table 3. Specification table for subcategory "Relative position of two lines"

Task number:	Bloom's taxonomy	Mathematics content	English (specialised terms)	Additional English expressions
9	Remembering	To understand the relationships between lines	square parallels intersecting lines perpendicular lines	determine relationship between
10	Analysing	To decide on the relationship between perpendicularity and parallelism	line parallel perpendicular	relationship
11	Applying	To determine the sizes of corresponding and alternate angles	angle	determine size
12	Applying	To determine the sizes of corresponding and alternate angles	angle	determine size

Tasks in subcategory "Relative position of two lines"

Figures 9, 10, and 11 present the tasks in the subcategory "Relative position of two lines".

Figure 9. 9th task in test

10) The line *a* is perpendicular to *b*, *b* is parallel to *c*. What is the relationship of *a* and *c*?

a) *a* is parallel to *c*

b) *a* is perpendicular to *c*

c) they are not in a relationship

d) we cannot say

Figure 10. 10th task in test

11) ***a* ‖ *b*. Determine the size of angle ω.**

a) 80° b) 90°

c) 100° d) 135°

12) ***a* ‖ *b*. Determine the size of angle *δ*.**

a) 30° b) 40°

c) 45° d) 50°

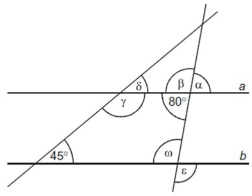


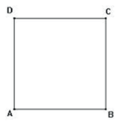
Figure 11. 11th and 12th task in test

RELATIVE POSITION OF TWO LINES

9) Determine the relationship between *AB* and *CD* in the square *ABCD*.

a) parallels b) intersecting lines

c) perpendicular lines d) we cannot say



If the student successfully completes task 9, they likely know the tested English terminology. If they answer task 10 incorrectly but task 9 correctly, they likely have a problem with mathematics. Tasks 11 and 12 primarily test mathematical knowledge.

Subcategory "Triangles"

Table 4. Specification table for subcategory "Triangles"

Task number:	Bloom's taxonomy	Mathematics content	English (specialised terms)	Additional English expressions
13	Remembering	To understand and find the altitude and median	triangle side altitude median bisector	determine
14	Analysing	To specify mathematical terms (altitude, median and side of the triangle) and the relationships between them	triangle side altitude median	the same length shorter
15	Applying	To apply the triangle inequality	triangle	
16	Applying	To calculate the size of internal angles in a triangle	triangle	

Tasks in subcategory "Triangles"
Figures 12, 13, 14, and 15 present the tasks in the subcategory "Triangles".

Figure 12. 13th task in test

TRIANGLES

13) Determine what *PR* and *SR* are called in this triangle.

a) side and bisector b) altitude and bisector

c) side and median d) altitude and median

Figure 13. 14th task in test

14) Determine what is true in the triangle.

a) An altitude and a median cannot have the same length.

b) An altitude can have the same length as a side.

c) All three sides of a triangle cannot have the same length.

d) All three medians can be shorter than all three altitudes.

Figure 14. 15th task in test

15) In triangle *ABC*: $a = 5$ cm, $b = 3$ cm. What can we say about c?

a) $c < 2$ cm b) $2 < c < 8$ cm c) $c = 5$ cm d) we cannot say

Figure 15. 16th task in test

15) In triangle *ABC*: $a = 5$ cm, $b = 3$ cm. What can we say about side c?

a) $c < 2$ cm b) $c < 8$ cm c) $c = 5$ cm d) we cannot say

16) In triangle *ABC*: $\alpha = 2\beta$, $\beta = 3\gamma$. Choose the right answer.

a) $\alpha = 106°$ b) $\beta = 56°$ c) $\beta = 36°$ d) $\gamma = 18°$

If a student answers task 13 correctly but task 14 incorrectly, they likely have a problem with mathematics. Tasks 15 and 16 primarily test mathematical knowledge.

Subcategory "Polygons"

Table 5. Specification table for subcategory "Polygons"

Task number:	Bloom's taxonomy	Mathematics content	English (specialised terms)	Additional English expressions
17	Remembering	To know the sum of internal angles in a polygon	interior angle regular polygon	formula sum
18	Understanding	To determine convexity To identify basic polygons (tetragon, hexagon)	convex tetragon	determine figure
19	Applying	To apply knowledge of the sum of internal angles in a polygon	interior angle regular hexagon	determine size
20	Applying	To apply knowledge about regular polygons	regular polygon line segment	determine length longest

Tasks in subcategory "Polygons"
Figures 16, 17, 18, and 19 present the tasks in the subcategory "Polygons".

Figure 16. 17th task in test

POLYGONS

17) A formula for the sum of interior angles in a convex polygon is:

a) $(n-3) \, . \, 180°$ b) $(n-2) \, . \, 180°$ c) $(n-1) \, . \, 180°$ d) 180°

Figure 17. 18th task in test

18) Determine which figure shows a convex tetragon.

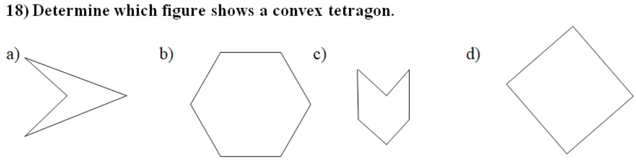


Figure 18. 19th task in test

19) Determine the size of the interior angle in a regular hexagon.

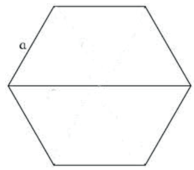
a) 30° b) 60° c) 120° d) 360°

Figure 19. 20th task in test

20) This is a regular polygon, $a = 5$ cm. Determine the length of the longest line segment in the picture.

a) 12 cm b) 10 cm

c) 5 cm d) we cannot say



If the student correctly identifies the quadrilateral in task 18, it is likely that they know the basic terminology for polygons. If they answer task 19 incorrectly but task 18 correctly, they likely have a problem with mathematics (as they likely understand the word "hexagon"). If they answer task 19 correctly but make a mistake in task 17, the problem, again, likely lies with mathematics, just as when they answer task 18 correctly but task 17 incorrectly. Task 20 primarily tests mathematical knowledge.

3.6. Pre-Experiment for Adjusting Distractors of Proposed Test

After preparing the test and designing the distractors for each task, a pre-experiment was conducted. In this pre-experiment, students of the same grade as those in the tested group worked on the test in Czech, without any provided answers. A total of 24 students, aged 16–17, participated in the pre-experiment. The time limit for the test was set at 10 minutes. The students were asked to complete the test and, if possible, write down their procedures, or add comments or explanations. The permitted tools were writing implements and a ruler.

The distractors contained in the test were adjusted based on the students' responses in the pre-experiment. This adjustment involved incorporating into the multiple-choice options the answers that were most frequently provided by students during the pre-survey tests. (The distractors in the test described in the previous part of this article are already adjusted.)

3.7. Experiment

The test was administered to students in two rounds – before the CLIL lesson, and after. The reason for conducting two rounds of testing and dividing students into those who wrote both tests in Czech, both tests in English, the first test in Czech and the second in English, and the first test in English and the second in Czech, was to verify the characteristics of the didactic test among different groups. Additionally, we wanted to compare the results of students in individual groups.

The test was conducted in class, students were given a paper test and worked on the test with a time limit of 15 minutes. They were asked to complete the test and were informed about the grading (one point for every correct answer, no negative points for incorrect ones). The permitted tools were writing instruments and a ruler.

The characteristics of the didactic test were verified from several perspectives. The first and second rounds of testing were assessed separately for each language. Then, the students' results and the changes in results in the created groups were assessed. Another tested aspect were the diagnostics of what the students have had difficulties with.

3.8. Statistical Analysis and Discussion of Results for Each Round of Testing

The statistical analysis was conducted using several statistical indicators according to Chráska (2007) – difficulty, selection of distractors, number of answers omitted, sensitivity, validity, and reliability and standardisation. We created Table 6, including some of these indicators as well as the number of students who participated in the test and the average score. Regarding the difficulty, tasks with a difficulty value (Q) higher than 80 are considered difficult, and tasks with a value lower than 20 are considered easy (Chráska, 2007). The selection of distractors is appropriate if the students' attention is not focused on any specific distractor. For students who do not know which answer is correct, all distractors should be equally acceptable (Chráska, 2007). Regarding the number of answers omitted, attention should be paid to tasks where more than 30-40% of students omit the answer. Sensitivity is calculated using the coefficient of sensitivity ULI (Chráska, 2007). If the coefficient reaches negative values, students who perform worse on the test get better results. No quantitative methods are used to determine validity (whether the test checks for what it is supposed to) (Chráska, 2007). According to Chráska (2007), reliability means that the test is reliable (i.e., repeating the test under the same conditions should yield the same or similar results) and accurate (relating to errors). Reliability can be determined, for example, by using the Kuder-Richardson formula, which is used for level tests composed of thematically homogeneous tasks (Chráska, 2007). This is the formula we used. The halving method is also used, but it requires items to be in order of increasing difficulty, which this test does not satisfy. According to Chráska (2007), the purpose of standardisation is to create a norm so that a student's results can be compared with those of other students. Several scales are used, such as the percentile scale, which determines what percentage of students scored lower.

Table 6. Analysis of statistical indicators in each round

	First Round of Testing in English	First Round of Testing in Czech	Second Round of Testing in English	Second Round of Testing in Czech
Number of students participating	10	10	7	8
Difficulty	The least challenging task was 6 (90% of students answering correctly). The most challenging task was 8 (no correct responses). Tasks 3, 4, 8, 17, and 19 are considered difficult. Task 8 with Q = 100 should be excluded from the test for potential future use.	The least challenging tasks were 5 and 9, as expected. No task had a difficulty value higher than 80. Task 6, with a difficulty value of 10, could also be described as not difficult.	Task number 19 was difficult. The least challenging tasks were tasks 2, 5, and 6.	The easiest tasks were tasks 1, 5, and 9. Tasks 6 and 10 were also easy. Tasks 15 and 19 are considered difficult.
Selection of Distractors	Appropriated	The selection of distractors in the Czech version is distributed differently than in the English version.[3]	appropriated	appropriated
Number of Answers Omitted	Tasks 16 and 17 are considered problematic. Task 17 tests formal knowledge, so the students are likely to have forgotten.	The students did not skip any tasks more frequently than others.	The students did not skip any tasks more frequently than others.	The students did not skip any tasks more frequently.
Sensitivity	The coefficient reaches negative values for tasks 1, 5, and 17. (However, due to the small number of participants, this represents a difference of one or two students.)	The sensitivity coefficient reaches negative values for tasks 6, 12 and 13. In contrast, the sensitivity coefficient is high for tasks 4 and 8.	The sensitivity coefficient achieves a negative value for task 3.	The sensitivity coefficient achieves a negative value for tasks 10, 13, and 15, but this constitutes a difference of only one person.
Reliability	The reliability is 0.73, which is average.	The reliability is 0.65, which is a low value.	The reliability is 0.81, which is average.	The reliability is 0.55, which is a low value.
Average score		In the Czech version, the students were more successful on average by 2.9 points than in the English version, which is due to the difference in language proficiency.	In comparison with the first round of the English version, the average increased by approximately 2.9 points.	In comparison with the first round of the Czech version, the average increased by only 1.4 points.

[3] For example, in task 3, half of the students selected the same incorrect distractor. In task 7, students selected the wrong angle size, which was often determined by students who were working on the open test in a pre-research.

4. Discussion

4.1. Discussion and Comparison of Results in Both Rounds for Individual Students

In Table 7 we observe a comparison of the students' results in each round. Only those students who took both tests are included in the comparison. This information was not directly relevant to any aims of the research, but we found it important to include as it indicates the results of individual students and therefore demonstrates the diagnostics of a teacher and students' progress regarding Czech and English language. We discussed individual test items to ensure that our diagnostic process was accurate.

Students are divided into rows by grades in English and mathematics, and into columns by the versions of the test they took. Each student's score in each round is indicated. (The table uses the notation 1st r for denoting the first round, and 2nd r for denoting the second round. ENG means English version of the test, Czech means Czech version of the test).

Table 7. Comparison of students' results in each round

ENG + ENG			ENG + Czech			Czech + ENG			Czech + Czech		
	1st r	2nd r		1st r	2nd r		1st r	2nd r		1st r	2nd r
Vítek	11	16	Michal	9	15	Aneta	10	9	Maruška	11	13
Dominika	15	15	Verča	7	9	Marcela	15	12	Jarda K.	16	15
			Jarda	5	10	Markéta	12	6	Míša P.	11	14
Zuzana	3	5	Míša	7	10	David	8	11	Eva	7	10

There are 3 students in the ENG + ENG group. One student neither improved nor worsened, while two students improved by 5 and 2 points, i.e., by 25% and 10%.

There are four students in the group with both Czech tests. These students improved by only 10% and 15%, and one student worsened. It appears that the students who wrote both versions in English improved more than those who wrote both versions in Czech. This may be due to the fact that the CLIL lesson was mainly focused on terminology with only a review of the mathematics portion. In the second round, the majority of students achieved similar results as their classmates.

When comparing students who wrote the first round in English and the second round in Czech, all of them improved – individually by 10%, 15%, 25%, and 30%. These students' results are also close to those of the other students in the second round, except for Michal, who showed a noticeable improvement.

In the group of students who took the first test in Czech and the second one in English, deterioration is evident. As Czech is their mother tongue, it is clear that it is easier for students to take a test in Czech – but we still find it surprising, especially for Markéta, who worsened by 30%. Considering that the students had already taken the test in Czech and had also learned the terminology, only minimal worsening or slight improvement would be expected, as was the case with David, where the deviation from his initial result equaled 5%.

Students who took the first round test in Czech got better results compared to those who took the second round test in English, which is not surprising, because Czech is their native language. We could ask if it is possible to compare results in Czech and English if Czech is their native language, with English being their second language.

We find the results surprising, as we would have expected a more significant improvement in the second rounds, given that the students took the same test twice and had a revision lesson between tests. we would have also expected improvement of all students in the group of Czech + Czech.

4.2. Analysis and Discussion of Students' Results in Terms of Identifying Areas of Difficulty

The test was designed to determine areas of difficulty for the students. In Table 8, we present an overview of the answers from individual students in both rounds of testing in English. Correct answers are highlighted in yellow, denoting tasks that may indicate whether the difficulties are in English or mathematics. Tasks primarily testing mathematical knowledge (not highlighted in yellow) are not included in the discussion. All students, including those who took only one test (did not take the second test due to their absence) are included in the table. In the following section, we describe the procedure of a teacher who wants to determine if the student has difficulties in math or English.

Table 8. Students' score in first and second round of testing in English

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Student/correct answer	B	B	C	D	C	B	C	B	A	B	C	C	D	B	B	C	B	D	C	B
1st round of testing																				
Dominika AJ	D	B	C	A	C	B	C	C	A	B	C	C	D	B	B	D	D	D	B	B
Vítek AJ	B	B	A	B	C	B	D	C	A	B	C	C	D	B	D	B	A	D	B	A
Michal AJ	B	B	B	D	B	B	C	D	B	C	C	C	B	C	D	D	D	D	C	C
Sabina AJ	C	B	C	D		B	B	C	A	A	B	C	C	C	B		D	A	B	B
Verča AJ	B	B	A	C	D	B	C	D	C	A	A	C	D	C	C			A	D	C
Lucka AJ	B	A	B	A	C	B	C	A	A	B	B	C	C	B	D	C	D	C	B	D
Míša T. AJ	B	B	B	A	C	B	B	D	A	D	A	A	C	B	D	C	D	D	A	C
Renata AJ	B	A	D	A	A	B	A	C	D	B	A	D	A	C	B	C	B	B	A	C
Jarda Š. AJ	B	C	A	A	C	B	B	D	B	D	C	C	B	C	C				A	A
Zuzana AJ	B	B		A	B	D	D	D	B	C	A	A	C		D			C		B
2st round of testing																				
Vítek AJ	B	B	B	C	C	B	D	B	A	B	C	C	D	B	B	D	C	D	A	B
Dominika AJ	B	B	C	D	C	B	C	C	A	B	C	C	D	B	D	C	B	D	D	B
Marcela AJ	A	B	B	C	C	B	C	B	C	A	C	B	D	B	B	D	B	A	D	C
David AJ		B	C	C	C	B	C	C	B	C	D	C	C	B	C	D	C		C	B
Aneta AJ	B	B	C	D	C	B	C	C	B	C	C	B	C	C	B	B	A	A	B	B
Markéta AJ	D	A	B	C	C	B	B	C	D	A	C	A	B	C	B	D	A	A	D	A
Zuzana AJ	A	B	C	D	C	D	D	D	A	A	B	A	B	C	D	B		A	A	B

Task 1 focused on determining a line and a line segment (i.e., primarily testing vocabulary). If a student answered task 1 correctly but task 3 and/or 4 incorrectly, we can conclude that there was a problem with mathematics. This is the case for 7 out of 17 students.

Task 5 also tested the terminology. If a student had the correct answer in this task but an incorrect answer in task 8, the problem was again with mathematics. Task 5 was not difficult; in the first round, five students had the correct answer, none of whom answered task 8 correctly, indicating a problem with mathematics. Renata, Sabina, Michal, and Verča struggled with English; whether they also had problems with mathematics cannot be determined.

Task 9 was a simple exercise to verify whether the students are familiar with the term "parallel lines" and could identify them. If a student answered task 9 correctly but task 10 incorrectly, there was likely a problem with mathematics. Sabina and Míša experienced difficulties with these tasks from a mathematical perspective. Six students probably had difficulties with English. Renata and Markéta chose answer d (we cannot say) for task 9, which suggests that they did not grasp the entire question, not just the term "parallel lines".

Regarding another part of the test, if a student answered task 13 correctly, they likely knew the terminology needed for task 14. It means that if a student had task 13 correct but task 14 incorrect, they had difficulties with mathematics. Both tasks were correctly answered by Vítek and Dominika in both rounds, as well as by Míša and Marcela. Verča had problems with mathematics.

In the last subcategory, we can observe the results of tasks 17, 18, and 19. If a student answered task 18 correctly but did not answer 17 and/or 19 correctly, they had difficulties with mathematics. This applied to Vítek in both rounds and Dominika and Míša in the first round. The source of difficulties for others could not be determined.

Comparing the difficulty levels of mathematics and English for each student, Vítek and Míša faced the most predominant difficulties with mathematics. Vítek likely understood a significant part of the test in the first round, even improving in the second, but he is not as strong in mathematics as he is in English. However, his overall results were good compared to other students; Vítek scored 10 points in the first round and 16 points in the second round. This aligns with his performance in English and mathematics, where he is among the top students in English and commendable in mathematics.

The most significant difficulties with English compared to mathematics were experienced by Sabina and Marcela. Whether these difficulties were solely with English or also with mathematics could be partially determined by comparing the results of the first test with the results of the second test if the second one was written in Czech. Both also had an average score compared to others. They are considered average students in both mathematics and English. Analysing the difficulty levels of mathematics and English for each student suggested that students had more difficulties with mathematics.

The highest score in the first round was obtained by Dominika (15 points), being strong in both mathematics and English compared to others. Interestingly, she scored the same number of points in the second round. However, studying the table of results reveals that she provided different answers, particularly in questions aimed at testing mathematical knowledge. Vítek achieved the highest number of points in the second round (16 points).

The lowest number of points in both rounds was obtained by Zuzana (3 and 5 points). Markéta, scoring 6 points, was also very weak in English and acquired the points mainly in more mathematically-focused tasks. At school, both students have unsatisfactory results in English, and Zuzana is weak in mathematics as well.

The research was supported by interviews with individual students along with an assessment of their general performance in both English and mathematics provided by their teachers. We discussed the individual tasks contained in the test with those students who seemed to have difficulties in specific concepts or terminology in order to acquire feedback and to be able to confirm or refute our conclusions about where the students' difficulties are. These interviews supported our conclusions and we can say that the test succeeded in distinguishing the source of the students' difficulties.

4.3. Discussion of Results of Student' Solution Analysis in General

The test analysis is influenced by the low number of participants in the experiment and likely by the fact that the students took the same test twice. One CLIL lesson was conducted between the two tests (as described in the section Methodology and Use of Research Tools). The CLIL lesson was mainly focused on terminology, the problems present in the test were not solved, and the students did not get to see their test after the first round.

The test contains several problematic items. In terms of difficulty, task 19 appeared challenging in almost all versions. If the test were to be used in further testing, this task should be omitted or replaced with another. In the first rounds, tasks 16 and 17 were also difficult, but in the second rounds, the solutions to these tasks did not appear challenging. On the other hand, tasks 5 and 6 were very easy. In the first round of testing in English, half of the students guessed the correct answer. When asked later how they knew the correct meaning of "acute" and "obtuse", students answered that they had guessed, going by the sound of the words. In the Czech rounds, task 9, which focuses on vocabulary, also appeared easy. Unexpectedly, some students had difficulty with task 1 in the first round of the Czech version, even though this task also focuses on English terminology, so it should be easy in Czech. This fact could be due to three students overlooking that the order of the pictures matters, even though it was noted in the assignment and emphasised when the test was administered. Regarding sensitivity, some tasks reach negative values; in two versions, tasks 14 and 15 are repeated, so it would be appropriate to reconsider the wording of the questions.

The reliability is 0.73 and 0.81 for the English versions, and 0.65 and 0.55 for the Czech versions. This is likely due to a significant difference in task difficulty in the Czech version due to the easy tasks in the English version which were focused on terminology. This also shows that, in terms of difficulty, a task given in English and the same task given in Czech can be considered a completely different task for the student.

Comparing different groups of students indicates that students who wrote both versions in English improved more than those who wrote both versions in Czech.

5. Summary

This article deals with the assessment of students learning with CLIL, integrating mathematics and English, and demonstrates a tool for identifying sources of difficulties in students during testing.

We stated the following research questions:

1. Which didactic tests are suitable tools for identifying difficulties in CLIL teaching when integrating mathematics and English?

   We believe alternative didactic tests to be suitable as tools, where by comparing the results of different tasks it can be determined whether the student has difficulties with mathematics or with English.

2. Is the test we have designed suitable for determining whether a student has difficulties in English or mathematics? How can the teacher diagnose this?

   We described several types of suitable tests which can be used by teachers during lessons and we performed an experiment with one test. We conducted an analysis, carried out an experiment with secondary school students, and discussed and analysed solutions from various perspectives. We showed how to determine where a student has difficulties using the test. Such tests could also be used in non-CLIL lessons taught in the native language. Here too, it is necessary to decide whether the students understand the terms and formulations in mathematics.

3. According to statistical indicators, is the test we have designed suitable for use in teaching?

   Analysis using statistics indicators showed that the proposed test might be suitable for use in lessons after omitting or replacing task 19, but the number of respondents was very small, and it is therefore necessary to conduct further research to be able to make the statistics meaningful. The results cannot be generalised. However, the experiment as well as interviews with students and their English and Math teachers showed that the proposed type of test is suitable for CLIL assessment. We assume that further research with more participants will follow the presented study.

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Estelle Szafran-Florian
European High School in Krakow

Chapter 4

MATHEMATISATION AND MODELLING – COMPARING THE PERFORMANCES OF IB DP AND POLISH PROGRAMME STUDENTS

Summary: The many differences between the Polish mathematical education programme and the IB DP (International Baccalaureate Diploma Programme) mathematics curriculum prompted a research regarding the differences in students' performances. The main focus is on the topic of functions and their use as a tool for mathematisation and modelling.
Mathematisation in literature is described as the process of adapting reality to a mathematician's needs (Freudenthal, 2002), while modelling is a commonly used tool in mathematisation, which can help build a bridge between the mathematical and non-mathematical world (Niss et al., 2007). Students in the IB DP following the Mathematics: Applications and Interpretation course are more often exposed to activities requiring them to use mathematics in a real-life context or to build a model than their peers following the Polish mathematics curriculum. It was therefore assumed that the IB DP students would achieve better results in exercises placed in a real-life context, while the Polish curriculum students would perform better in exercises placed in an entirely mathematical context.
The assumptions above have been verified by a study in which the author asked both IB DP and Polish curriculum students to solve a research questionnaire composed of nine questions from the domain of functions, which are based either in a real-life or mathematical context and require the use of various skills. The author then compared the results and came to the conclusion that differences in skills are not entirely representative of the differences in the curricula. It is possible that the differences that did appear are a reflection of the differences in the entrance performance of the students, or their motivation to learn mathematics.
In the report the differences between curricula will be shortly discussed, as well as the structure and methodology of the study and the results and conclusions obtained.

Keywords: mathematisation, modelling, functions, application of mathematics, secondary school students, Polish and IB DP curriculum.

1. Introduction

This chapter regards the topic of mathematisation (Krygowska, 1979, p. 48) and modelling (Niss et al., 2007, p. 4), or more specifically, the ability of students to carry out those processes with the use of functions. The focus of the chapter will be on an empirical study conducted in 2022 by the author. The study required the participation of high school students, who studied either in the Polish national curriculum or were part of the IB DP (*International Baccalaureate Diploma Programme*) and thus, studied mathematics according to the international curriculum of the programme.

The motivation behind this study was to better understand how differences in curricula may affect a student's skill set and how different approaches to teaching mathematics impact a student's thinking. Teachers should always hope to optimise their teaching in order to achieve better results with their students and comparing different methods of teaching might be one of the ways to achieve this goal. Moreover, modelling is a key ability in mathematics (Blum, 1993).

The IB group students who participated in the study are all students in the course Mathematics: Applications and Interpretation (AI), which is one of the two mathematics subjects proposed by the IB DP. It was selected over the course Mathematics: Analysis and Approaches (AA), as the AA curriculum is much more similar to the Polish national curriculum and thus, it is likely that less differences would be observed. Anytime the IB syllabus is mentioned onwards, it will always refer to the curriculum of the Mathematics AI subject.

The main difference observed between the IB and Polish national curricula is an entirely different approach to the context given to mathematics. IB prefers a much more realistic approach, with exercises set in a real-life setting, in which mathematics is only a tool to draw conclusions significant in the given context. On the other hand, the Polish national curriculum has a more analytical approach, with problems rarely having any context other than the required mathematical information. Apart from analysing the content of the respective syllabuses, those differences can be observed when comparing questions from the programmes' respective end-of-course examinations. Examples are shown below.

Figure 1. Example question from the Polish end-of-course examination

Source: Matura examination 2022, standard level

Zadanie 11. (0–1)

Miejscem zerowym funkcji liniowej f określonej wzorem $f(x) = -\frac{1}{3}(x+3)+5$ jest liczba

A. (-3) **B.** $\frac{9}{2}$ **C.** 5 **D.** 12

Text translation

The zero of the linear function *f*, given with the equation $f(x) = -\frac{1}{3}(x+3)+5$ is the number

In the question above, the student is asked to choose the given function's zero among the proposed answers. No real-life context is deemed necessary – this is a strictly algebraic question, which verifies whether the student knows how to identify the zero of a function (or how to find it).

Figure 2. Example question from the AI end-of-course examination

(Source: Examination for AI SL, May 2021, Paper 1, standard level)

4. [Maximum mark: 7]

The price of gas at Leon's gas station is \$1.50 per litre. If a customer buys a minimum of 10 litres, a discount of \$5 is applied.

This can be modelled by the following function, L, which gives the total cost when buying a minimum of 10 litres at Leon's gas station.

$$L(x) = 1.50x - 5,\ x \geq 10$$

where x is the number of litres of gas that a customer buys.

(a) Find the total cost of buying 40 litres of gas at Leon's gas station. [2]

(b) Find $L^{-1}(70)$. [2]

The price of gas at Erica's gas station is \$1.30 per litre. A customer must buy a minimum of 10 litres of gas. The total cost at Erica's gas station is cheaper than Leon's gas station when $x > k$.

(c) Find the minimum value of k. [3]

The question above also concerns linear functions, but is set in a real-life context. In part (a) and (c), it doesn't ask the student to find the value of the function for the argument 40 or find the intersection point of the graphs of two functions. Rather, it asks a question in the given context, and the student has to understand by themself what this means in a mathematical context. Since it is postulated that translating information from an extra-mathematical context to a mathematical one is the first step of any modelling activity (Niss et al., 2007, p. 4), this means that IB students are very often subject to mathematical modelling, as opposed to Polish curriculum students. With this information in mind, the methodology of the study was designed.

2. Methodology

The study took place in April of 2022. The research group consisted of students following the IB DP and students following the Polish national curriculum. In total, 149 senior students from 10 different schools in Cracow and Warsaw, Poland, filled out the research questionnaire. The structure of the research group (with division into subgroups) is shown in the table below.

Table 1. Structure of the research group

	SL	HL
PL	60	53
AI	32	4

The research group was divided into four subgroups:

- PL SL – standard level students following the Polish national curriculum
- PL HL – higher level students following the Polish national curriculum
- AI SL – students in the IB DP following the Applications and Interpretation course at the standard level
- AI HL – students in the IB DP following the Applications and Interpretation course at the higher level

Unfortunately, the AI HL group consisted only of 4 students, which would result in a very low statistical significance. As shown in the "results" part of this chapter, their answers were analysed and included in the general results of the whole research group, but not analysed separately.

As mentioned, the study focused on the strategies of solving questions from the topic of functions implemented by IB DP and Polish curriculum students. The aim of the study was to identify the differences between and the correctness of the methods used by these students.

The main research question was: **Does the ability to solve tasks from the topic of functions differ between IB DP and Polish programme senior (grade 12) students?**

As the topic of functions is broad, it was decided to limit the study to the matters of domain and range of functions, given in algebraic or verbal form, as well as the topic of linear functions.

The main research question was later supported with further, more detailed research questions:

1. What are the mathematisation abilities of students in the compared groups in terms of transforming and using information in a:
   a. Mathematical context,
   b. Real-life context?
2. What are the modelling abilities of students in the compared groups in terms of building and using a model based on information given in a:
   a. Mathematical context,
   b. Real-life context?
3. What are the abilities of students in the compared groups in terms of recognising functions, specifically:
   a. Deciding whether a relation given in algebraic form is a function,
   b. Deciding whether a relation given in verbal form is a function,
   c. Using theoretical knowledge about relations and functions?

There is not much pre-existing research on the investigated topic, hence the hypotheses were formulated based on personal experience with the Polish and IB programmes, and a thorough analysis of textbooks and curricula (Dobrowolska et al., 2019; Haese et al., 2007; Kurczab et al., 2019; International Baccalaureate Organisation, 2017). The hypotheses formulated for the study are shown below.

1. The students following the Polish national curriculum will achieve better results in terms of the abilities to:
   a. Transform and use information in a mathematical context,
   b. Build and use a model based on information given in a mathematical context,
   c. Decide whether a relation given in algebraic form is a function.
2. The students following the IB DP curriculum will achieve higher results in terms of the abilities to:
   a. Transform and use information in a real-life context,
   b. Build and use a model based on information given in a real-life context,
   c. Decide whether a relation given in verbal form is a function.

No hypothesis was formulated to accompany hypothesis 3.c., as the analysis of textbooks and curricula did not provide enough information to suspect one of the groups would perform better in terms of using their theoretical knowledge of functions.

The next step in the study was to formulate the variables used and their indicators. For an independent variable, the research used the programme followed by the student (Polish or IB), as well as the student's course level (standard level or higher level). The dependent variables used relate directly to the detailed research questions. The indicators refer to the research questionnaire, which is briefly described below the table.

Table 2. Variables and indicators

Main dependent variable	Indicator
The ability to solve questions from the topic of functions	The sum of points obtained for all questions
Detailed dependent variables	**Indicators**
The ability of mathematisation in a mathematical context	The points obtained for the question verifying the ability to transform and use information given in a mathematical context (question 2)
The ability of mathematisation in a real-life context	The sum of points obtained for questions verifying the ability to transform and use information in a real-life context (questions 3, 8)
The ability of modelling in a mathematical context	The points obtained for the question verifying the ability to build and use a model based on information given in a mathematical context (question 9)
The ability of modelling in a real-life context	The points obtained for the question verifying the ability to build and use a model based on information given in a real-life context (question 1)
The ability to recognise functions among relations given algebraically	The points obtained for the question verifying the ability to decide whether a relation given in algebraic form is a function and justifying their answer (question 4)
The ability to recognise functions among relations given verbally	The points obtained for the question verifying the ability to decide whether a relation given in verbal form is a function and justifying their answer (question 7)
The ability to use theoretical knowledge about functions	The sum of points obtained for questions verifying the ability to use knowledge on functions to answer theoretical questions in a mathematical context (questions 5, 6)

The research method used was a survey, and the research tool was a questionnaire (Nowak, 1981, p. 106). A key part of building the questionnaire and selecting the topic of the study was to accommodate the differences between curricula. The AI SL students are allowed to use, throughout their course and during examinations, a graphic display calculator (GDC), often being expected to use it above classic algebraic methods. For example, AI SL students are not required to know how to solve quadratic equations algebraically, as they are expected to use their GDC to solve them. It is a powerful tool that provides a lot of help, and allowing IB students to use it while denying it for the Polish group, would cause a large imbalance between the two groups. It was necessary to choose a topic which the IB group is used to working on without the use of their GDC. This is why, as previously mentioned, the selected topic is the domain and range of functions given in algebraic or verbal form, as well as linear functions. The students were also asked to identify whether a relation is a function at all, when provided as an equation (for example: $x = 4 - 0.75y$) or in verbal form (for example: "To a student in your class are assigned the names of their siblings").

The research questionnaire consisted of 9 questions (mathematisation – 3 questions, modelling – 2 questions, function knowledge – 4 questions). The students were expected to solve the questionnaire in 45 minutes (it was necessary to accommodate the duration of periods in Poland – this is the typical duration of one lesson). The students could not use a calculator. Each of the exercises in the questionnaire corresponds to one of the detailed research problems shown above. A few example questions from the questionnaire were selected for the purpose of a more detailed description in this chapter and are shown below. A full version of the questionnaire is provided as the appendix to this chapter. The example questions below are numbered according to the numbering in the questionnaire.

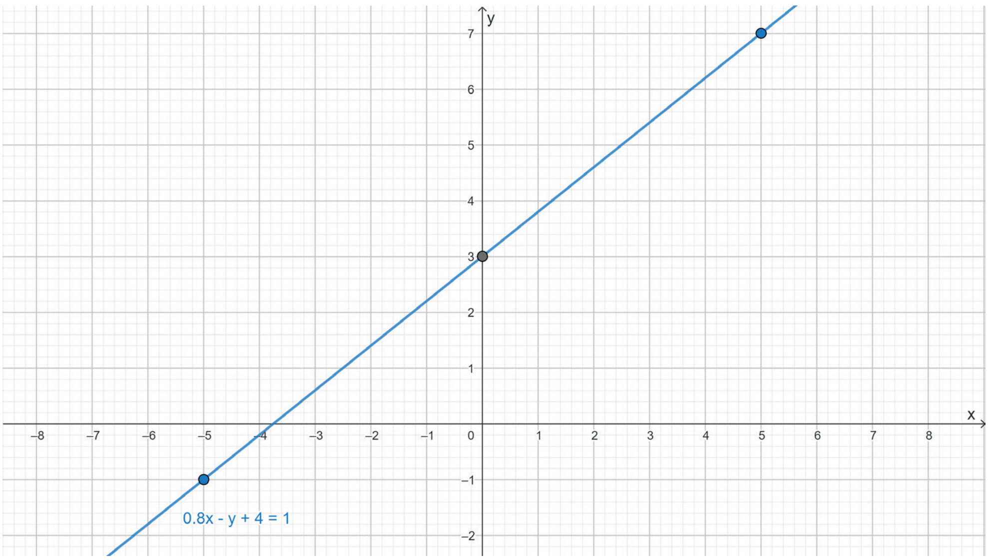
Question 2 – Transforming and Using Information Given in a Mathematical Context

2. Consider the line with equation $0.8x - y + 4 = -4$
   a. Draw this line in the coordinate system below. Write down all necessary calculations. (In the questionnaire a coordinate system was provided with the question)
   b. Find the equation of a linear function whose graph is parallel to this line.
   c. Find the equation of a linear function whose graph is perpendicular to this line.

The main difficulty in the exercise above was the form in which the line equation was given. The students are usually familiar with the gradient-intercept form $y = mx + b$, where m is the gradient and b is the y-intercept of the graph of the function. IB students are also accustomed to the point-slope form of a line equation $y - y_0 = m(x - x_0)$, where m is the gradient of the line and (x_0, y_0) are the coordinates of any point belonging to the graph of the function. The general form $Ax + By + C = 0$, where $A^2 + B^2 \neq 0$ is also used on a few occasions. Since the equation from the question was not given in any of these forms – some students did not recognise this function as linear.

Ad. (a) The students could, but did not have to, decide to transform the equation of the line to gradient-intercept or point-slope form to find the gradient of the line. There are infinitely many possibilities in terms of the points the students could decide to find in this question in order to draw the graph of the given line. Common points include (0,3), (5,7) and (-5,-1) due to their integer coordinates. A correct graph of the given line is shown below:

Figure 3. Solution to question 2, part (a)



Ad. (b) If the student did not do this in part (a), this is where they should transform their line equation, for example to a gradient-intercept form, to find the value of the gradient:

$$0.8x - y + 4 = 1$$
$$y = 0.8x + 3.$$

As can be seen, the given line has the gradient 0.8. Other options to find the gradient are possible – such as using the points found in part (a), but none of the students decided to use this method. Part (b) of this question aimed to verify the students' knowledge about the properties of gradients – specifically, that when two lines are parallel, their gradients are equal. Four different types of answers were expected in this question, shown below, with examples.

Table 3. Solutions to question 2, part (b)

Exact value for the *b* coefficient, different than 0 or 3, i.e.:	Coefficient $b = 0$	General *b* coefficient	Line overlapping with the given line
$y = 0.8x + 4$ $y = 0.8x - 3$	$y = 0.8x$	$y = 0.8x + b$	$y = 0{,}8x + 3$

Ad. (c) Similarly to part (b), finding the correct solution depends on finding the appropriate gradient and the students' knowledge about the gradients of perpendicular lines. When two lines are perpendicular, their gradients satisfy the equation $m_1 \cdot m_2 = -1$. Following this equation, the student could identify the correct gradient for a line perpendicular to the one given.

$$m_2 = \frac{-1}{0.8} = \frac{-1}{\frac{4}{5}} = \frac{-5}{4}(= -1.25)$$

Again, four different types of correct answers could be expected, as shown below.

Table 4. Solution to question 2, part (c)

Exact value for the *b* coefficient, different than 0 or 3, i.e.:	Coefficient $b = 0$	General *b* coefficient	Same *b* coefficient as in the given line
$y = \frac{-5}{4}x + 4$ $y = \frac{-5}{4}x - 7$	$y = \frac{-5}{4}x$	$y = \frac{-5}{4}x + b$	$y = \frac{-5}{4}x + 3$

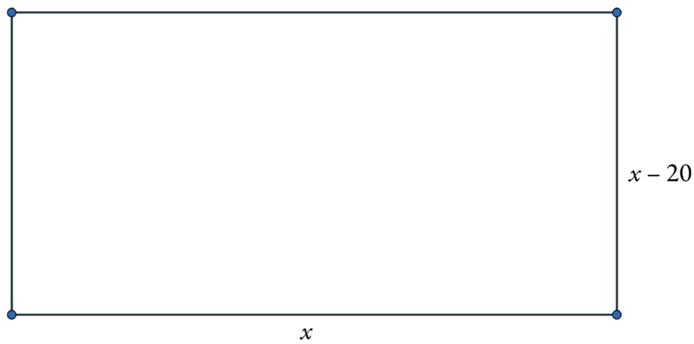
Question 9 – Building and Using a Model in a Mathematical Context

9. A rectangle has a length of x and its width is 20 units shorter than the length.
   a. Write down the equation of a function describing the perimeter of this rectangle in terms of x.
   b. What is the domain of this function?
   c. Is it possible to find the smallest perimeter of this rectangle? Circle the right answer (YES / NO) and justify. If you circle YES, calculate the smallest perimeter.
   d. Is it possible to find the largest perimeter of this rectangle? Circle the right answer (YES / NO) and justify. If you circle YES, calculate the largest perimeter.
   e. Write down the range of this function.

The question was formulated by the author specifically for this study. Its aim was to verify the students' performance in modelling within a purely mathematical context, with a simultaneous use of linear functions. Constructing the equation of the function should not be too challenging for the students. The hardest aspect might be part (c) – the student can be drawn to designating 20 as the smallest value of x, which is not possible, as the width would then be 0 units. Moreover, question 9 is the last one in the questionnaire, which means that the students may be tired or unmotivated when solving this question after already solving eight demanding problems.

Ad. (a) The rectangle described in the question can be sketched as follows:

Figure 4. Rectangle described in question 9



After sketching the figure, it can be easily calculated that the perimeter P of the rectangle is:

$$P(x) = 2x + 2(x - 20) = 2x + 2x - 40 = 4x - 40.$$

Ad. (b) To find the domain it is important to consider three conditions: the lengths of both sides and the perimeter of the rectangle must all be positive. The following results are obtained:

$$x > 0, \qquad \begin{gathered} x - 20 > 0 \\ x > 20, \end{gathered} \qquad \begin{gathered} 4x - 40 > 0 \\ 4x > 40 \\ x > 10. \end{gathered}$$

By finding the intersection of all three conditions, the domain of the function can be determined: $x \in (20, +\infty)$.

Ad. (c) Per the function $P(x) = 4x - 40$, which describes the perimeter of the rectangle as a linear function, the minimum perimeter can be found when the minimum value of argument x is known. The domain states that such a minimum value does not exist, hence it is not possible to find a minimum perimeter for this rectangle.

Ad. (d) Analogically to part (c), when searching for a maximum perimeter, the maximum value of argument x should be considered. Such a value cannot be defined in the domain, hence it is not possible to find a maximum perimeter.

Ad. (e) Despite not being able to find either a maximum or minimum volume, it is possible to find a range which matches the domain, as the function is linear and increasing.

$$P(20) = 4 \cdot 20 - 40 = 80 - 40 = 40$$

As mentioned in Ad. (c), 40 is not a value of this function, it is, however, a lower bound for the range. Furthermore, as the domain does not have an upper bound, the range will not have it either. The range of function P is then $P \in (40, +\infty)$.

Question 1 – Building and Using a Model in a Real-Life Context

1. At a sushi restaurant, nigiri costs \$4.50 per serve and sashimi costs \$9.00 per serve. Hiroko spent a total of \$45 buying x serves of nigiri and y serves of sashimi.
   a. Find a function describing the relation between the number of nigiri serves (x) and the number of sashimi serves (y) bought by Hiroko.
   b. If Hiroko bought 4 serves of *nigiri*, how much *sashimi* did she buy?
   c. If Hiroko bought 1 serve of *sashimi*, how much *nigiri* did she buy?
   d. Sketch the graph of the function you found in "a". Mark two points on your graph to indicate your answers to "b" and "c".

The question is a modified version of a question found in an IB textbook, which is shown below:

Figure 5. The question which formed the basis of question 1 in the questionnaire
(Source: Haese et al., 2019a, p. 28)

6 At a sushi restaurant, *nigiri* costs \$4.50 per serve and *sashimi* costs \$9.00 per serve. Hiroko spent a total of \$45 buying x serves of *nigiri* and y serves of *sashimi*.

a Explain why $4.5x + 9y = 45$.

b If Hiroko bought 4 serves of *nigiri*, how much *sashimi* did she buy?

c If Hiroko bought 1 serve of *sashimi*, how much *nigiri* did she buy?

d Draw the graph of $4.5x + 9y = 45$. Mark two points on your graph to indicate your answers to **b** and **c**.

The first modification applied to the base question was to change part (a) so that the function is not provided. This was done in order to give the students more freedom in choosing a way to approach the question while being able to investigate their ability to build a function. The question as a whole is very interesting from a modelling perspective. The graphic representation required in part (d) also shows whether the student understands the model they built.

Ad. (a) The main difficulty is to understand that the model needs to bind the number of serves bought to the amount of money spent. A correct answer is $4.50x + 9y = 45$ or any equivalent representation, such as the gradient-intercept form $y = 5 - 0.5x$.

Ad. (b) This part of the question is not of high difficulty and mainly correct answers should be expected. It is interesting to observe whether the students used the model built in part (a), or used a different method to answer the question. The correct answer to this question is three serves, which can be calculated with or without the use of the model, as shown below:

Table 5. Solution to question 1, part (b)

With use of the model	Without the use of the model
$x = 4$ $4.50 \cdot 4 + 9y = 45$ $18 + 9y = 45$ $9y = 27$ $y = 3$ OR $x = 4$ $y = 5 - 0.5 \cdot 4$ $y = 5 - 2$ $y = 3$	Total amount spent on sashimi and nigiri: \$45 Amount spent on 4 serves of nigiri: 4 · \$4.50 = \$18 Amount of money left: \$45 – \$18 = \$27 Amount of serves of sashimi: \$27 : \$9 = 3 serves.

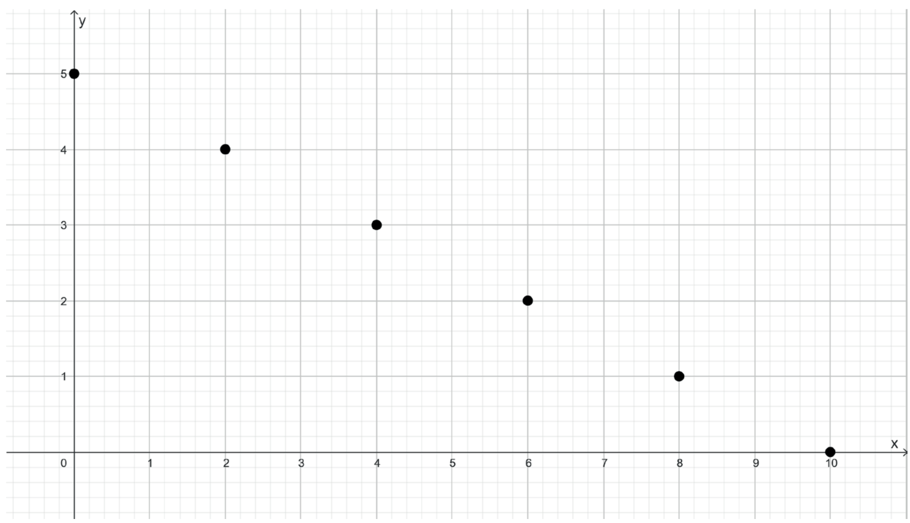
Ad. (c) Analogically as in part (b), this part also did not have a high level of difficulty and mostly correct answers are expected. Again, the student could calculate the correct answer with or without the use of their model, as shown below.

Table 6. Solution to question 1, part (c)

With use of the model	Without the use of the model
$y = 1$ $4.50x + 9 \cdot 1 = 45$ $4.50x + 9 = 45$ $4.50x = 36$ $x = 8$ OR $y = 1$ $1 = 5 - 0.5x$ $-4 = -0.5x$ $x = 8$	Total amount spent on sashimi and nigiri: \$45 Amount spent on 1 serve of sashimi: 1 · \$9 = \$9 Amount of money left: \$45 – \$9 = \$36 Amount of serves of sashimi: \$36 : \$4.50 = 8 serves

Ad. (d) The last part of the first question is probably the hardest part of this problem. The students are expected to draw their model in a coordinate system. In part (a) they should observe that the relation is of a linear type. The difficulty is in identifying a correct domain and range for their function. It is impossible to purchase a negative or non-integer amount of serves. From parts (b) and (c) the student already knows two points lying on the graph of the function: (4,3) and (8,1). The next step is not to draw a line through these two points, but to fill the graph with points representing other possibilities that still suit the context of the question – (0,5), (2,4), (6,2), and (10,0). It is worth observing that the value, the amount of serves of nigiri can only be an even number. This is due to the range of the function – if Hiroko tries to buy an uneven amount of serves of nigiri, she will have to buy a non-integer amount of serves of sashimi, or not spend the whole $45. Both situations do not fit the conditions of the question. A correct graph of the function is shown below:

Figure 6. Solution to question 1, part (d)



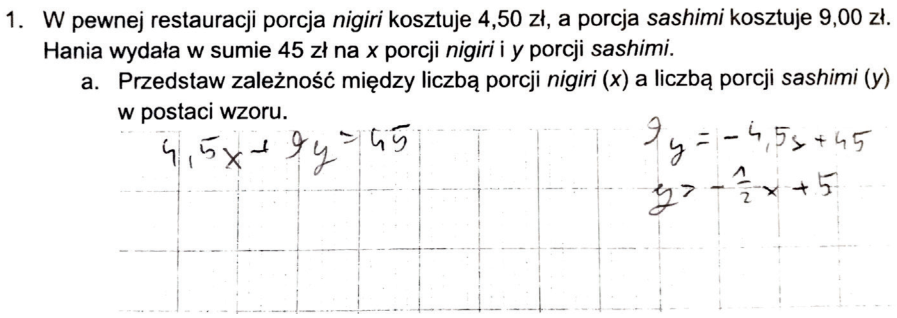
3. Results

When analysing the questionnaire as a whole, it is worth noting that the questions used were not standard, textbook questions. They required the students to use their knowledge in an advanced, non-algorithmic way. The questions also required the students to frequently justify their answers, which can be a very demanding activity, to which they may not be accustomed. For the purpose of demonstrating the process of analysing the answers provided by the students, a detailed description of the results of question 1, part (a) is shown below. Later, more general results are discussed.

Question 1 – Results

Ad. (a) Firstly, the answers provided by the students were analysed and categorised. The answer was either labelled as correct (1 point awarded) or incorrect (no points awarded). Incorrect answers were further categorised depending on the type of mistake made by the student. An example of a correct answer provided by a student is shown below. The questionnaires were available both in Polish and in English to accommodate the students' needs, which is why on the figures, the question and/or answer provided might be in either of the languages. The questions in Polish are an exact translation of the questions shown above in English.

Figure 7. A correct answer provided by a student to question 1, part (a)


1. W pewnej restauracji porcja *nigiri* kosztuje 4,50 zł, a porcja *sashimi* kosztuje 9,00 zł. Hania wydała w sumie 45 zł na *x* porcji *nigiri* i *y* porcji *sashimi*.
   a. Przedstaw zależność między liczbą porcji *nigiri* (*x*) a liczbą porcji *sashimi* (*y*) w postaci wzoru.

$4,5x + 9y = 45$

$9y = -4,5x + 45 \quad /:9$

$y = -\frac{1}{2}x + 5$

In this question, the student first wrote down the 4.5x + 9y = 45 form of the equation, then decided to transform it to gradient-intercept form.

The table below shows the average result achieved by each of the research subgroups.

Table 7. Average score achieved in question 1, part (a)

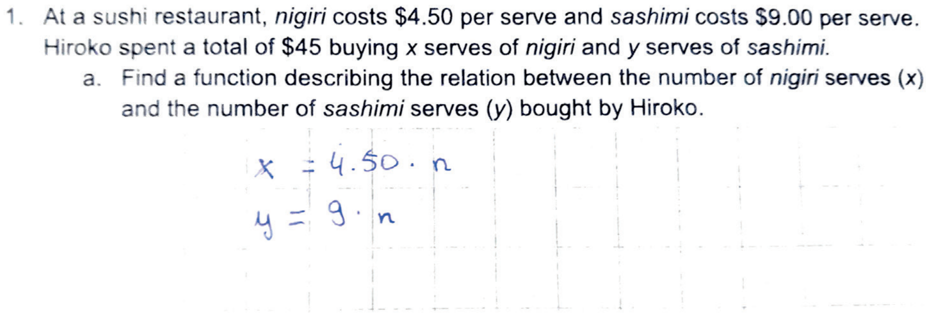
	AI SL	PL SL	PL HL	General
Average score	68,75%	66,67%	86,79%	74,50%

In terms of incorrect answers, 4 types of mistakes were identified. They are described below.

- **No function equation**

An answer was put in this category if a student showed some work, but did not write down an equation connecting the variables. Four students committed this mistake. An example is shown below.

Figure 8. Incorrect answer to question 1, part (a) – no function equation


1. At a sushi restaurant, *nigiri* costs $4.50 per serve and *sashimi* costs $9.00 per serve. Hiroko spent a total of $45 buying *x* serves of *nigiri* and *y* serves of *sashimi*.
   a. Find a function describing the relation between the number of *nigiri* serves (*x*) and the number of *sashimi* serves (*y*) bought by Hiroko.

$x = 4.50 \cdot n$

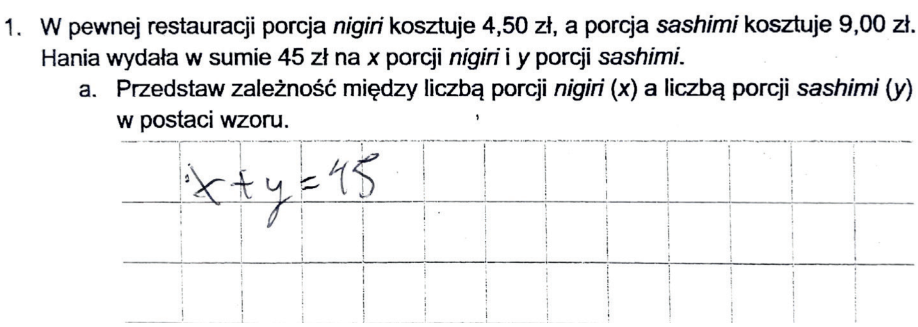
$y = 9 \cdot n$

In the example above, the student did not use variables *x* and *y* as indicated in the question. Their work implies that here, *x* would mean the amount of money spent on nigiri, and *y* the amount of money spent on sashimi. Furthermore, the student did not include information regarding spending $45 on the food in total and did not include a relation between the two variables. They also falsely assumed that the same amount of serves of sashimi and nigiri was bought. The answer is incorrect.

- **Price not included or false**

A student's answer was labelled with this type of mistake if they did not include the price of serves in their model or if they included incorrect prices. Five students committed this type of mistake. An example is shown below.

Figure 9. Incorrect answer to question 1, part (a) – price not included


1. W pewnej restauracji porcja *nigiri* kosztuje 4,50 zł, a porcja *sashimi* kosztuje 9,00 zł. Hania wydała w sumie 45 zł na *x* porcji *nigiri* i *y* porcji *sashimi*.
   a. Przedstaw zależność między liczbą porcji *nigiri* (*x*) a liczbą porcji *sashimi* (*y*) w postaci wzoru.

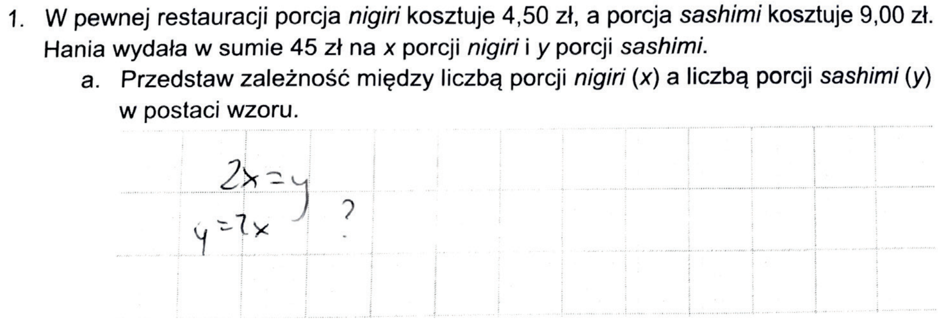
$x + y = 45$

The model shown in Figure 9 would suggest that 45 is the number of serves bought, not the amount of money spent on those serves. The answer is incorrect.

- **Equation describing the relation between the prices**

A student committed this type of mistake if instead of showing a model relating the number of serves bought to each other and the total amount of money spent, they built a model connecting the prices of serves to each other. Fourteen students committed this error. An example is shown below.

Figure 10. Incorrect answer to question 1, part (b) – relation between prices of nigiri and sashimi


1. W pewnej restauracji porcja *nigiri* kosztuje 4,50 zł, a porcja *sashimi* kosztuje 9,00 zł. Hania wydała w sumie 45 zł na *x* porcji *nigiri* i *y* porcji *sashimi*.
   a. Przedstaw zależność między liczbą porcji *nigiri* (*x*) a liczbą porcji *sashimi* (*y*) w postaci wzoru.

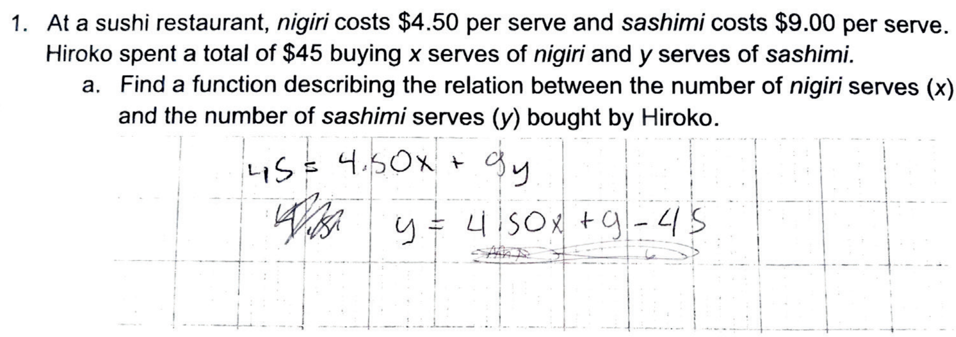
2x = y
y = 2x ?

In the example above, the student misused the variables provided in the questions – it could be concluded that *x* represents the price of a serve of nigiri, while *y* represents the price of a serve of sashimi. As the prices are constant and provided in the question, there is no purpose to build a model like this one – the answer is incorrect.

- **Calculation error**

This type of mistake happened if a student used a right method, but made a calculation error in their work. Four students made such a mistake. An example is provided below.

Figure 11. Incorrect answer to question 1, part (a) – calculation error


1. At a sushi restaurant, *nigiri* costs $4.50 per serve and *sashimi* costs $9.00 per serve. Hiroko spent a total of $45 buying *x* serves of *nigiri* and *y* serves of *sashimi*.
   a. Find a function describing the relation between the number of *nigiri* serves (*x*) and the number of *sashimi* serves (*y*) bought by Hiroko.

45 = 4.50x + 9y
y = 4.50x + 9 - 45

In the example above, the student provided a correct model, but then failed to transform it to the gradient-intercept form. Because of their subsequent work being wrong, and because the student underlined the transformation as their final answer, this answer was deemed incorrect.

The table below shows the types of mistakes made by the students in each of the subgroups.

Table 8. Error types in question 1, part (a)

Type of error	AI SL	PL SL	PL HL	Sum
No function equation	2	2	0	4
Price not included or false	1	0	4	5
Equation describing the relation between the prices	2	10	1	13
Calculation error	4	0	0	4
No answer provided	1	8	2	11
Sum	10	20	7	37

As can be observed, the students from the AI SL group most often committed calculation errors, while the students in the PL SL group most often built equations describing the relation between prices. Among the students in the PL HL group with incorrect answers, not including the price was the most common mistake.

The table below shows the results achieved by the students in question 1 (all parts)

Table 9. Average results in question 1

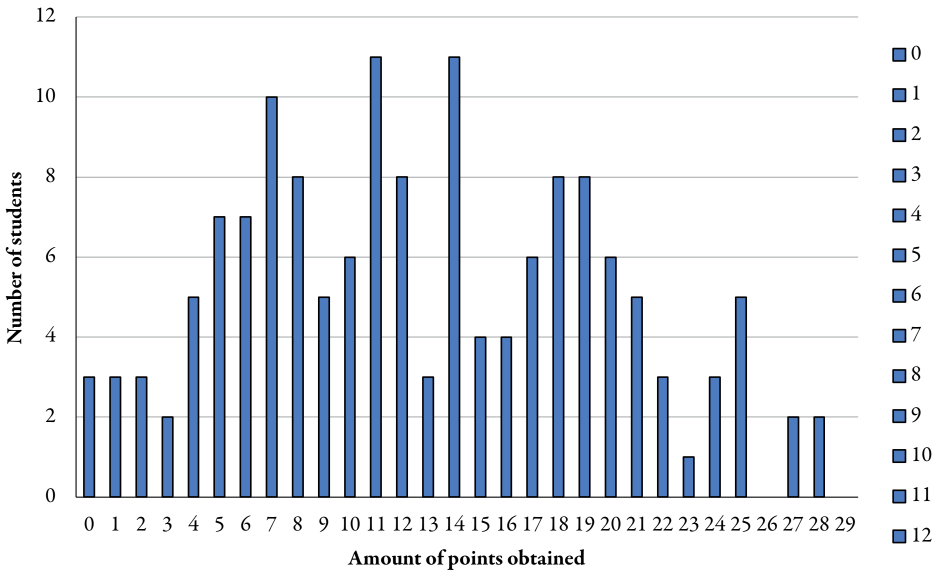
	AI SL	PL SL	PL HL	General
Average score	60,25%	59,5%	70,75%	63,5%

This question showed an interesting structure. As assumed, the results achieved in parts (b) and (c) were quite high, but the average score for the question as a whole was greatly lowered by the answers provided to part (d). In part (d), only 8 students from the entire research group provided a correct answer. It is important to note that the assumed difficulty of the last part was much higher than of the previous parts, as the students are rarely used to drawing graphs with a discrete domain. Regarding the results of question 1, one can notice that the higher-level group obtained the highest average score, while the AI SL and PL SL groups achieved very similar scores.

General Results

A student could obtain a maximum of 29 points for all correct answers throughout the questionnaire. The histogram below shows the results obtained by all 149 students who took part in the experiment.

Figure 12. Score obtained by each student



The scores observed are relatively evenly distributed. Only four students (2.68% of the group) achieved a score higher than 26 points (90%). A relatively large number, 11 students (7.38% of the group) achieved scores lower than 4 points (13.8%). Most students got either 11 (37.93%) or 14 points (48.28%). The next most frequent score was 7 points (24.14%), achieved by 10 students (6.71% of the group). The lowest score was 0 points, achieved by 3 students (2.01% of the group), while the highest score was 28 points, achieved by 2 students (1.34% of the group). No student achieved the maximum possible score.

The table below shows the average scores achieved by each of the subgroups.

Table 10. Average scores with respect to subgroups

	AI SL	PL SL	PL HL	General
Average score	30,71%	36,55%	61,68%	44,06%

The lowest average score was observed in the AI SL group. The PL SL group achieved a slightly higher result, while the highest result, as expected, was observed in the PL HL group.

Detailed Results

What is more interesting than the results achieved by the subgroups in the whole questionnaire are the detailed results for each of its parts. Analysing those results allows us to address the hypotheses stated at the beginning of this chapter. The table below shows the results obtained by each group in the parts regarding mathematical contexts.

Table 11. Summary of indicators – questions 2, 9, 4

	AI SL	PL SL	PL HL
Question 2 (Hypothesis 1.a)	27,07%	56,11%	**79,87%**
Question 9 (Hypothesis 1.b)	19,38%	26%	**68,68%**
Question 4 (Hypothesis 1.c)	7,81%	15,83%	**41,04%**
General	17,45%	30,14%	**62,26%**

In all three questions, the PL SL group obtained significantly higher results than the standard-level groups. When comparing only the standard-level groups, in these questions, PL SL students obtained, on average, higher scores than their IB peers. The overall score of the PL SL group for this part of the questionnaire is almost twice the score of the AI SL group. The overall results of the SL groups are very low.

It is possible to confirm hypotheses 1.a, 1.b, 1.c based on the results of the study. Students following the Polish national curriculum achieved better results in all three investigated aspects.

The table below shows the indicators linked to hypotheses 2.a, 2.b., 2.c, which regarded mathematical performance in a non-mathematical context.

Table 12. Summary of indicators – questions 1, 3, 7, 8

	AI SL	PL SL	PL HL
Questions 3 and 8 (Hypothesis 2.a.)	49,55%	46,90%	**66,31%**
Question 1 (Hypothesis 2.b.)	60,16%	59,58%	**70,75%**
Question 7 (Hypothesis 2.c.)	4,69%	12,5%	**31,13%**
General	40,41%	41,11%	**58,11%**

Again, in all three aspects the PL HL groups achieved the best results. Comparing only the SL groups, the only significant difference can be observed in question 7 (identifying a function among relations given in verbal form). Both results for this question are very low, but the AI SL group managed to achieve a score more than twice lower than the score of the PL SL group. The overall scores of both groups are comparable.

In relation to the hypotheses, it can be stated that the Polish higher level group achieved higher results than the standard-level groups. There is insufficient evidence to state whether IB students achieved better results in this part of the questionnaire, as the results are too similar. It could be possible to state that the Polish students have better abilities in the domain of identifying functions given in verbal form, but, due to the very low score of both SL groups, this would be a very risky statement.

The last investigated variable concerned the use of theoretical knowledge of functions. The results of the indicators are shown below.

Table 13. Summary of indicators – questions 5 and 6

	AI SL	PL SL	PL HL
Questions 5 and 6 (Ability to use theoretical knowledge about functions)	18,75%	18,33%	**52,83%**

In this part of the questionnaire, the PL HL group achieved the highest results as well, placing more than 30 percentage points higher than the two SL groups. Comparing only the SL groups, the difference between them is insignificant.

Apart from stating that the HL group clearly performed better in using their theoretical knowledge of functions than the SL groups, it is not possible to draw any conclusions comparing the IB group to the PL group for this part of the experiment.

4. Conclusions

After analysing the answers provided by the students, it was possible to answer the research questions and draw conclusions from the study.

For the detailed research questions, it was concluded that students following the Polish national curriculum show better abilities in terms of modelling and mathematisation in a mathematical context, as well as in recognising functions given in algebraic form, than their IB peers. Both standard-level groups showed very similar, quite low, achievements in terms of modelling and mathematisation in a real-life context, as well as in recognising functions given in verbal form. The higher-level group showed a significantly better performance in using their theoretical knowledge of functions, while the SL groups, again, showed very similar abilities.

It was therefore possible to answer the research question: **Does the ability to solve function exercises differ between IB DP and Polish programme senior (grade 12) students?** A satisfying answer seems to be – partially, yes. Higher-level students showed, unsurprisingly, much better results throughout the questionnaire. But the most important comparison is between the two SL groups – in some aspects, the Polish students showed better abilities than the IB students, while in other aspects, their results were very similar. The average scores from the whole questionnaire seem to indicate a small advantage of the PL SL group in comparison to the AI SL group.

Apart from answering the research questions, other general conclusions can be drawn as well:

- The level of skill shown by the students from the research group was alarmingly low. The study does not investigate the reason for such low results, but to some extent, it is possible to make assumptions based on the timing of the study. The low results can either be a result of remote learning, or the unwillingness of senior students to work on something non-mandatory and non-graded near the end of their school year (the study took part mid-April, and senior students in Poland finish their school year at the end of April to start final examinations in May). It is also possible that students are not used to the type of questions used in the questionnaire. Most questions called for a justification of their work and some were more theoretical than what is typically used in school.
- It is not possible to efficiently compare the PL HL group, which studies mathematics on a higher level than the PL SL and AI SL groups, which study mathematics on a standard level. To improve the conclusions of this study, better participation would be required in the AI HL group in order to collect statistically significant results. The results obtained by the AI HL group would be a suitable comparison to the PL HL group.

Acknowledgement

This chapter was written on the basis of a master thesis written in the year 2022 under the supervision of Dr. Assoc. Professor Mirosława Sajka at the Department of Mathematics, University of the National Education Commission (previously: Pedagogical University) in Krakow. This endeavour, both in the form of the full master thesis and this chapter, would not have been possible without Professor Sajka's invaluable feedback and assistance.

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Appendix – Research Questionnaire

RESEARCH QUESTIONNAIRE

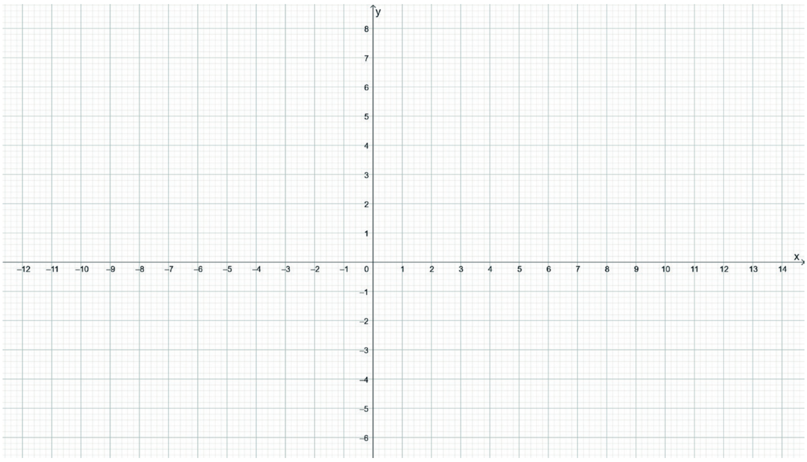
Name: ______________________ School: ______________________

1. At a sushi restaurant, *nigiri* costs \$4.50 per serve and *sashimi* costs \$9.00 per serve. Hiroko spent a total of \$45 buying x serves of *nigiri* and y serves of *sashimi*.
   a. Find a function describing the relation between the number of *nigiri* serves (x) and the number of *sashimi* serves (y) bought by Hiroko.

   b. If Hiroko bought 4 serves of *nigiri*, how much *sashimi* did she buy?

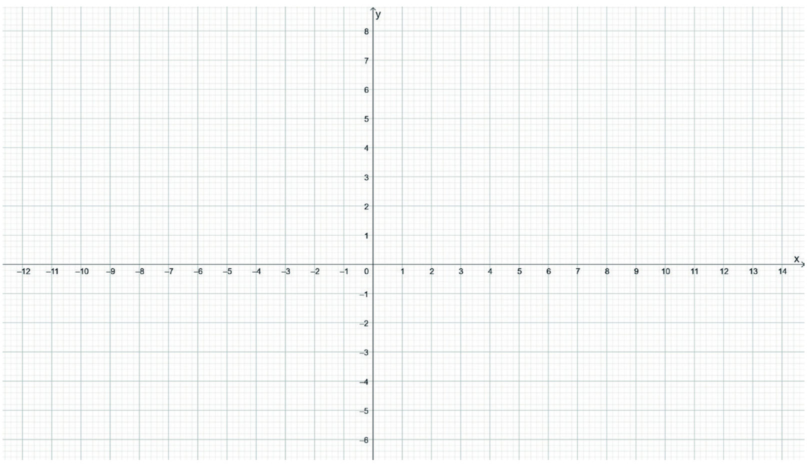
   c. If Hiroko bought 1 serve of *sashimi*, how much *nigiri* did she buy?

d. Sketch the graph of the function you found in “a”. Mark two points on your graph to indicate your answers to “b” and “c”.



2. Consider the line with equation $0.8x - y + 4 = -4$.

 a. Draw this line in the coordinate system below. Write down all necessary calculations.



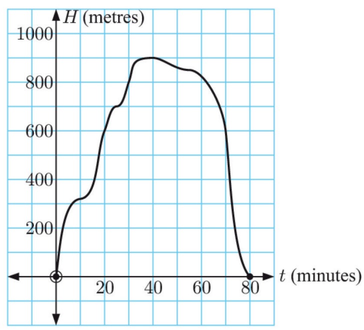
Calculations:

b. Find the equation of a linear function whose graph is parallel to this line

c. Find the equation of a linear function whose graph is perpendicular to this line

3. For a hot air balloon ride, the function gives the height of the balloon after minutes. Its graph is shown alongside.

   a. Find $H(30)$ and explain your answer in the context of the hot air balloon ride.



   b. Find the values of t such that $H(t) = 600$. Explain your answer in the context of the hot air balloon ride.

c. What range of heights was recorded for the balloon?

d. How long was the balloon ride?

e. Can you read the distance travelled by the balloon from the graph of function H? Circle YES or NO below and justify your answer.
YES / NO, because ....
....
....

4. Determine whether these relations are functions of variable x. Circle the right answer: YES (the relation is a function) or NO (the relation is not a function). Justify your answers.
   a. $y = x^2 - 9$ YES / NO, because ....
   ....
   b. $2x - 5y = 1.7$ YES / NO, because ....
   ....
   c. $4x + y^2 = 1$ YES / NO, because ....
   ....
   d. $x = 4 - 0.75y$ YES / NO, because ....
   ....

5. Is it possible for the graph of a function of a variable to have more than one y-intercept? Circle the right answer and justify.
   YES / NO, because ....
   ....
   ....

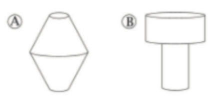
6. Is a straight line always the graph of a linear function? Circle the right answer and justify.
YES / NO, because ....
....
....

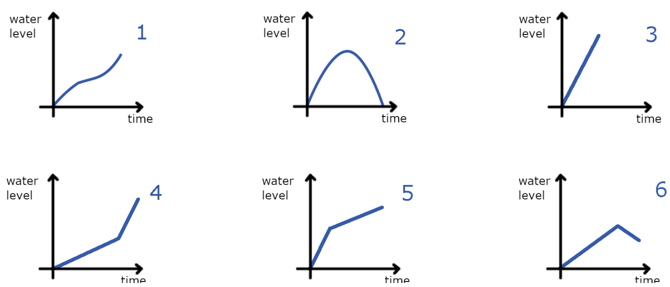
7. Decide whether the relations described below are functions. Circle the right answer: YES (the relation is a function) or NO (the relation is not a function) and justify.
   a. The table shows the price of entering an amusement park depending on age.
   YES / NO, because ....
   ....
   ....

Age	*Cost*
0 - 2 years (infants)	$0
2 - 16 years (children)	$20
16+ years (adults)	$30

   b. The *area* of an equilateral triangle is assigned to the *length of its side*.
   YES / NO, because ....
   ....
   c. To the *number of pages* in a mathematics textbook is assigned the *number of sentences* on that page.
   YES / NO, because ....
   ....
   d. To a student in your class are assigned the names of their siblings.
   YES / NO, because ....
   ....

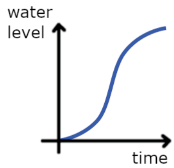
8. The graphs below represent how the water level changes in a vessel as it is being filled. Assume the water stream is steady.
   a. Assign the appropriate graph to the vessels below.





   Vessel A – graph number ....
   Vessel B – graph number ....

b. Draw a vessel which could be assigned to the graph below.



9. A rectangle has a length of x and its width is 20 units shorter than the length.
   a. Write down the equation of a function describing the perimeter of this rectangle in terms of x.

   b. What is the domain of this function?

   c. Is it possible to find the smallest perimeter of this rectangle? Circle the right answer and justify. If you circle YES, calculate the smallest perimeter.
   YES / NO, because ....
   ....
   ....

   Calculations:

d. Is it possible to find the largest perimeter of this rectangle? Circle the right answer and justify. If you circle YES, calculate the largest perimeter.
YES / NO, because ....
....
....

Calculations:

e. Write down the range of this function.

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PART II

Critical Thinking as an Integral Component of the Mathematical Cognition of Students

Bożena Maj-Tatsis
University of Rzeszów

Konstantinos Tatsis
University of Ioannina

Chapter 5

PERSPECTIVES ON YOUNG STUDENTS' MATHEMATICAL REASONING

Summary: According to relevant studies, children at all ages have the potential to engage in mathematical reasoning. What remains under discussion is what qualifies as mathematical reasoning, which relates to methods of identifying mathematical reasoning in children's activities. In the article, we present different ways to view mathematical reasoning, with a focus on children in the first years of primary school. Following relevant studies, we suggest ways to identify and analyse instances of mathematical reasoning among young students, with a focus on communication. We also present two significant factors that seem to affect the extent of mathematical reasoning, namely the role of the teacher and task characteristics. Finally, we present our own study on mathematical reasoning with young children to showcase the implementation of the selected analytical frameworks.

Keywords: mathematical reasoning, mathematical thinking, young students, task design.

1. Introduction

The involvement of children in meaningful tasks or, more generally, within meaningful contexts is a prerequisite for their engagement in important mathematical activities, even at a young age (Clements & Sarama, 2020). Powerful mathematical ideas, such as mathematisation, connections, argumentation, number sense and mental computation, algebraic reasoning, spatial and geometric thinking, or data and probability sense can be accessible to young children (Perry & Dockett, 2002). Among these ideas, the notion of *mathematical reasoning* prevails among many studies on mathematical thinking. It is, however, noteworthy that although the term "is widely used with the implicit assumption that there is universal agreement on its meaning", "most mathematicians and mathematics educators use this term without any clarification or elaboration" (Yackel & Hanna, 2003, p. 228). A common idea that accompanies many studies is that mathematical reasoning is actually deductive reasoning (Duval, 1991). At the same time, we can find studies that stress the importance of abductive reasoning in mathematical inquiry (Reid, 2003). Our own reading of papers related to mathematical reasoning has led us to studies that perceive reasoning as a subordinate activity of an overarching activity (e.g., Stylianides, 2008), as a discursive construct that possesses a dual nature of structure and process (Jeannotte & Kieran, 2017), or as an epistemologically sound notion that may be further categorised into various styles (Kollosche, 2021). Given the importance of mathematical reasoning, the diversity of approaches in mathematics education studies, as well as our own interest in young children's reasoning, we present various approaches in a comprehensible manner, with an aim to suggest ways to identify and analyse instances of mathematical reasoning among children. We will focus mainly on analytical frameworks, namely those that come with concrete methods on analysing data; however, no method can exist without a sound theoretical background, therefore we will also refer to the theories that underlie these approaches.

2. Perspectives on Students' Mathematical Reasoning

2.1. Reasoning Complementing or Constituting Mathematical Thinking

A large strand of research on mathematical thinking contains references to mathematical reasoning. The majority of these studies view reasoning as an element of the process of generalisation or conjecturing (Ellis, 2007a; Mason, 1982; Stylianides, 2008). Blanton and Kaput (2002) claim that "justification in any form is a significant part of algebraic reasoning because it induces a habit of mind whereby one naturally questions and conjectures in order to establish a generalisation" (p. 25). In most of these studies, reasoning is closely tied to proving; for instance, Stylianides (2008) uses the term "reasoning-and-proving" to denote the overarching activity that encompasses "'identifying patterns', 'making conjectures', 'providing non-proof arguments', and 'providing proofs'" (p. 9). Due to the significance given to other processes, there is no clear definition of mathematical reasoning in these studies, and, actually, in some cases, reasoning is conflated with argumentation and justification (Simon & Blume, 1996).

2.2. Reasoning as a Content-Based Notion

Studies on mathematical reasoning sometimes focus on its manifestation within a particular mathematical field. The most common case is algebraic reasoning. According to Kaput

(2008), the first core aspect of algebraic reasoning is "generalisation and the expression of generalisations in increasingly systematic, conventional symbol systems" (p. 10), whereas the second core aspect "is syntactically guided action on symbols within organised systems of symbols". The generalisations mentioned in the first aspect may refer to "arithmetic operations and their properties and reasoning about more general relationships and their forms" (p. 12) or even to mere computational strategies. However, there is also another possibility in which generalisation "can be thought of as describing systematic variation of instances across some domain" (p. 13). This leads to the notion of function, therefore some studies use the term *functional reasoning* to denote this specific strand of algebraic reasoning. We need to note at this point that although the term 'functional' may seemingly relate to higher-order mathematical concepts and advanced thinking, there is evidence that children as young as six years old "are capable of making sense of functional relationships, of representing these relationships in sophisticated ways, and of reasoning with symbolic relationships in novel situations" (Blanton et al., 2015, p. 545).

Other instances of content-oriented reasoning are: spatial and numerical reasoning (Battista et al., 2017), additive and multiplicative reasoning (Simon & Blume, 1996), and combinatorial reasoning (Batanero et al., 1997). Spatial and numerical reasoning may refer to measurements in two- or three-dimensional space, therefore can be related to angle, length, area and volume measurements and sense-making (Battista et al., 2017). Additive and multiplicative reasoning also involve measurements but are usually enriched with additive and multiplicative notions and understandings. As for combinatorial reasoning, research has shown that children, even during preschool and the first years of primary school, are able to articulate and work with combinatorial concepts, or, in other words, engage in combinatorial reasoning (Batanero et al., 1997; English, 1991; Fesakis & Kafoussi, 2009). In line with this, Maher and Martino (1996) describe the progression in a student's justifications from a trial-and-error approach to a more proof-oriented written justification. English (1991) identified six solution strategies in her study with 4 to 9-year-old children, ranging from random selection to a systematic pattern; she also noticed that although younger children did not manage to move beyond trial-and-error procedures, the "7- to 9-year-olds discovered systematic combinatorial procedures" (p. 471). This approach has been useful in one of our studies, as we will show later.

In addition to the above, Kollosche (2021) suggests a framework based on styles of reasoning, namely: postulation style, experimental style, modelling style, taxonomic style, statistical style, and genetic style. The author acknowledges the fact that these styles do not suffice to cover all types of mathematical reasoning, e.g., functional reasoning, but they still provide a comprehensive framework for the analysis of mathematical reasoning.

Summing up, we do acknowledge the specificities involved in each mathematical field or style; at the same time, we believe that mathematical reasoning consists of some general mathematical properties which can be the object of analysis, as we show next.

2.3. Reasoning as the Object of Analysis

Theoretical approaches to mathematical reasoning lead to the adoption of specific analytical frameworks for identifying and categorising related actions. These frameworks mainly differ in the way they perceive reasoning. The first view that appears in the literature is that reasoning has a structural and a process aspect (Jeannotte & Kieran, 2017): the structural aspect is static and refers to elements such as deduction, induction, abduction, data, claim, warrant, qualifier, and backing (Toulmin, 2007), while the process aspect refers to "processes related to the search for similarities and differences, or the processes related to validating" (Jeannotte & Kieran, 2017, p. 9). This distinction leads to two different analytical procedures, since in the structural view the focus is on the form and the structure of statements 'locally', while in the process view, the analytical lenses 'zoom out' in order to capture more global and contextual information. For instance, Boero et al. (2018) stress the importance of considering historical-epistemological and anthropological aspects of argumentation and proof situations. In line with this, Simon and Blume (1996) support the "idea that mathematical justification is a cognitive and a social process, the process of working within socially constituted and accepted modes of establishing validity to collectively determine what is cognitively compelling" (p. 28). This in turn, means that the researchers need to focus on interactions, in order to identify the social norms regulating the acceptable perceptions of justification and validation. The 'activity perspective' to reasoning has a similar basis, according to which we "are inclined to consider students' ways of doing, thinking, and talking about mathematics as fundamental" (Kaput, 2008, p. 9).

In line with the process approach, a fruitful way to view and analyse mathematical reasoning is by viewing it as a *communication process* with others or with oneself, and as a process "that allows for inferring mathematical utterances from other mathematical utterances" (Jeannotte & Kieran, 2017, p. 7). By putting discourse at the core of reasoning, we can deploy discursive methods to identify and analyse mathematical reasoning within discursive practices. Such methods have led Jeannotte and Kieran (2017) to identify nine processes, eight of which belong to two overarching categories – namely, the search for similarities and differences, and validating. The search for similarities contains generalising, conjecturing, identifying a pattern, comparing, and classifying, while validating contains validating, justifying, proving, and formal proving. The ninth process, that of exemplifying, supports the other processes. The advantage of this framework is that it contains functional definitions of concepts that are sometimes hard to identify. In the relevant literature, we can find other studies offering analytical schemes; the main difference lies in their focus. For instance, Ellis (2007b) suggests an actor-oriented taxonomy (focused on students' actions) for categorising generalisations. This taxonomy contains two main categories: generalising actions, which "describe learners' mental acts as inferred through the person's activity and talk" (p. 233) and reflection generalisations, which contain students' actual statements. Generalising actions include relating, searching, and extending, while reflection generalisations contain identifications or statements, definitions, and influence, denoting the "influence of a previously developed generalisation on new activity" (p. 249). Lannin et al. (2011) have also suggested nine essential understandings related to mathematical reasoning:

- developing conjectures
- generalising to identify commonalities
- generalising by application
- conjecturing and generalising using terms, symbols, and representations
- investigating why
- justifying based on already-understood ideas
- refuting a statement as false
- justifying and refuting the validity of arguments
- validating justifications.

We see the above framework as more encompassing than Jeannotte and Kieran's (2017), therefore it can be applied to a variety of situations involving reasoning processes. This framework has been effectively deployed in our studies involving eight-year-old students (Maj-Tatsis & Tatsis, 2019; 2023), as we will show later. Following other relevant studies, we have acknowledged that it is also important to consider the role of the teacher (or the researcher) in proposing appropriate tasks and orchestrating discussions. These two topics will be discussed in the next sections.

2.4. The Teacher's Role

Mathematical reasoning is impossible to be established without the help of the teacher or a more knowledgeable adult or peer. A number of studies have investigated the conditions that enhance reasoning in the mathematics classroom or during peer interactions. Most of these studies agree that mathematical reasoning (together with other mathematical processes) can be promoted by the establishment of relevant mathematical norms and by working on particular tasks; both of these elements are considerably affected by the teacher. For example, Simon and Blume (1996) stress the taken-as-shared nature of knowledge that results from justification and reasoning within a community. Mata-Pereira and da Ponte (2017) refer to an exploratory approach, in which the students work on non-trivial tasks that require some kind of investigation and interpretation. During this process, they are requested to present and justify their reasoning, assisted by particular teacher actions. The authors claim that in a mathematics classroom, the teacher engages in two types of actions: those related to mathematical processes, and those related to classroom management:

> Regarding actions related to mathematical processes, *inviting actions* aim to trigger a whole-class discussion or a discussion segment, where the teacher encourages students to participate or share their responses. Then, the teacher relies mostly on *informing/suggesting actions* to provide information to students or to validate their statements; on *supporting/guiding actions* to lead students to present information; and on *challenging actions* to encourage students to go further than their previous knowledge. In those three sets of actions, the authors refer to several mathematical processes that are involved, not necessarily disjointed, such as representing, interpreting, reasoning, and evaluating (Mata-Pereira & da Ponte, 2017, p. 172, emphasis in the original).

This reminds us of the 'knowledge quartet', introduced by Rowland et al. (2005) – particularly the contingency unit, referring to classroom situations which cannot be anticipated or planned. It consists of the "readiness to *respond to children's ideas* and a consequent preparedness, when appropriate, to *deviate from an agenda* set out when the lesson was prepared" (p. 263, emphasis in the original).

In the same line, Drageset (2014) has suggested a detailed analytical framework containing redirecting, progressing, and focusing teacher actions. These categories are related to Wood's (1998) funnelling and focusing actions as well as the IRE (initiation–response–evaluation) sequence, in which the teacher creates makes the questions, the students are expected to respond and then the teacher evaluates the responses (Cazden, 2001). Concerning young children, we agree with Yackel and Hanna (2003) that:

> … students as early as the primary grades of elementary school, given a classroom environment constituted to support mathematics as reasoning, can and do engage in making and refuting claims, use both inductive and deductive modes of reasoning, and generally treat mathematics as a sense-making activity – that is, they treat mathematics as reasoning. However, these studies also demonstrate clearly that creating a classroom atmosphere that fosters this view of mathematics is a highly complex undertaking that requires explicit effort on the part of the teacher (p. 234).

Most of the mentioned studies stress the importance of choosing or designing tasks appropriate for implementation in the classroom. This is an important factor which has been at the focus in some of our own studies as well.

2.5. Task Design with a Focus on Reasoning

The importance of task design has been highlighted in many studies in mathematics education, including studies which focus on mathematical reasoning. For instance, Francisco and Maher (2005) claim that mathematical reasoning in problem solving can be promoted by establishing the following conditions: the "role of basic ideas, complex tasks, strands of problems, students' ownership of their mathematical activity, justification of ideas and student collaborative work" (p. 371). According to the authors of the study, although mathematics contains complex relationships between concepts, mathematical reasoning can emerge within basic concepts as well. Concerning task design, Francisco and Maher (2005) acknowledge that although sometimes complex tasks are broken down into simpler sub-tasks, it is beneficial to present the whole task to the students first. Additionally, the authors advocate the advantages of offering a strand of problems to the students, that is a "series of related tasks designed around identified mathematical concepts with comparable levels of difficulty and similar problem-solving structure" (p. 366). They claim that:

> the opportunity to revisit the same concepts in different, but related problem situations, helps students build rich and durable forms of mathematical understandings of mathematical concepts. It also provides a way of enhancing reasoning without the need to tell or show students what to do (p. 371).

2.6. Our Studies on Task Design and Reasoning Actions

In one of our studies, we have investigated the influence of task characteristics on the reasoning of young students (Maj-Tatsis & Tatsis, 2019). Two eight-year-old students were provided with seven tasks. Our choice of tasks was based on their potential to stimulate students' interest, be solvable, or at least approachable, in more than one way and without the use of tricks, illustrate important mathematical ideas, serve as first steps towards mathematical explorations, and be extensible and generalisable (Schoenfeld, 1994). Additionally, we agreed that the "problems must be accessible, inviting and worthwhile to solve" and the "students must have the opportunity to give their own answers in their own words" (van den Heuvel-Panhuizen, 2005, p. 3). We also acknowledged that pictures play an important role as context-bearers and may serve multiple functions (van den Heuvel-Panhuizen, 2005). Based on these assumptions, we firstly performed an a priori analysis of the tasks in order to identify: a) the mathematical concepts involved and whether they matched those of the intended curriculum, b) the role of the images, based on van den Heuvel-Panhuizen's (2005) picture functions (motivator, situation describer, information provider, action indicator, model supplier, and solution and solution-strategy communicator) and c) the function of context, based on de Lange's (1999) assessment framework; we also categorised each task as open or closed.

Our a posteriori analysis focused on the interactions that took place among the students and the researcher and was based on Lannin et al.'s (2011) framework, supplemented by three more activities: reformulating conjectures or justifications, monitoring each other (a form of collaborative reasoning), concluding and explaining (a form of informal justification). Based on the results of this analysis, the presence of images in the tasks did not affect the solutions. However, the context of the tasks influenced the students' reasoning. Certain tasks were found to be more effective in promoting reasoning than others. Open tasks with a limited number of solutions and closed tasks with multiple solution options were easy for the students. However, they faced difficulties in shifting from geometric structures to numerical ones. On the other hand, they showed a range of reasoning processes when working on a geometrical task that did not require a shift to numerical properties.

In a second study (Tatsis & Maj-Tatsis, 2023), which focused on a combinatorial task, Lannin et al.'s (2011) framework was used, this time enriched with a framework related to solution strategies in combinatorial tasks (English, 1991). Our students, despite their age (8 years old), manifested all levels of reasoning actions, from comparing and contrasting to generalising and justifying. They were also able to progress from randomly selecting colours (they were engaged in a colouring task) to a systematic strategy. In this study, we have observed the importance of the researcher's role in supporting the students' reasoning actions by posing specific questions.

3. Discussion

In this article, we have highlighted the basic aspects of mathematical reasoning as they appear in the literature, focusing on analytical frameworks, suggested teacher roles, and task design principles. Based on our research interest on early years students' reasoning, we have briefly presented our own studies, in which particular analytical frameworks have been used. Our approach to mathematical reasoning in these studies can be characterised as discursive (Jeannotte & Kieran, 2017), since our focus was on the verbal exchanges that took place between the students and between the students and the researcher. We have seen the young students' reasoning evolving during the interactions, strongly affected by the provided tasks, but also by the researcher's interventions, usually in the form of questions. It seems crucial for the teacher in the mathematics classroom to establish specific social and sociomathematical norms that allow for fruitful and meaningful exchanges among students while working on tasks that enable them to express, discuss, and elaborate mathematical ideas, including mathematical arguments. These are the core elements of mathematical reasoning which should characterise all interactions in a mathematics classroom. A series of carefully chosen tasks by the teacher may also assist in establishing an atmosphere of meaningful interactions where all students have the opportunity to express their mathematical ideas, acknowledging the norms that are at place.

We believe that there is room for further adapting or enriching the existing frameworks, possibly by considering group dynamics in cases when more than two students interact. When considering the teacher's role and their possible interventions it might be useful to consider the issues of authority and politeness that affect the outcome of teacher-student interactions in the classroom (Tatsis & Wagner, 2018). We also might need to consider particularities related to specific mathematical content, such as the cases of combinatorics or geometry. Such content-bounded frameworks may offer important insights into the processes related to mathematical reasoning and to mathematical thinking in general.

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Esperanza López Centella
University of Granada

Chapter 6

ANALYSIS OF TASKS FROM A HEJNÝ METHOD MATHEMATICS TEXTBOOK FOR THE SIXTH GRADE

Summary: In the recent years, studies have presented evidence of the positive effects on pupil learning of the so-called Hejný Method for teaching mathematics. This chapter presents a study where mathematical tasks from a Hejný Method electronic textbook for sixth grade are examined from a qualitative perspective and by using content analysis techniques. Tasks are analysed according to the NCTM learning expectations they contribute to, the type of PISA contexts in which they are set, and the systems of representation involved in their statements and promoted in their resolution and implementation. The analysis reveals a great contribution of the tasks for the achievement of NCTM learning expectations and others related to logic and algebraic activity, a predominance of the use of scientific and personal contexts versus societal and occupational contexts, the implication of a diversity of systems of representation in the tasks, and a constant invitation to reflection and discussion frequently accompanied by manipulation.

Keywords: analysis of mathematical tasks, final elementary education/initial secondary education, Hejný Method, National Council of Teachers of Mathematics.

1. Introduction

Hejný Method is a popular name for a specific way of teaching mathematics – scheme-based education (Hejný, 2012) or genetic constructivism (Kvasz & Pilous, 2020) – based on the constructivist paradigm. Following the principles of this method, a series of mathematics textbooks for primary school were written by Milan Hejný and his team at the Faculty of Education at the Charles University in the Czech Republic. These textbooks were approved by the Czech Ministry of Education, Youth, and Sports and, by 2015, they had been adopted by 20% of Czech schools: more than 750 out of the 4100 Czech schools on the elementary and lower-secondary level. Since then, studies have provided quantitative and qualitative evidence of the positive effects of the Hejný Method for teaching mathematics on pupil learning (e.g., Chytrý et al., 2020; Greger et al., 2022; López Centella et al., 2021; Papadopoulos et al., 2017). Hejný Method is an elaborate teaching method in which textbooks and their accompanying teaching manuals explaining task implementation in the classroom play an important role. Thus, a question arises about the characteristics of the tasks included in the textbooks.

Regarding curricular aspects, Hejný et al.'s textbooks are designed with the Czech educational curriculum in mind, whose current version in force is the one proposed by the Ministry of Education, Youth, and Sports in 2017 (MECS, 2017) with subsequent revisions. At the international stage, the National Council of Teachers of Mathematics (NCTM) is a recognised driving force in mathematics education reform whose principles and standards have inspired mathematics education worldwide. Its framework has been used in studies of teaching and learning mathematics, including the studies of textbooks. NCTM proposes the curricular organisation of mathematics education from pre-kindergarten to grade 12 through five blocks of mathematical content and five mathematical processes (NCTM, 2000). As for the content, it distinguishes numbers and operations, algebra, geometry, measurement, and data analysis and probability. As for the processes, it considers problem solving, reasoning and proof, communication, connections, and representation. According to the NCTM, selecting appropriate tasks and having pupils solve them is a way to reach the learning expectations.

Moreno and Ramírez (2016) propose six descriptors to characterise a mathematical task: its goal, its formulation, the materials and resources it involves, the types of grouping it foresees, the forms of interactions it promotes, and its timing. In this study, we focused on the first three descriptors.

This study presents the results of the content analysis of the electronic textbook "Mathematics-A (6th grade)" (Hejný et al., 2018) in which 252 tasks were identified and analysed to answer the following questions:

RQ1. Which NCTM learning expectations are supported by the tasks?
RQ2. What are the main characteristics of the tasks in terms of the resources required for their solution, the context in which they are set, and the systems of representation involved in their statement and/or promoted in their resolutions?

2. Theoretical Framework

We will first briefly describe the Hejný Method and its basic principles, and then elaborate on the mathematics tasks and their descriptors which are at the centre of our attention in this study.

2.1. Hejný Method

According to Hejný (2012), "scheme-oriented education is based on the construction of different schemes that interlink, combine and form a dynamic network of a pupil's mathematical knowledge and skills" (p. 47). As described by Gerrig (1991), the term *schemes* was coined by theorists "to refer to the memory structure that incorporate clusters of information relevant to comprehension... A primary insight to scheme theories is that we do not simply have isolated facts in memory. Information is gathered together in meaningful functional units" (pp. 244–245). More precisely, Hejný Method is anchored in the *theory of generic models* (Hejný, 2012), a five-stage model of knowledge acquisition. It starts with *motivation* and contemplates two mental shifts: the first, called generalisation, leads from concrete knowledge (*isolated models*) to generalised knowledge (*generic model*); the second, called abstraction, leads from generic to *abstract knowledge*. The permanent part of this process of gaining knowledge is *crystallisation*, which consists of integrating new knowledge into an already existing mathematical structure.

Grounded in these ideas, the Hejný Method moves away from teaching structured strictly in blocks of mathematical content. Instead, the notion of what Hejný calls the *environment*[1] becomes crucial. An environment contains a sequence of connected tasks that revolve around the same theme and involve a variety of mathematical phenomena, encouraging experimentation and discovery. Some environments build on children's daily experiences and others focus on children's preferences and the activities they enjoy doing. A scheme is not understood as a result of learning a particular curricular topic, but as a result of everyday experience in a given environment (Hejný, 2012). Working in environments is one of 12 key principles proposed by the authors to characterise the method. The list of these principles is completed by the interconnection of topics, the development of children's character and independent thinking, true motivation, real-life experience, the enjoyment of mathematics, the promotion of personal knowledge, the teacher's role as a guide and mediator of discussion, working with errors, the different levels of difficulty for appropriate challenges, and the support for collaboration and teamwork.

The Hejný Method has sparked interest beyond the Czech Republic and is being implemented in a number of alternative schools and in home-schooling. It is introduced to student teachers in primary education programmes of the Charles University in Prague and at the University of Ostrava. Some of the textbooks based on the Hejný Method created for elementary and lower-secondary education have been translated into other languages, such as Polish, Slovak, and English. There is growing evidence of the effects of the method on pupil learning.

1 h-mat.cz/en/principles/environments These environments meet the requirements of substantial learning environments as introduced by Wittmann (1995, 2021).

Greger et al. (2022) conducted a secondary analysis of data from the 2015 and 2019 editions of the Trends in International Mathematics and Science Study (TIMSS). In addition to the data usually available in the TIMSS, information regarding the teaching method used in the individual classes involved in tests in the Czech Republic (verified by the Czech School Inspectorate) was considered. The analysis made use of results from 5202 pupils, 265 classes, and 135 published tasks from TIMSS 2015, and 4692 pupils, 263 classes, and 70 tasks from TIMSS 2019. Greger et al.'s report shows that pupils from classes that signed up to teach mathematics using the Hejný Method did slightly better in the TIMSS tests and solved complex problems more successfully than pupils from non-Hejný Method classes. Chytrý et al. (2020) carried out a study to compare the mathematical self-efficacy and mathematical problem-solving of 1133 fifth grade students in 36 schools in the Czech Republic taught with four different teaching methods: Dalton, Hejný, Montessori, and the ordinary teaching method. Among its conclusions, it stands out that students taught in accordance with the Hejný Method achieved better results in both aspects than those from ordinary primary schools.

From a qualitative perspective, some research highlights the mathematical abilities shown by elementary school students taught with the Hejný Method when exposed to contextualised non-standard tasks. López Centella et al. (2021) report the algebraic thinking and capacity to argue shown by fifth graders in a task involving different representations of equations. Papadopoulos et al. (2020) show functional thinking and the ability to generalise put into play by fourth graders when addressing a contextualised generalising task that involves covariant quantities.

Other studies focus on contributions to learning mathematics from particular environments from Hejný Method targeting elementary school students. Jirotková et al. (2013) describe how the use of the environment called the "Bus" helped first graders to "lose their tendency to use irrelevant aspects of language and effectively select and structure those that are key in a given situation" (p. 978). Using this same environment with first graders in Greece, Papadopoulos et al. (2017) conclude that it empowered "both their early conceptualisation of natural numbers and their development of important mathematical skills such as the mental operations and the meaningful organisation of arithmetical data" (p. 303). Hejný et al. (2006; 2013) show pupils' gains in conceptual learning and building a schema in relation to the solution of equations through the use of the environments called "Father Woodland" and "Additive triangles". Kloboučková et al. (2013) report how the "Cube buildings" environment, along with its corresponding materials, promoted the development of spatial sense and problem-solving strategies in first and second grade pupils, leading to interactions that contributed to the acquisition of geometric knowledge and skills.

2.2. Mathematics Tasks

Mathematics tasks are at the heart of mathematics teaching and learning: "In effective teaching, worthwhile mathematical tasks are used to introduce important mathematical ideas and to engage and challenge students intellectually. Well-chosen tasks can pique students' curiosity and draw them into mathematics" (NCTM, 2000, p. 18). According to Moreno and Ramírez (2016), a "school mathematical task is a structured demand for action that the teacher provides to students on an intentional basis" (p. 67). The quality of tasks used in mathematics lessons influences the knowledge and skills the pupils can acquire and develop (Larsen & Bartlo, 2009; Margolinas, 2013; Powell et al., 2009); thus, tasks have been at the centre of research attention in mathematics education for decades.

In this study, we rely on the work of Moreno and Ramírez (2016), in which they propose six descriptors to characterise a mathematical task. We use three of them to describe the nature of tasks in Hejný et al.'s textbook. The goal summarises the purposes that the teacher assigns to the task: the learning expectations to which it is intended to contribute and those errors and difficulties that the task is expected to help overcome. The materials and resources refer to any means that can be used or are proposed to be used in the task to learn a specific mathematical concept or procedure (even if it has not been specifically designed for it). The formulation of the task signifies the text (including any picture, graph, etc.) or instruction that the teacher provides to the students which specifies what they are expected to do. In particular, this includes the context in which the task is set and the systems of representations involved in its statement. The remaining three characteristics (the types of grouping the task foresees, the forms of interactions it promotes, and its timing) will not be followed as they are more related to the actual implementation process as part of a lesson.

Below, we will elaborate on each task characteristic used in our analysis.

Content and Learning Expectations

As already stated, the NCTM provides an internationally recognised framework for the content and learning expectations. For our work, learning expectations for grades 3-5 are relevant. They are listed below, whereas the specifications for each can be found in Appendix A.

Number and Operations (NCTM, 2000, p. 148)

NO1. Understand numbers, ways of representing numbers, relationships among numbers, and number systems.

NO2. Understand meanings of operations and how they relate to one another.

NO3. Compute fluently and make reasonable estimates.

Algebra (NCTM, 2000, p. 158)

A1. Understand patterns, relations, and functions.

A2. Represent and analyse mathematical situations and structures using algebraic symbols.

A3. Model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions.

A4. Analyse change in various contexts.

Geometry (NCTM, 2000, p. 164)

G1. Analyse characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.

G2. Specify locations and describe spatial relationships using coordinate geometry and other representational systems.

G3. Apply transformations and use symmetry to analyse mathematical situations.

G4. Use visualisation, spatial reasoning, and geometric modeling to solve problems.

Measurement (NCTM, 2000, p. 170)

M1. Understand measurable attributes of objects and the units, systems, and processes of measurement.

M2. Apply appropriate techniques, tools, and formulas to determine measurements.

Data Analysis and Probability (NCTM, 2000, p. 176)

DAP1. Formulate questions that can be addressed with data and collect, organise, and display relevant data to answer them.

DAP2. Select and use appropriate statistical methods to analyse data.

DAP3. Develop and evaluate inferences and predictions that are based on data.

DAP4. Understand and apply basic concepts of probability.

Context

For the analysis of the context, we employ the framework of the Programme for the International Students Assessment (PISA), which understands it as the "aspect of an individual's world in which the problems are placed" (PISA, 2022, pp. 29–30). In general, the choice of mathematical strategies to solve a problem depends on the context in which it arises, and it is necessary to use knowledge of the context in developing a solution. In this way, the PISA emphasises involving a variety of contexts in problems and tasks. It distinguishes the following four categories of contexts (pp. 29–30).

Personal. Tasks in this category focus on activities of oneself, one's family or one's peer group (e.g., food preparation, shopping, games, personal health, personal transportation, sports, travel, personal scheduling and personal finance).

Occupational. Tasks in this category are centred on the world of work (e.g., measuring, costing and ordering materials for building, payroll/accounting, quality control, scheduling/inventory, design/architecture and job-related decision making).

Societal. Tasks in this category focus on one's community (local, national or global; e.g., voting systems, public transport, government, public policies, demographics, advertising, national statistics and economics). Although individuals are involved in all of these things in a personal way, in the societal context category, the focus of problems is on the community perspective.

Scientific. Tasks in this category relate to the application of mathematics to the natural world and issues and topics related to science and technology (e.g., weather or climate, ecology, medicine, space science, genetics, measurement and the world of mathematics itself).

Systems of Representations

Representations are indispensable for the learning of mathematics: we use them to think, communicate and operate on mathematical ideas (Hiebert & Carpenter, 1992; Rico et al., 2000). As stated by Duval (1999), its use is essential for mathematical thinking because "unlike the other fields of knowledge (botany, geology, astronomy, physics), there is no other ways of gaining access to the mathematical objects but to produce some semiotic representations" (p. 4). In the words of Goldin and Shteingold (2001), the "fundamental goals of mathematics education include representational goals: the development of efficient (internal) systems of representation in students that correspond coherently to, and interact well with, the (external) conventionally established systems of mathematics" (p. 1). As defined by these authors, a representation is typically a sign or a configuration of signs, characters, or objects that can stand for (symbolise, depict, encode, or represent) something other than itself. Systems of representations include the conventional symbol systems of mathematics, such as numeration, formal algebraic notation, the real number line, the Cartesian coordinate representation, etc. (Goldin & Shteingold, 2001).

Converting representations is a crucial task in the learning of mathematics. As pointed out by Rico et al. (1996), there is no representation system that completely expresses the complexity of each mathematical concept: each of them emphasises some properties and blurs others. From a didactical point of view, according to Duval (1999), only students who can switch from one representation system to another do not confuse a mathematical object with its representation and can transfer their mathematical knowledge to other contexts different from the one of learning. In this sense, the diversification of representations in tasks and the promotion of students' ability to work with each of them and move from one to another is considered positive for the development of their mathematical competence (Cordero-Siy & Ghousseini, 2022; Gagatsis & Shiakalli, 2004; Heinze et al., 2009; Moseley, 2005; Tripathi, 2008).

Materials and Resources

Multiple studies point out the effects of using concrete materials (manipulatives or virtual) in mathematics teaching and learning processes, generally concluding in favour of their use (e.g., Carbonneau et al., 2013; Uribe-Flórez & Wilkins, 2017). In particular, different authors (Bonilla & Rojano, 2012; Figueira-Sampaio et al., 2009; Otten et al., 2019), explore children's learning in solving equations through the use of concrete models (virtual scale, hanging mobile, etc.). Such studies report significant achievement in transferring actions and strategies (restructuring, isolation and substitution) from the sign language of the model to the actions of the algebra sign system, as well as a better understanding of the properties of equality. For example, Larbi and Mavis (2016) and Kablan (2016) describe how students who were taught mathematics through the use of manipulatives performed better and scored significantly higher in the post-test than those who were taught through a "talk and chalk"/"lectures and exercises" method without involving further materials. These studies suggest that the experiences that students gain in the embodied learning environment with certain materials provide a basis for mathematical thinking, which appears to support them when addressing problem solving in other contexts.

3. Methodology

We conducted a qualitative and descriptive analysis (Vasilachis, 2009) using content analysis techniques from the Didactic Analysis (Rico Romero, 2013). We consider as units of analysis the 252 tasks included in the sixth-grade electronic textbook "Mathematics-A (6th grade)" (Hejný et al., 2018). The book is structured through 21 environments (briefly described in the Appendix B), opening with an introductory section "Appetizer" and ending with an "If there is time left" section.

3.1. Content and Learning Expectations

In the first phase of the analysis, we focused on each task's demand and solving processes in order to identify the NCTM learning expectations that the work on each task contributes to achieving. NCTM learning expectations are set according to the educational stage: from pre-K to grade 2, grades 3-5, 6-8, and 9-12. The textbook under analysis (Hejný et al., 2018) is primarily intended for sixth grade students in the Czech education system (11 to 12-year-old). We considered the following to pick a relevant set of NCTM learning expectations. Firstly, both elementary and secondary education in the Czech Republic and the United States are organised differently. While in the Czech Republic it consists of 13 grades, typically finishing at the age of 18-19 years, in the United States, it consists of 12 grades, typically finishing at the age of 17-18 years. Secondly, NCTM learning expectations are understood as a body of skills on mathematical content that all the students should have developed by the end of the corresponding educational stage. Based on this, the NCTM learning expectations established for grades 9-12 can be related to Czech grades 10-13, the ones established for grades 6-8 to Czech grades 7-9, and the ones for grades 3-5 to Czech grades 4-6. This motivated us to consider the NCTM learning expectations for grades 3-5 in our analysis.

3.2. Context, Systems of Representation, Resources, and Materials

In the second phase of the analysis, we focused on identifying the type of PISA context (PISA, 2022) in which each task is set, the systems of representation, and the materials and resources involved in each task. Both phases required careful reading of the statements and resolution of the tasks. For contexts and learning expectations, we used the categories presented in section 3.1. For systems of representation and resources and materials, categories originated in an inductive-deductive way (Tables 1 and 2) – some categories are based on literature (e.g., text, number line), others were particular to the textbook (e.g., grid). For both, we considered task statements along with the recommended materials and implementation suggestions provided for teachers, included in the book.

Table 1. Systems of representation

Category	Description
Text	Tasks that include verbal statements and communicate information through words
Numerical	Tasks that use numbers in their statements and in which numbers play a major role in their resolution
Number line	Tasks that show a number line in their statements or that invite to draw and work on a number line in their resolution
Algebraic and symbolic notation	Tasks that include algebraic or symbolic notation whose meaning has been previously presented or that promote the use of notational conventions
Geometric construction	Tasks in which geometric constructions (like drawing the heights of a triangle, building a certain structure using cubes, etc.) play an important role in their statements or resolution
Picture/Drawing[2]	Tasks that rely on a picture or drawing to formulate their questions or that invite to make a drawing for their resolution
Grid	Tasks that formulate questions that involve a pattern or structure made from horizontal and vertical lines crossing each other to form squares
Graph	Tasks that prresent information through a table, pictograph, bar graph, line graph, pie chart, or request to represent data using any of the above
Diagram	Tasks that invite to work with some kind of schematic structure, possibly involving boxes or blanks to fill in, in accordance with a set of described rules

Table 2. Resources and materials

Category	Description
Manipulative materials	Tasks that encourage using hands-on materials and manipulation of objects for mathematical exploration
Pencil and paper	Tasks that are essentially aimed at being solved through the use of pencil and paper
Calculator	Tasks that explicitly encourage the use of a calculator to provide or check answers
Digital tools	Tasks whose statement or completion explicitly require the use of specific mathematical software or applets (for symbolic calculation, graph representation, dynamic geometry, data management, programming, etc.)

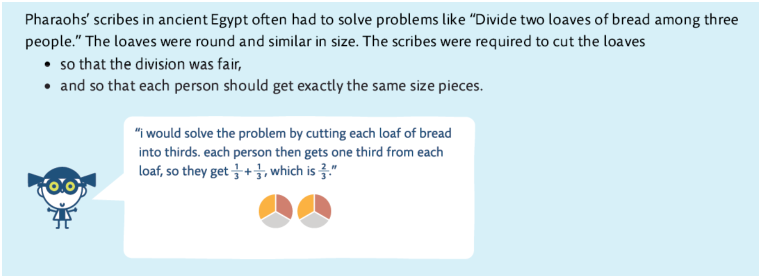
3.3. Examples of Data Analysis

Below we present an example of how tasks from the textbook were analysed and classified according to the criteria and categories described above. The selected task belongs to the environment *Egyptian division of bread* (II). Figures 1, 2, and 3 consist of excerpts from the

[2] Illustrations with purely decorative purposes were not considered.

textbook showing, respectively, a brief explanation of this form of division, a convention on the optimality of the division, and the statement of the selected task.

Figure 1. Brief explanation of the ancient *Egyptian division of bread* (Hejný et al., 2018)


Pharaohs' scribes in ancient Egypt often had to solve problems like "Divide two loaves of bread among three people." The loaves were round and similar in size. The scribes were required to cut the loaves

- so that the division was fair,
- and so that each person should get exactly the same size pieces.

"i would solve the problem by cutting each loaf of bread into thirds. each person then gets one third from each loaf, so they get $\frac{1}{3} + \frac{1}{3}$, which is $\frac{2}{3}$."

Figure 2. Convention on the optimality of the *Egyptian division of bread* (Hejný et al., 2018)

Division of bread

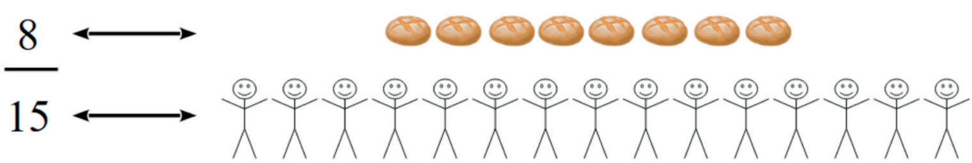
Let's agree: If we have two different solutions of tasks on division of loaves of bread, the better one is the one that requires fewer cuts.

Figure 3. Task from the *Egyptian division of bread* environment (Hejný et al., 2018)

Find two different methods of dividing 8 loaves among 15 people. For both methods, each person should only get two pieces. Then decide which method is better.

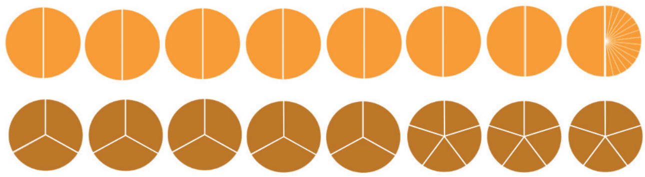
Regarding NCTM learning expectations, the task shown in Figure 3 is considered to contribute to the achievement of the third, fourth, and fifth specifications of learning expectation NO1 (see Appendix A). Indeed, the use of fractions in the task context contributes to understand fractions as representations of parts of unit wholes and divisions of whole numbers (learning expectation NO1.3), as illustrated in Figure 4.

Figure 4. Fraction as division of whole numbers



Furthermore, the task requirement invites the use of an area model (learning expectation NO1.4) to identify different ways of cutting bread according to the task criteria, as represented in Figure 5.

Figure 5. Use of an area model



The portions of bread obtained by a person based on two valid divisions (according to the task criteria) can and must be checked to be equivalent (learning expectation NO1.5), as shown in Figure 6.

Figure 6. Equivalent representations of the same fraction

$$\frac{1}{2} + \frac{1}{30} = \frac{8}{15} = \frac{1}{3} + \frac{1}{5}$$

$$\frac{15}{30} + \frac{1}{30} = \frac{16}{30} = \frac{10}{30} + \frac{6}{30}$$

Note: in the area model, equality is established between the parts coloured in orange, red, and brown

As for the systems of representation of the task, this involves text and numbers – present in its own statement – and drawings – promoted in the resolution of the task.

The PISA context of the task is considered personal because it involves handling and sharing food at a household level.

Concerning resources and materials, we include the task in the categories of pencil and paper and manipulatives, since its completion invites both manual calculations and manipulation of materials such as, for example, a set of fraction circles.

To further illustrate our analysis and the categories considered, Table 3 shows examples of tasks (from Figures 7 and 8, numbered from 1 to 9) for each of them.

Table 3. Examples of tasks for the categories of context, systems of representations and resources

Characteristics	Category	Task number
Context	Scientific	1-6
	Personal	7
	Occupational	8
	Societal	9
Systems of representations (Note: in most tasks, more than one system of representation was identified)	Number line	1
	Algebraic notation	2
	Diagram	3
	Geometric construction	4
	Grid	5
	Graph (table)	6
	Text and numbers	7
	Picture	8
Resources and materials	Pencil and paper	1-9
	Manipulative materials	4, 5, 6, 8, 9
	Calculator	1

The main manipulatives were sticks in the *Wooden sticks* environment (to build figures for geometric reflections and pattern identification), paper in the *Origami* environment (to fold and cut it for working with geometric elements and reflecting on their properties), cubes in the *Cubic shapes* environment (to build solids and work on their views and two-dimensional representations), geoboard in the *Grid* environment (to create figures and reflect on their properties), coins in the *Coins* environment (to work with equations), and measuring tape.

Figure 7. Sample of tasks (Hejný et al., 2018)

1

Sort the numbers from the smallest to largest using a calculator:

0.3 $\frac{2}{5}$ 0.5 $\frac{1}{2}$ $\frac{1}{4}$

Place the numbers on the number line.

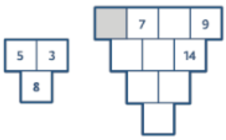
2

Draw these equations as scale diagrams and then solve them.

a) $k + 24 = 5 \times k + 4$
b) $4 \times a + 3 = 13$
c) $20 = 4 \times (y + 2)$
d) $5 \times z - 6 = 3 \times z$
e) $2 \times (a + 5) = 1 + 5 \times a$
f) $4 \times k + 1 = 6$
g) $6 \times a + 1 = 2 \times (a + 1)$
h) $3 \times (5 + a) = a + 11$

3

In the picture, there is an addition triangle. To solve the triangle, fill in the spaces so that the sum of two neighbouring numbers is written in the space below. (See the example with numbers 5, 3 and 8).



Solve the addition triangle in the picture if the following is true:

a) there is the number 5 in the grey space
b) the sum of the numbers in the second row is 37
c) the sum in the third row is 48
d) the sum of all 10 numbers is 124

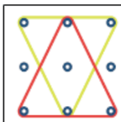
4

Make the solid in the picture from cubes. Place it on a table in such a way that on the first storey there are the following number of cubes:

a) two **b)** three **c)** four **d)** one

Sketch your solutions.

5



a) In the picture, there are two isosceles triangles made from rubber bands. Make more isosceles triangles; they should be different to the ones you have already made.
b) Repeat the same problem using righ-angled triangles.
c) Now try to make triangles which are not isosceles and not right-angled.

Sketch all the triangles into a square grid.

Figure 8. Sample of tasks (Hejný et al., 2018)

6

Make more double windows like the previous problem and put the numbers of sticks used in the table below. How many sticks do you need to make **a)** 5, **b)** 7, **c)** 16, **d)** 50 double windows?

Double windows	2	3	4	5	6	7	...	16	...	50
Sticks	12									

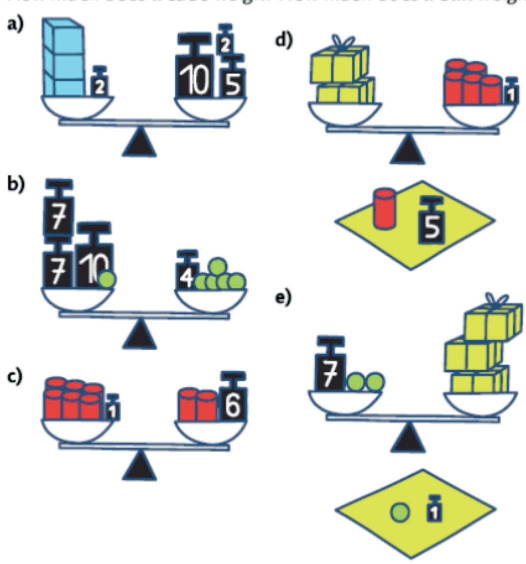
7

We are going to Grandma's. We have 24 km to go. How far away does Grandma live from our house if we have already travelled this fraction of the journey?

a) a half **b)** a third **c)** a fifth

8

How much does a cube weigh? How much does a ball weigh? And how much does a cylinder weigh?



9

In England, there are six coins with a value less than £1: 1, 2, 5, 10, 20 and 50 pence.

a) How many different ways are there to pay for something that costs 10p?

b) Which amounts can be paid by using up to two coins?

c) What is the lowest sum of money which cannot be paid by using up to two coins?

d) What is the lowest sum of money which cannot be paid by using up to three coins?

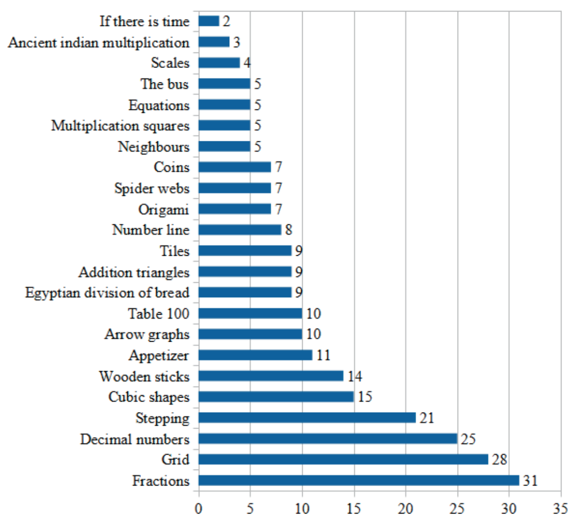
4. Results

4.1. Book Structure and Organisation

The book is structured through 21 environments. Each environment does not focus on concepts from a single NCTM mathematical content block, but rather a single environment can promote work on, for example, numbers and operations, algebra, measurement, geometry, etc. Figure 9 shows the number of tasks included in each environment. Some environments are revisited several times throughout the book: for example, *Fractions* (I) and *Fractions* (II), *Grid* (I), *Grid* (II), and *Grid* (III), etc. are interspersed between the other environments. Furthermore, at the end of the work of an environment, one to three tasks unrelated to it or related to another environment are typically provided. It emphasises the way of working promoted by the Hejný Method: content is not intensively studied for a short period and then replaced by another group of content, but is continuously explored, with the attention focused on different aspects each time it is revisited. The Hejný Method tries to contribute to the creation of schemes in this way. In Hejný's words (2012), the scheme that each of us have of our flat

> is not the result of learning the curricular topic 'The furnishing of our flat' in September, 'Lamps and carpets' in October, 'The Kitchen' in November, etc. at school. The scheme is an outcome of our everyday experience in the given environment. In some activities our attention focuses on some part of the flat (we hang up a picture, clean windows, move furniture, tidy up, ...) but all of these particulars are perceived as parts of one whole (p. 46).

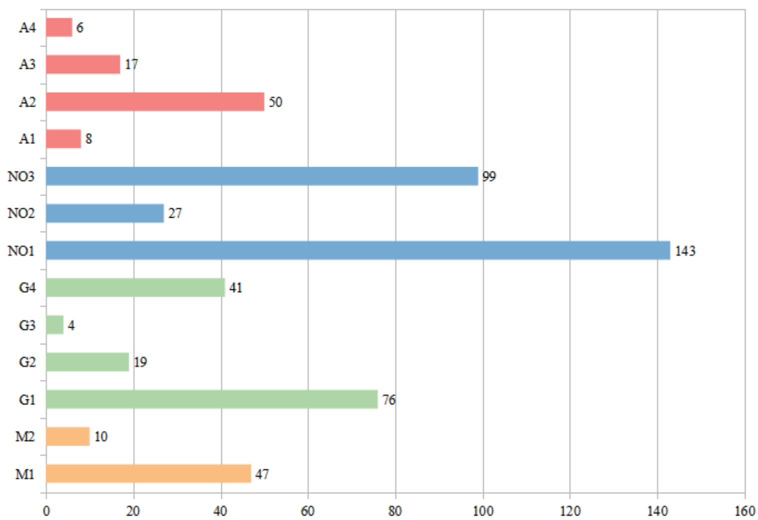
Figure 9. Number of tasks included in each environment



4.2. Learning Expectations

Figure 10 shows the number of times each NCTM learning expectation was identified during the analysis of the 252 tasks. As shown in the bar chart, most frequently identified are those related to understanding numbers, ways of representing numbers, relationships among numbers and number systems (NO1), and those aimed at computing fluently and making reasonable estimates (NO3). The learning expectations related to the mathematical content titled Numbers and operations represent 49.2% of the total NCTM learning expectations identified in the tasks. The second most frequent group of learning expectations is related to geometry, which involves identifying, comparing, and analysing the attributes of two- and three-dimensional shapes and developing the vocabulary to describe the attributes (G1). Of the total of NCTM learning expectations identified, the ones related to geometry represent 25.6%. Most of learning expectations connected to algebraic activity are related to representing the idea of an unknown quantity using a letter or a symbol, and express mathematical relationships and model problem situations using equations (A1). The learning expectations related to algebra represent 14,8% of the total NCTM learning expectations identified in the tasks.

Figure 10. Frequency of NCTM learning expectations identified in the tasks



NCTM learning expectations about measurement represent 10,4% of the total identified. Most of them are related to understanding measurable attributes of objects and the units, systems, and processes of measurement (M1).

The number of identified NCTM learning expectations about data analysis and probability was low (specially compared to other content) and was not represented in the bar chart of Figure 10. Many of them were related to the use of tables to represent data or basic notions of combinatorics. It is worth mentioning in this regard that in the Czech curriculum (MŠMT, 2017), the main curricular reference of the textbook, these contents are addressed at higher educational levels (and, therefore, addressed primarily in textbooks for other grades). Remarkably, most tasks contribute to achieving more than one NCTM learning expectation, and often two, three, or four. Table 3 shows the number of tasks that promote the work on a certain number of NCTM learning expectations. Some tasks' contributions are beyond the learning expectations established by the NCTM and for which no NCTM learning expectations were identified. Some of these are related, for example, to tessellations, logic, and reasoning about the veracity of a statement.

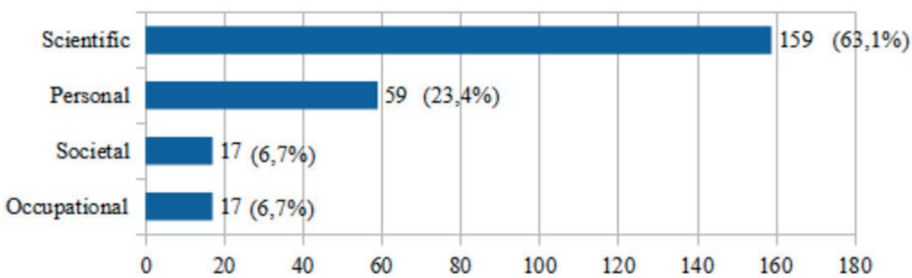
Table 3. Number of tasks that contribute to the achievement of a certain number of NCTM learning expectations

No. of NCTM learning expectations	0	1	2	3	4	5	7
No. of tasks	16	67	73	58	34	3	1

4.3. Contexts

Figure 11 shows the absolute and relative frequencies of each PISA context in the tasks. A predominance of scientific and personal contexts versus societal and occupational is observed.

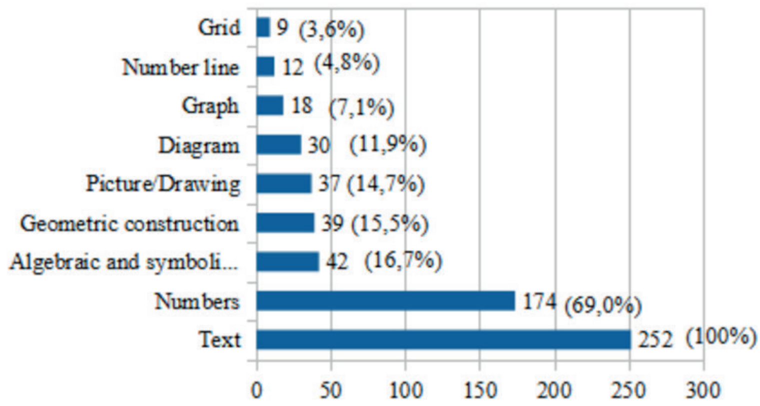
Figure 11. Frequencies of PISA contexts identified in the tasks



4.4. Systems of Representation

Figure 12 shows the number of tasks involving each system of representation in their statements or in what could be considered a natural or supposedly expected resolution.

Figure 12. Frequencies of systems of representation identified in the tasks

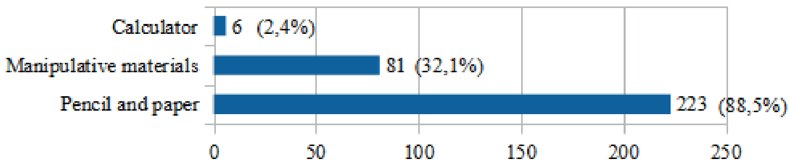


In addition to the use of the ordinary verbal and numerical systems of representation, the bar chart reveals that a significant number of tasks involve algebraic and symbolic notation, geometric constructions, and pictures and drawings. Likewise, the tasks encourage pupils to handle various systems of representations like diagrams, graphs (mainly tables), the number line, the grid. Each task involves more than one system of representation and multiple tasks request a transfer of information from one representation system to another. Such tasks can be found, for example, in the environments *Scales* and *Coins*, requesting the transcription of pictorially represented equations into algebraic language.

4.5. Resources and Materials

Figure 13 shows the number of tasks identified as particularly designed to be implemented using pencil and paper, manipulative materials, and calculator, as well as their corresponding percentages in the total of tasks.

Figure 13. Frequencies of resources and materials



According to the chart, it can be said that manipulation is emphasised in the textbook. Manipulatives are usually used for the first discoveries in a certain environment and stop being used when they are no longer necessary for a pupil. Through manipulation, students are invited to explore different mathematical facts and properties that are then worked on more abstractly.

Calculator might be used in several tasks and its use is explicitly recommended in some of them. No explicit mentions of working with other mathematical software were identified.

More generally, regarding implementation, it is worth noting that many tasks are planned to be discussed orally and in work groups, encouraging collaboration and argumentation between peers, regardless of whether they are subsequently addressed with pencil and paper.

5. Discussion and Conclusions

In this study, we explored the mathematical tasks included in the electronic textbook "Mathematics-A (6th grade)" (Hejný et al., 2018) with two objectives: first, analysing the NCTM learning expectations that they contribute to achieving; and second, analysing the PISA contexts in which they are set, the systems of representation they involve, and the resources and materials used for their completion and resolution.

Regarding our first objective, most of the NCTM learning expectations associated with numbers and operations, algebra, geometry, and measurement were identified in the 252 tasks of the textbook. For each learning expectation there are frequently several tasks designed to work towards its fulfilment; and, in general, each task contributes to achieving several NCTM learning expectations. While there are tasks that allow pupils learn about basic combinatorics and practise using tables to organise data, NCTM learning expectations on data analysis and probability were much less frequently identified in the tasks. The Czech curriculum (MŠMT, 2017), of reference for the textbook, includes them at higher educational levels and, therefore, they are primarily addressed in textbooks for other grades.

Concerning frequencies of the NCTM learning expectations identified in the tasks, the ones related to numbers, operations, and geometry are significantly higher than the others, without this meaning a deficit of NCTM learning expectations on algebra or measurement (there are many tasks that work on them as well).

On the other hand, some learning expectations beyond the ones established by NCTM were detected in the tasks. These are mainly related to the use of logic and deeper algebraic activity. In particular, we highlight the use of different environments to work on the notion of equality, equation, inverse operation, etc. through diagrams where some missing terms must be calculated. Positive effects on the development of mathematical reasoning processes for equations have been reported in research studies where similar types of calculation structures were used (Nührenbörger & Schwarzkopf, 2016).

Concerning our second objective, the systems of representations employed in tasks statements or their solving processes are multiple and seem balanced. The use of algebraic and symbolic notation, geometric constructions, computational numerical diagrams and tables stands out. This contributes to widening the opportunities provided to pupils to read, use, interpret, represent, and communicate mathematical knowledge.

As for the contexts, most tasks are posed in an intramathematical context, considered scientific within the PISA framework. Nevertheless, a significant number of tasks are also set in personal contexts, and a smaller number of them in societal or occupational contexts.

From a didactic point of view, and based on the performed analysis, it is worth mentioning the richness of the textbook tasks in terms of encouraging mathematical thinking, reflection, and the development of mathematical sense (Lupiáñez Gómez & Rico Romero, 2015). For future studies, it would be interesting to carry out a similar analysis of tasks focused on identifying the NCTM learning expectations for mathematical processes: problem solving, reasoning and proof, communication, connections, and representation. Such results would contribute to completing the tasks analysis presented in this chapter.

Finally, regarding the limitations of the study, it must be pointed out that the analysis of the tasks was based solely on the statements of the tasks and the related information included in the aforementioned electronic textbook. Naturally, each task could involve other systems of representation, materials, and resources when implemented in class, with the teacher as the guide and facilitator of the learning process. Likewise, variations in the implementation of tasks could have an impact in the learning expectations they help to achieve. Differences in the introductory information between the English electronic version and the Czech print version of the textbook may impact the interpretation of the tasks.

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Appendix A

Specifications of the NCTM Learning Expectations for grades 3-5 (NCTM, 2000)

NO1.

1. Understand the place-value structure of the base-ten number system and be able to represent and compare whole numbers and decimals.
2. Recognise equivalent representations for the same number and generate them by decomposing and composing numbers.
3. Develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers.
4. Use models, benchmarks, and equivalent forms to judge the size of fractions.
5. Recognise and generate equivalent forms of commonly used fractions, decimals, and percents.
6. Explore numbers less than 0 by extending the number line and through familiar applications.
7. Describe classes of numbers according to characteristics such as the nature of their factors.

NO2.

1. Understand the effects of multiplying and dividing whole numbers.
2. Identify and use relationships between operations, such as division as the inverse of multiplication, to solve problems.
3. Understand and use properties of operations, such as the distributivity of multiplication over addition.
4. Develop fluency with basic number combinations for multiplication and division and use these combinations to mentally compute related problems, such as 30×50.

NO3.

1. Develop fluency in adding, subtracting, multiplying, and dividing whole numbers.
2. Develop and use strategies to estimate the results of whole-number computations and to judge the reasonableness of such results.
3. Develop and use strategies to estimate computations involving fractions and decimals in situations relevant to students' experience.
4. Use visual models, benchmarks, and equivalent forms to add and subtract commonly used fractions and decimals.
5. Select appropriate methods and tools for computing with whole numbers from among mental computation, estimation, calculators, and paper and pencil according to the context and nature of the computation and use the selected method or tool.

A1.

1. Describe, extend, and make generalisations about geometric and numeric patterns.
2. Represent and analyse patterns and functions, using words, tables, and graphs.

A2.

1. Represent the idea of a variable as an unknown quantity using a letter or a symbol.
2. Express mathematical relationships using equations.
3. Model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions.

A3.

1. Model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions.

A4.

1. Investigate how a change in one variable relates to a change in a second variable.
2. Identify and describe situations with constant or varying rates of change and compare them.

G1.

1. Identify, compare, and analyse attributes of two- and three-dimensional shapes and develop vocabulary to describe the attributes.
2. Classify two- and three-dimensional shapes according to their properties and develop definitions of classes of shapes such as triangles and pyramids.
3. Investigate, describe, and reason about the results of subdividing, combining, and transforming shapes.
4. Explore congruence and similarity.
5. Make and test conjectures about geometric properties and relationships and develop logical arguments to justify conclusions.

G2.

1. Describe location and movement using common language and geometric vocabulary.
2. Make and use coordinate systems to specify locations and to describe paths.
3. Find the distance between points along horizontal and vertical lines of a coordinate system.

G3.

1. Predict and describe the results of sliding, flipping, and turning two-dimensional shapes.
2. Describe a motion or a series of motions that will show that two shapes are congruent.
3. Identify and describe line and rotational symmetry in two- and three-dimensional shapes and designs.

G4.

1. Build and draw geometric objects.
2. Create and describe mental images of objects, patterns, and paths.
3. Identify and build a three-dimensional object from two-dimensional representations of that object.
4. Identify and build a two-dimensional representation of a three-dimensional object.
5. Use geometric models to solve problems in other areas of mathematics, such as number and measurement.
6. Recognise geometric ideas and relationships and apply them to other disciplines and to problems that arise in the classroom or in everyday life.

M1.

1. Understand such attributes as length, area, weight, volume, and size of angle and select the appropriate type of unit for measuring each attribute.
2. Understand the need for measuring with standard units and become familiar with standard units in the customary and metric systems;
3. Carry out simple unit conversions, such as from centimeters to meters, within a system of measurement.
4. Understand that measurements are approximations and how differences in units affect precision.
5. Explore what happens to measurements of a two-dimensional shape such as its perimeter and area when the shape is changed in some way.

M2.

1. Develop strategies for estimating the perimeters, areas, and volumes of irregular shapes.
2. Select and apply appropriate standard units and tools to measure length, area, volume, weight, time, temperature, and the size of angles.
3. Select and use benchmarks to estimate measurements.
4. Develop, understand, and use formulas to find the area of rectangles and related triangles and parallelograms.
5. Develop strategies to determine the surface areas and volumes of rectangular solids.

DAP1.

1. Design investigations to address a question and consider how data-collection methods affect the nature of the data set.
2. Collect data using observations, surveys, and experiments.
3. Represent data using tables and graphs such as line plots, bar graphs, and line graphs.
4. Recognise the differences in representing categorical and numerical data.

DAP2.

1. Describe the shape and important features of a set of data and compare related data sets, with an emphasis on how the data are distributed.
2. Use measures of center, focusing on the median, and understand what each does and does not indicate about the data set.
3. Compare different representations of the same data and evaluate how well each representation shows important aspects of the data.

DAP3.

1. Propose and justify conclusions and predictions that are based on data and design studies to further investigate the conclusions or predictions.

DAP4.

1. Describe events as likely or unlikely and discuss the degree of likelihood using such words as certain, equally likely, and impossible.
2. Predict the probability of outcomes of simple experiments and test the predictions.
3. Understand that the measure of the likelihood of an event can be represented by a number from 0 to 1.

Appendix B

Environments in the electronic textbook *Matematika A, učebnice pro 2. stupeň ZŠ a víceletá gymnázia* (Hejný et al., 2015)

Fractions. It allows to learn about meanings of fractions (part-whole, group-set, ratio, division) in several contexts (geometric, metric, numerical) and procedures to split different kind of objects in equal parts; to identify the fraction that a part represents in a whole; to compare fractions by using the fraction wall; etc.

Decimal numbers. It proposes to reflect on the meanings of measures expressed through decimal numbers; to perceive the relationships between different measurement units of the same quantity and their equivalences; to estimate specified amounts of quantities of objects; to use different systems of representation (pictorial, verbal, fraction, decimal number) to express a quantity; to continue a sequence of decimal numbers in context in order to better understand the decimal number system; to operate with decimal numbers; to compare and sort decimal numbers; to locate and mark them on a number line; etc.

Cubic shapes. It promotes the development of visualisation skills through the work with 3D objects (cubes), the identification and interpretation of positions and locations of solids in the three-dimensional space, the correspondence between three-dimensional bodies and two-dimensional representations of them, the characterisation of solids by means of their two-dimensional representations.

Coins. It fosters the practice of decomposition of numbers into addends with some restrictions; the understanding of the algebraic equivalence through equations relating the total value of two different sets of coins pictorially represented; the reflection about the existence of solution of these kind of equations in the corresponding range of values (those of the specified coins) and their resolutions where appropriate; the transference between an equation posed in a coins context to an equivalent one in a context of weights; the algebraic writing of an equation pictorially presented.

Egyptian division of bread. It aims to make children reflect about the meanings of fractions, the divisions, the use of models for fair sharings; to recognise equivalent representations for the same quantity; to discuss the optimality of two different procedural solutions for divisions; etc.

Wooden sticks. It promotes the development of visualisation abilities through the work with segments (sticks); the consolidation of the knowledge of properties of flat geometric figures; the identification of regularities from geometric constructions; the exploration of geometric figures subject to given conditions (e.g. fixed perimeter); the construction of some geometric figures from others; etc.

Arrow graphs. It reinforces numerical computation; understanding of equivalence, of composition of arithmetic operations and of inverse of operations; the use of properties of decimal number system; the decomposition of a number in factors subject to conditions.

Addition triangles. It promotes the understanding of equalities and operations (addition, subtraction) with structures of computing-terms; the finding of numbers subject to conditions related to their decomposition in addends and composition as addends of others.

Stepping. It facilitates the knowledge of negative numbers, suggests the use of specific verbal terminology and symbolic notation (arrows) to represent whole numbers and their addition (vertical bars); resolution of equations expressed in the previous notation.

Equations. It focuses on solving linear equations of one unknown; writing algebraically an equation expressed in certain symbolic notation (e.g., arrows).

Tiles. It includes tasks consisting of tiling flat geometric figures using polyominoes; reflecting on the different possibilities of tiling the same floor with various combinations of polyominoes.

Neighbours. It proposes to operate with non-negative decimal numbers.

Ancient Indian multiplication. It shows and proposes to practise the ancient Indian algorithm for multiplication.

Table 100. It presents a 10×10 table to perform operations with natural numbers less than 100, it introduces symbolic notation to express such operations using properties of the decimal number system; it asks for finding a value such that a symbolic expression in the introduced notation satisfies a specific condition.

Grid. It invites to identify and name flat figures presented in a grid; to estimate and measure the length of segments on a grid; to discuss the veracity of statements on the length of specific segments on a grid; to use symbolic notation to express that the value of a measure is exactly/more than/less than certain value; to discuss the paralellism of sides of certain polygons defined through their vertices in a grid.

Spider webs. To find out the positive numbers that need to be consistently added to others so that several equalities hold in a diagram; to operate with positive numbers; interpret information expressed by colours.

The bus. It encourages reading, interpreting and elaborating tables as an useful system of representation of information (in particular, number of passengers that get on and get off a bus); operating with natural numbers to solve word problems.

Origami. It includes tasks focused on creating geometric figures subject to conditions by folding, cutting and overlapping paper; showing lines and geometric constructions involving the notions of paralellism, mid-segment, etc. by folding paper; configuring a set of geometric figures on a paper to cut them using as few cuts as possible.

Scales. It is aimed at promoting the understanding of the algebraic equivalence through equations relating total value of two set of weights pictorially represented; the algebraic writing of an equation pictorially presented and vice-versa.

Number line. Tasks in this environment encourage working on proportional reasoning, representing numbers on a number line, operating with fractions, interpreting geometric notions (midpoints, segment, endpoints, distance, etc.).

Multiplication squares. It promotes understanding operations (multiplication, division) and their different effects, algebraic equivalence, working with structures of computing-terms with whole and decimal numbers; finding numbers subject to conditions related to their decomposition in factors and composition of others by means of their factors.

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Edyta Juskowiak
Adam Mickiewicz University in Poznań

Chapter 7

Ways of Solving Mathematical Tasks by Students Aged 14–15 as Manifestations of Critical Thinking

Summary: The study aims to examine methods of inference and argumentation, alongside approaches to geometric proof tasks, which present atypical challenges for 7th-grade students in Polish elementary schools. Beyond finding solutions, students were required to justify their answers and verify the correctness of their solutions. Grade 7, approximately 14 years old, marks the final stage of primary education in Poland and serves as a pivotal period for students to initiate independent critical assessments of mathematical situations. The chapter provides the analysis of solutions and solution attempts for one of these tasks. The research was initially conducted shortly before the COVID-19 pandemic and was repeated in spring 2022 following the return to in-person classes after over a year of remote learning. The analysis emphasises aspects of critical thinking, with additional consideration of how lesson organisation influences the development of critical thinking skills among students.

Keywords: critical thinking, problem solving, task analysis, geometry, inference and argumentation.

1. Introduction

Today's schoolchildren are growing up in an environment of constant technological, economic, political and communicative change. They are surrounded by a multitude of visual and auditory signals and stimuli. Whatever the job or activity, whatever the age, quick and specific, complete and correct responses or feedback are expected today. Speed is not a supportive factor in the ability to make mature and thoughtful decisions. This is all the more difficult when one is not trained in the ability to analyse situations requiring decision-making. These facts should influence the organisation of the learning-teaching process in which the student is constantly put in a situation requiring analysis and evaluation of his or her own actions. On the other hand, teachers and principals should be expected to be aware of the competences and skills with which a graduate of a modern school should be equipped. At the same time, one should not forget not to lose oneself in the process of change, perhaps weakening the quality and correctness of the substantive message, the depth of understanding, and the readiness of learners to undertake challenges that require thinking, including mathematical thinking.

2. Theoretical Background to the Research Carried Out

2.1. Critical Thinking as a Competence Needed in Today's World

Critical thinking is formed through mental effort, which should be organised from an early age. "Critical thinking is a type of realistic thinking directed towards the specific goal of evaluation. (...) The aim of critical thinking is to evaluate, reliably and realistically, the relevant aspects of a person's intellectual activity", Nęcka defines in Cognitive Psychology[1]. Spector (2019a; 2019b), on the other hand, argues that critical thinking encompasses a range of cumulative and related abilities, dispositions and other variables such as motivation, criteria, context and knowledge. The formation of critical thinking is based on experiences, e.g. observing something unusual or out of the ordinary, and then through various forms of enquiry that involve observation, inference, argumentation, proof, testing of conclusions, and reflection arriving upon the formulation of conclusions and responses.

The development of critical thinking often starts with simple experiences such as observing differences, encountering puzzling questions or problems, questioning someone's statements, which then leads to more complex experiences using higher-order mathematical thinking skills, i.e., logical reasoning, questioning assumptions, considering and evaluating alternative explanations.

In order to provoke and shape this type of thinking, knowledge and motivation for development are required. If a person is not interested in what needs to be observed or investigated, there is usually not even an attempt to solve the problem. Therefore, creative reasoning and critical thinking require motivation and an inquisitive disposition.

Among the many publications and contributions by educators, researchers in mathematics education and teaching practitioners, i.e., teachers and principals, the voice of Wagner, an internationally recognised expert in the field of education, has resonated very strongly in recent years. He is a long-time Harvard University faculty member, secondary school teacher, principal, university professor of teacher education, frequent speaker at national and international conferences, and author of two bestselling publications, Creating Innovators and The 7 Survival Skills for Work, Learning, and Citizenship in

[1] https://web.swps.pl/strefa-psyche/blog/relacje/18957-myslenie-krytyczne-jak-sie-go-nauczyc-i-wlasciwie-po-co?dt=1672145078588

the 21st Century. In his speeches and publications, Wagner pays special attention to the seven competencies needed by today's students to become effective and efficient citizens in the 21st century. They are:

- Problem solving and critical thinking,
- Working together in different groups,
- Flexibility and ability to adapt to new conditions,
- Initiative and entrepreneurship,
- Effective communication – written and oral,
- Evaluation and analysis of information,
- Curiosity about the world and imagination (Wagner, 2015).

It is indisputable that the first of these competences is, or should be, part of the process of mathematical development at each educational level. The other six competences can also stem from a student's participation in mathematics lessons, as long as he or she participates in a properly organised process of mathematical cognition. Curiosity, imagination, critical thinking, planning, verifying hypotheses, the ability to correctly select the tools of mathematics and, finally, proper communication using the language of mathematics, both verbally and in writing, should be evoked while solving mathematical problems or tasks on inference and proof. A detailed analysis and interpretation of the way in which students behave when working on such tasks independently or in teams can provide an insight into the organisation of the process of shaping students' mathematical knowledge and activities, ultimately invoking mathematical thinking.

2.2. The Relationship between Critical Thinking and Mathematical Thinking

Wagner, already quoted, lists *Problem Solving and Critical Thinking* as the first competence needed by the modern man. Properly implemented mathematics education has an important place in the formation of this competence. The proper formation of mathematical thinking fosters the development of critical thinking – a competence very much expected in the modern world. It is the opposite of automatic and schematic thinking.

It is difficult to have a single definition of this specific activity of the mind. Mathematical thinking is characterised by a whole set of mental activities undertaken by a person solving a mathematical task. The manifestations of mathematical thinking include:

- Spotting and using analogies,
- Schematisation and mathematisation,
- Defining, interpreting a given definition and applying it rationally,
- Deduction and reduction,
- Coding, construction and rational use of mathematical language,
- Algorithmisation and the rational use of algorithms (Krygowska, 1986).

The Regulation of the Ministry of Education in Poland (MEN) of 23 December 2008 on the core curriculum for pre-school education and general education in particular types of schools states that mathematical thinking is the ability to use the tools of mathematics in everyday life and to formulate judgements based on mathematical reasoning (MEN, 2008). Mason (2005), on the other hand, writes in his book that mathematical thinking is a dynamic process that expands our understanding as it allows us to deal with increasingly complex ideas.

The effectiveness of mathematical thinking is strongly influenced by:

1. The ability to use processes used in mathematical research,
2. The mastery of mental and emotional states and the ability to use them,
3. The understanding of the relevant area of mathematics (Mason, 2005, p. 143).

The same author also provides suggestions on how to influence the effective formation of mathematical thinking. Among these directives are:

4. Improving mathematical thinking. This can be achieved by concretising, generalising, making hypotheses, justifying,
5. Provoking mathematical thinking. Conducive activities include: creating a gap type of challenge, surprise, contradiction, perceived gap,
6. Fostering mathematical thinking. Here, the effect will be achieved by asking questions, posing and challenging, reflecting,
7. Sustaining mathematical thinking. We understand it as the development of an awareness of process flow, of one's own involvement, of mental states (Mason, 2005).

Each of the above is a guideline addressed to the teacher aware of his/her responsibility for the development of mathematical thinking in students. The training of all the aforementioned competences and activities and other manifestations of mathematical thinking should happen in parallel with the formation of new concepts or with working on other elements of knowledge.

Such recommendations are reflected in official documents regulating work in Polish schools (MEN, 2008). The Polish core curriculum for teaching mathematics includes, in the list of general requirements, such activities and actions which, when evoked and shaped in a student, will foster the development of critical mathematical thinking. These include:

II. Verification and interpretation of the results and assessment of the reasonableness of the solution.
V. Reasoning and argumentation.

1. Conducting a simple reasoning, providing arguments justifying the correctness of the reasoning, distinguishing a proof from an example.
2. Noticing regularities, similarities, and analogies, and drawing conclusions based on them.
3. Applying strategies stemming from the content of the task, devising strategies to solve the problem also in multistage solutions and those requiring an ability to combine the knowledge of different fields of mathematics (MEN, 2008).

2.3. Analysis of the Process of Solving Mathematical Tasks in Research

One way to analyse the phenomenon of critical thinking in students is to analyse their approaches to solving mathematical problems. Research in mathematics education regarding the analysis and description of the process of solving mathematical tasks by both students and teachers has been conducted for many years. It is part of the interdisciplinary cognitive science, and studies in this field are geared towards learning about the actual human learning process. It is primarily a way of getting to know the thinking process qualitatively and not just quantitatively. This means that they focus not only on quantitative results, but primarily on understanding the deep thought processes that occur when solving mathematical problems. Achieving a fuller understanding of this process is crucial for improving teaching methods and developing effective educational strategies.

Research in mathematics education concerns various aspects, such as thinking strategies, problem-solving approaches, cognitive barriers or the development of logical or, ultimately, critical thinking skills. Researchers also try to identify factors influencing the effectiveness of mathematics teaching and identify best practices. One important aspect of didactic research is also the analysis of the role of teachers in the process of imparting mathematical knowledge. Such research focuses on the teaching methods used by educators, their approach to the development of students' mathematical skills and their ways of dealing with possible learning difficulties.

From an interdisciplinary perspective, this research broadens our understanding of cognitive processes by integrating knowledge from cognitive psychology, neuroscience, and other disciplines. As a result, they contribute to a better adaptation of teaching methods to individual students' needs, which is crucial for effective mathematics education and the development of thinking skills among students.

Here are examples of research in mathematics education on the analysis of the mathematical task-solving process[2] :

Research involving the analysis of the written work of:

- participants in mathematics Olympiads (Callejo, 1994; Ciosek, 1978),
- mathematics degree candidates (Ciesielska et al., 2004),
- students graduating from secondary school, mathematics teachers, and mathematics students solving the same matriculation task (Powązka, 2004),
- students in mathematics education and teachers solving the same set of tasks regarding the concept of function (Kortus, 2006),
- students solving tasks about the application of mathematics (Treliński, 1985),
- groups of students, from the perspective of the heuristic issues involved in solving a mathematical problem (Schoenfeld, 1979),
- students solving mathematical tasks using a computer or calculator (Ratusiński, 2003; Herma, 2004; Kąkol & Ratusiński, 2004; Juskowiak, 2004; Duda, 2018),
- students, in the scope of occurrences of critical thinking (Novakova, 2021; Juskowiak, 2021; Kiss & Konya, 2021; Ponte, 2022; Pytlak, 2022).

Research involving observation of individual work:

- students solving an atypical task (Żeromska, 2001),
- students solving a text-based task (Ćwik, 1990).

A detailed analysis of the process of solving a mathematical task allows one to learn about working methods and strategies, makes it possible to clarify what triggers thinking and what becomes the cause of difficulties and obstacles. Conclusions of such research often become the basis for the implementation of new educational solutions both at the level of the lesson unit, the organisation of which is the responsibility of the teacher, and the whole system and learning process. Analysis of students' solutions due to manifestations of critical thinking, i.e. argumentation or inference, can certainly become the basis for draw-

[2] Examples and categories of research are partly taken from Ciosek (2005, pp. 28–29), entitled "Level of task solving at different levels of mathematical knowledge and experience".

ing conclusions about, among others, the types of tasks solved during lessons, the way they are solved, the role of the student and the teacher in the process of undertaking the creative act of justification, or the readiness (substantive and methodical) of the student to solve tasks of the justification type.

2.4. Selection of Tasks Provoking Critical Mathematical Thinking in Polish Textbooks

There are also studies (often conducted as part of master's theses) devoted to qualitative-quantitative analyses of the most frequently used textbooks by Polish mathematics teachers in grades 7 and 8 in terms of the number of tasks aimed at provoking inference, augmentation and proof competences. The study by Szalbierz (2022) shows that there are few such tasks therein viable to become a tool for provoking critical mathematical thinking and shaping all the aforementioned mathematical activities and competences. The textbooks analysed were: "Matematyka z plusem", publishing house: Gdańskie Wydawnictwo Oświatowe, grade 7 (Bolałek et al., 2020) and grade 8 (Bolałek et al., 2021), and "Matematyka z kluczem", publishing house: Nowa Era, grade 7 (Braun et al., 2020) and grade 8 (Braun et al., 2021).

Table 1. Analysis of mathematics textbooks in terms of the number of tasks that provoke inference and argumentation

Textbook	Total number of tasks	Number of tasks provoking inference and argumentation
„Matematyka z plusem" [eng. „Mathematics with plus"], publishing house: Gdańskie Wydawnictwo Oświatowe, grade 7 (Bolałek et al., 2020)	1403	32 (3%)
„Matematyka z plusem" [eng. „Mathematics with plus"], publishing house: Gdańskie Wydawnictwo Oświatowe, grade 8 (Bolałek et al., 2021)	1080	64 (6%)
"Matematyka z kluczem" [eng. "Mathematics with key"], publishing house: Nowa Era, grade 7 (Braun et al., 2020)	964	39 (4%)
"Matematyka z kluczem" [eng. "Mathematics with key"], publishing house: Nowa Era, grade 8 (Braun et al., 2021)	1044	140 (13%)

The data presented in the table and other analyses conducted by students writing their master's theses at the Faculty Centre for the Didactics of Mathematics and Informatics under the supervision of the author of this article show that there are too few open tasks, problem-type tasks, tasks with excessive and insufficient data, i.e. tasks that provoke observation, searching, experience, argumentation and research. There is a lack of atypical tasks, i.e. tasks that are new to the student and that do not follow familiar patterns.

Examples of tasks that provoke inference and argumentation are:

1. Justify that the product of two two-digit numbers cannot be a five-digit number. (Bolałek et al., 2020, grade 7).
2. In a quadrilateral ABCD, the points E, F G, H are marked successively on the sides AB, BC,CD and DA in such a way that AE=BF=CG=DH. Prove that the quadrilateral EFGH is a rhombus. (Braun et al., 2021, grade 8).

3. Research

3.1. Critical Thinking and the Pandemic

This chapter presents an excerpt from a study on a qualitative analysis of the ways in which students in grades 7 and 8 solved non-standard, proof, inference and argumentation tasks, concerning the way in which students approached the challenge presented to them, the path of reasoning taken, the correctness and the way in which they argued the provided result (Juskowiak, 2019).

The pandemic, a period of more than a year of remote work, has completely changed the way lessons are conducted (Juskowiak, Vetulani 2022; Jaskulska 2021), at first disorganising the learning-teaching process and reducing it in many cases to simply sending students work to be done. Over time, synchronous remote classes started to be arranged on learning platforms. This has definitely changed the definition of lesson delivery. Numerous new remote work tools and applications were introduced to support the introduction of mathematical concepts, lessons either became lectures with elements of demonstration, or a time of discussion and chat between students and teachers. Few teachers were able to engage all of their students and review their progress on an ongoing basis, and there were times when students did not participate at all.

It was suspected, and at the same time expected, that reinforcing the role of student autonomy, openness to different sources of information, and the inclusion of a wide range of IT tools in task work would enhance the development of critical thinking. A need therefore arose to see to what extent these hypotheses were confirmed by the students' performance.

4. Methodology

The article presents and discusses exemplary solutions to the same geometric task that was performed by Polish students in seventh grade (aged 14-15). The discussion will concern the features specific to critical thinking found in the solutions. The presented works come from more comprehensive research conducted by the author of this article on a group of almost two thousand students aged 14 who solved 6 geometrical tasks requiring justification. The main aim of the research was to test students' readiness to demonstrate formal mathematical thinking and the ability and means to justify their judgments.

Only an excerpt of the findings is included in this paper, with additional discussion focused on analysing the students' solutions to one of the worksheet tasks in the scope of manifestations of critical thinking.

The research fragment described here addresses the following research question:

> Whether and how the implementation of remote work tools during the pandemic influenced the course and effect of the process of forming students' reasoning and argumentation skills, using manifestations of formal thinking.

Studying the performance of students in a specific school context allows for a better understanding of the effectiveness of the educational process. Comparing the results of students using the same textbook, working with the same teachers, on the basis of the same working methods and concepts, allows the effectiveness of the given educational material to be assessed and can help identify strengths and areas for improvement. The time of the pandemic, as mentioned earlier, provided an opportunity to incorporate a number of new remote working tools, the use of which was expected to change the concept of working with students, to strengthen the role of student autonomy, and to enhance the development of critical thinking.

The results of the analysis relate to:

- Two groups of students attending the same school (students in grades 7 & 8, aged 14-15), taught by the same teacher and using the same textbook,
- Solutions to one task from the test sheet[3],
- Analyses of work sets from two periods – pre-pandemic and post-pandemic (June 2022)[4].

[3] The publication's limitations do not allow for the presentation of results and conclusions from the analysis of the solutions of all tasks.

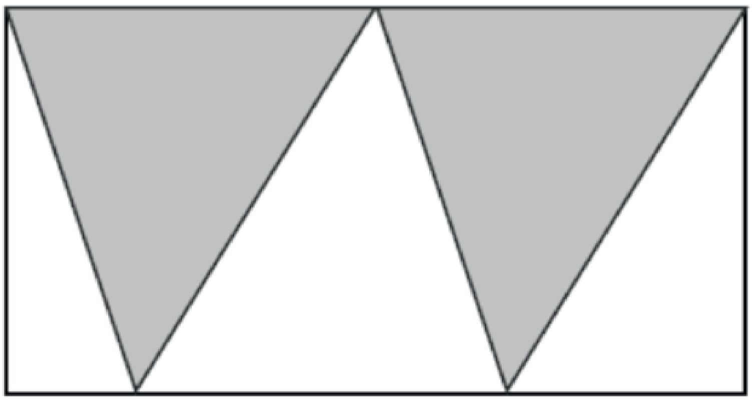
[4] The author of this article continues to conduct research using the above work sheets. The studies have been extended to secondary schools and future mathematics teachers.

The students had 45 minutes to solve the tasks, but the time limit could be extended if necessary, upon the teacher's agreement. They were allowed to use geometric instruments and calculators. The teachers, as supervisors, were given detailed instructions on the organisation and conduct of the test.

The research tool was the following task

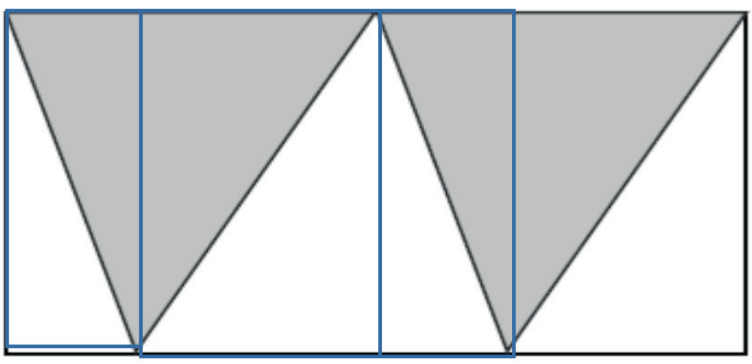
Content of the task[5]:

Check what is the ratio of the area of shaded figures to the area of the rectangle.



It is possible to use a variety of ways to look for the answer to this question. One expected approach might be to divide the whole rectangle into parts, in which the shaded and white parts have equal areas:

Figure 1. Drawing proof showing the equality of the shaded and white surfaces across the rectangle



5 This task is taken from the textbook for grade 1 of lower secondary school, Gdańskie Wydawnictwo Oświatowe.

5. Results and Short Discussion

5.1. Examples of Solutions

The analysis identified the dominant ways of working on the task. These are:

- Measurement and billing,
- Inference using the symbolic language of mathematics,
- Solution as a description of the proceedings,
- Transformation.

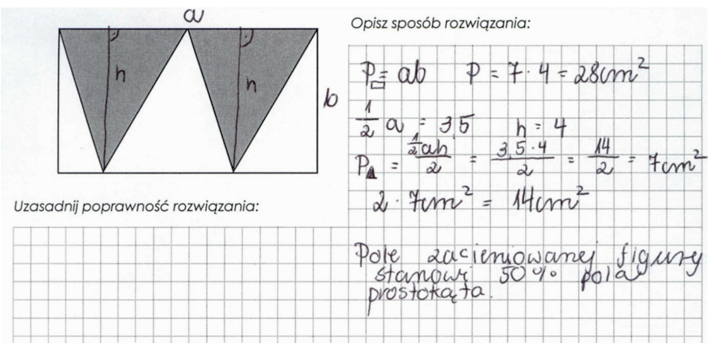
In addition, distinctions were made:

- No solution and no comment,
- No solution but with commentary,
- No solution but with the correct result,
- No solution, wrong result.

One example of a student's solution for each of the approaches is presented below, accompanied by short didactic commentary.

5.1.1. Measurement and Account

Figure 2. Example of a solution by measurement and account



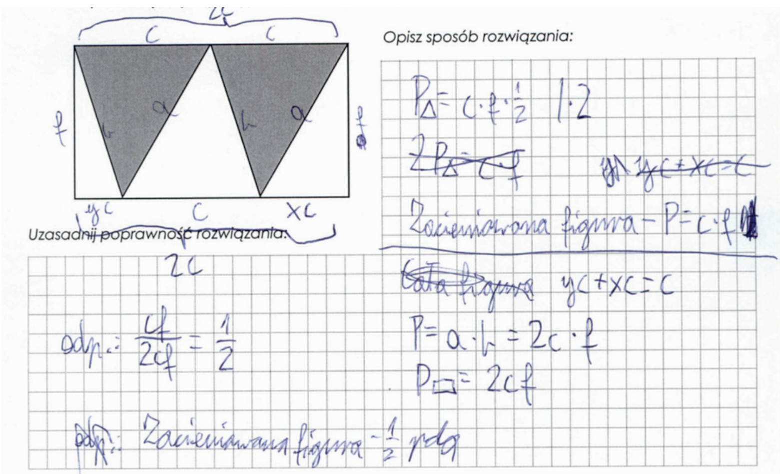
Text translation

The area of the shaded figure constitutes 50% of the area of the rectangle.

The student took on the challenge of dealing with a new situation – a task that lacked numerical data. He introduced a line of reasoning (successive steps of reasoning) directly related to the formulated question. The initial entries show that he intended to first determine the size of the whole rectangle, then the size of the areas of two triangles, treated without justification as congruent triangles, and then to compare the values so obtained. He attempted to carry out this concept by calculation, unfortunately making an incorrect assumption at the start of his work regarding the length of the base of the shaded triangle. The content of the task did not provide any information on the length of any of the sides. The student measured the lengths of the sides of the rectangle and the length of the base of the triangle using a ruler. He used the correct formulae and correctly deduced that if the value of the area of the shaded triangles equals half the area of the rectangle, then this value is 50% of the area of the rectangle.

5.1.2. Inference Using the Symbolic Language of Mathematics

Figure 3. Example of a solution by inference using the symbolic language of mathematics



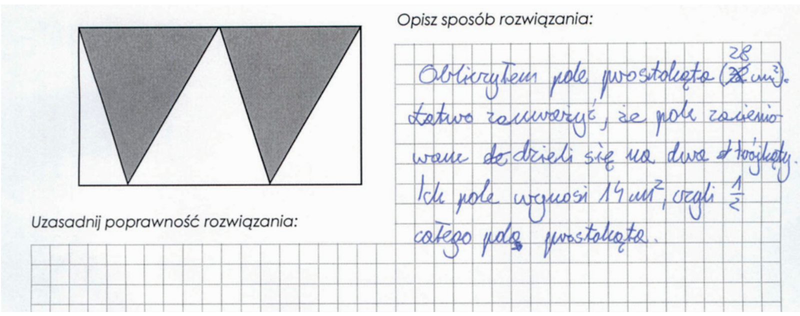
Text translation

Shaded figure – ½ field

The student, noticing the lack of data in the content of the task, independently determined the lengths of all the segments appearing in the figure. He assumed that the lengths of the bases of the shaded rectangles and the white triangle between them are the same, without justifying this step. Furthermore, by describing the sides a and b with the same symbols, he assumed that they were of the same length and were parallel, an action he also did not justify. However, he used these symbols twice for two different purposes, once describing the lengths of the sides of the triangle that he had introduced, and in the second case using the commonly used formula for the area of a rectangle. He correctly answered the question in the task using the ratio of the sum of the areas of the triangles to the area of the rectangle using the symbolic notations introduced. Finally, he created a ratio between the fields thus described using the symbolism he had created himself: cf/2cf, which led him to the fraction ½. This result was the basis for the verbal answer that the shaded figure was a ½ field.

5.1.3. Solution as a Description of the Proceedings

Figure 4. Example of a solution by description of proceedings



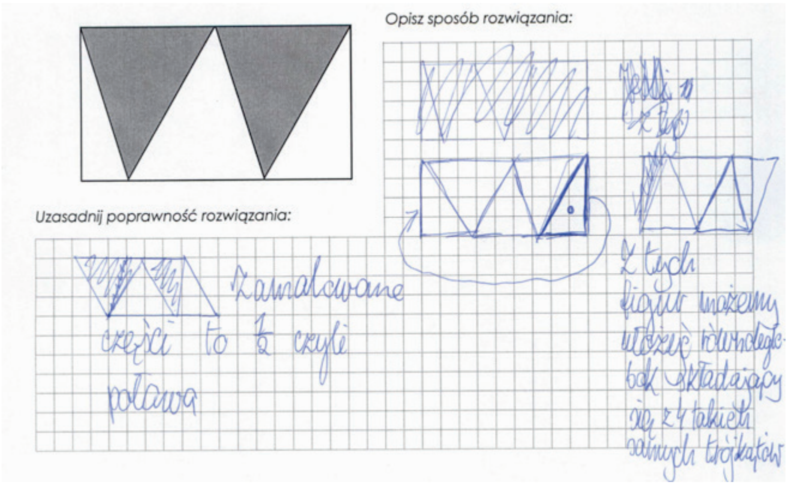
Text translation

I calculated the area of the rectangle (28). It is easy to see that the shaded areas are divided into two triangles. Their area is 14 or 1/2 of the whole rectangle.

The student has not included any calculations, although the description states that he has calculated the area of the rectangle and the areas of the shaded triangles. He describes the values of the areas with specific numbers, and therefore presumably measured the lengths of the sides needed to do the calculations. As above, he assumes that the lengths of the bases of the shaded triangles are the same. From the statement ‚it is easy to see' it can be inferred that the assumptions made are the result of his observation and analysis of the drawing. The student did not justify the correctness of the assumptions and conclusions made.

5.1.4. Transformation

Figure 5. Example of a solution by transformation



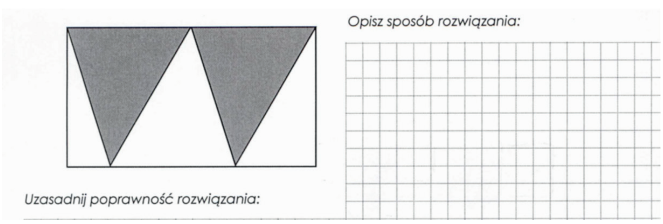
Text translation

From these figures, we can form a parallelogram consisting of 4 of the same triangles.

The last of the students' approaches consisted of implementing the observation of the relationships of the objects in the drawing. It can be assumed that the student intended to use the concept of transforming the drawing by using parallel displacement, although the first attempts to create an adequate drawing failed. The student assumed the parallelism and equal length of the corresponding pairs of sides of the resulting triangles, which is evident from the description, and not from the accompanying drawings. The author of the solution states that the newly created figure is a parallelogram, but bases the justification for his solution on the fact that the entire area is made up of 4 identical triangles, two of which are shaded. However, there is no justification of the assumptions made and no reference to theory.

5.1.5. No Solution and No Comment

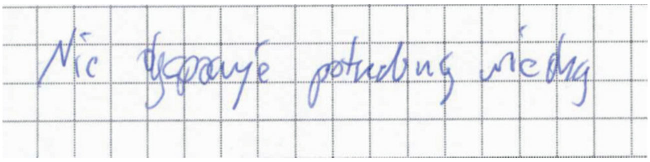
Figure 6. Example of no solution and no comment



This group of responses came from students who skipped this task completely – presumably they had no idea what could be done with such a task at all or did not want to think about it.

5.1.6. No Solution But with Commentary

Figure 7. Example of no solution but with commentary



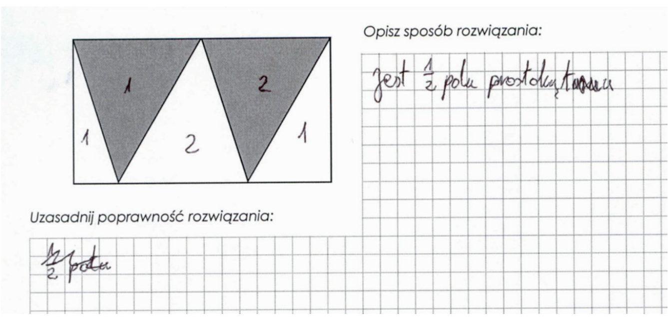
Text translation

I do not have the necessary knowledge.

The students often included the following type of entry. The theory of the areas of plane figures is introduced in grade 6 of primary school – it can be assumed that the problem was not knowledge or lack thereof, but a deficiency in the skills involved in dealing with a completely new and unusual situation of missing numerical data in an otherwise typical task and the need to justify the performed operations. However, it must be assumed that the student did not have the knowledge needed to solve this task.

5.1.7. No Solution But with Correct Result

Figure 8. Example of no solution but with correct result



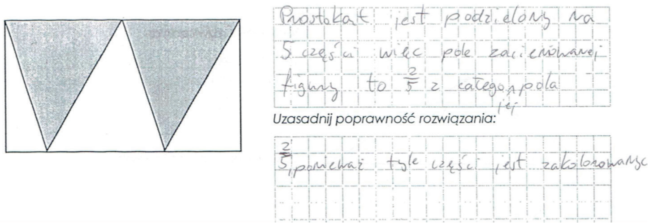
Text translation

It's a half of rectangle's field.

The student only wrote down the answer. From the markings introduced in the figure, it can be suspected that the student assumed the equality of the corresponding areas made up of triangles labelled with the same numbers, which in turn are made up of the same two triangles. No calculus, no description, no justification of the reasoning carried out. This reasoning is intuitive. It can be assumed that the student noticed (assumed) that the triangles with areas denoted by the value 2 are congruent (as this can be perceived), so their areas are the same, but the further stage of reasoning required the application of mental manipulation and justification. The student's work does not provide a basis for assessing whether what he presented as the result does stem from a 'guess'.

5.1.8. No Solution, Wrong Result

Figure 9. Example of no solution and wrong result



Text translation

The rectangle is divided into 5 parts so the area of the shaded figure is 2/5 of the whole area.

2/5 because these parts are coloured in

The above answer that two parts out of five were marked was not an isolated answer. There were also responses that the shaded triangles represented two thirds of the area of the rectangle.

Asking which part of the rectangle is the painted area may certainly not have triggered geometric thinking in the students, instead invoking an association to tasks from grade four, where questions of the same type involved verifying their knowledge and skills in the area of 'simple fractions'. Applying this reasoning when we cannot see or do not know that the rectangle/figure has been divided into exactly the same congruent figures will not allow the student to get the correct answer.

5.2. Impact of the Pandemic on the Students' Approach to the Task

The table below shows the results of the analysis of the solutions to the presented task, broken down by the distinguished types of response. It provides the amount of ways in which the surveyed students acted in the time before and after the pandemic.

Table 2. Summary of approaches to working on the task, before and after the pandemic period

Solution methods	Pre-pandemic research Amount of solutions	Post-pandemic research Amount of solutions
Measure and calculate	14	2
Reasoning with symbols	4	1
Verbal solution	2	1
Transformation	4	2
No solution, without comment	2	7
No solution, with comment	4	0
No solution, with correct answer	1	6
No solution, with incorrect answer	0	3
Catfish	31	23

Although the comparison is between two small groups, with 31 students before the pandemic and 23 after the pandemic, one can take the liberty of noting and formulating some regularities:

- Before the pandemic, the dominant way of working was to use measurement and calculus strategies; after the pandemic, all four ways of working on the task (measurement and calculus, inference using the symbolic language of mathematics, solution as a description of the procedure, rectangular transformation) were used a similarly low amount of times,
- Before the pandemic, a significant number of students attempted to solve the task, whereas after the pandemic a significant number of students did not carry out or write down any reasoning.

6. Summary and Conclusions

Critical thinking is a desirable competence in all disciplines of life, but despite provisions in various documents, such as curricula or core curricula, concerning the development of manifestations of critical thinking, activities and topics focusing on this issue are rare, even during mathematics lessons. As already mentioned, an effectively stimulated process of developing mathematical thinking results in the development of critical thinking. The analysis of the solutions to the task presented in this article shows the difficulties students may encounter in implying mathematical operations as part of the process of solving unusual mathematics tasks, choosing effective, in their understanding, ways of reasoning, justifying their judgements, reflecting, choosing the appropriate methods or making comments, and, finally, asking questions. These skills should be acquired by students in parallel with the process of acquiring mathematical knowledge, and they should face different challenges and new situations at each level of their intellectual development. Upon deeper analysis, it can be seen that students make attempts at analysis and reflective observations. They also introduce their own assumptions to trigger a planned and familiar reasoning process – this is, unfortunately, a familiar pattern of behaviour that they like to implement somewhat forcefully in unfamiliar cases. Students make correct attempts to use symbolic language, the language of mathematics, and the properties of concepts.

In the case of the task that forms the basis of the analysis presented in this article, it has already been recognised that students:

- Mostly give the correct answer,
- When providing answers, they overwhelmingly do not use formal reasoning,
- Mostly use specific measurements with a ruler,
- Do not see the need to justify their statements,
- Apply faulty reasoning,
- Are unable to separate their reasoning from the specific drawing provided with the task.

Despite many interesting ideas for solving the task, students are unable to carry out complete and correct reasoning at the level of formal or even empirical thinking.

A comparison of the students' approaches to solving the task before and after the pandemic period shows significant differences. These differences are both positive and negative. The positives include the fact that the strategies used by the students after the pandemic are more varied. Perhaps the fact that they were forced to be more independent in their work triggered their desire for independent exploration. On the other hand, too many students gave up on solving the task. This may be because they are not persistent in their search, the lack of an immediate idea discouraging them from working further.

There is a need for more open tasks, problem tasks, tasks that provoke observation, exploration, experience, argumentation, and research. There is a lack of atypical tasks, i.e., tasks that are new to the student and do not follow familiar patterns. On the other hand, it is the teacher's responsibility to provoke, develop, and support mathematical thinking in students.

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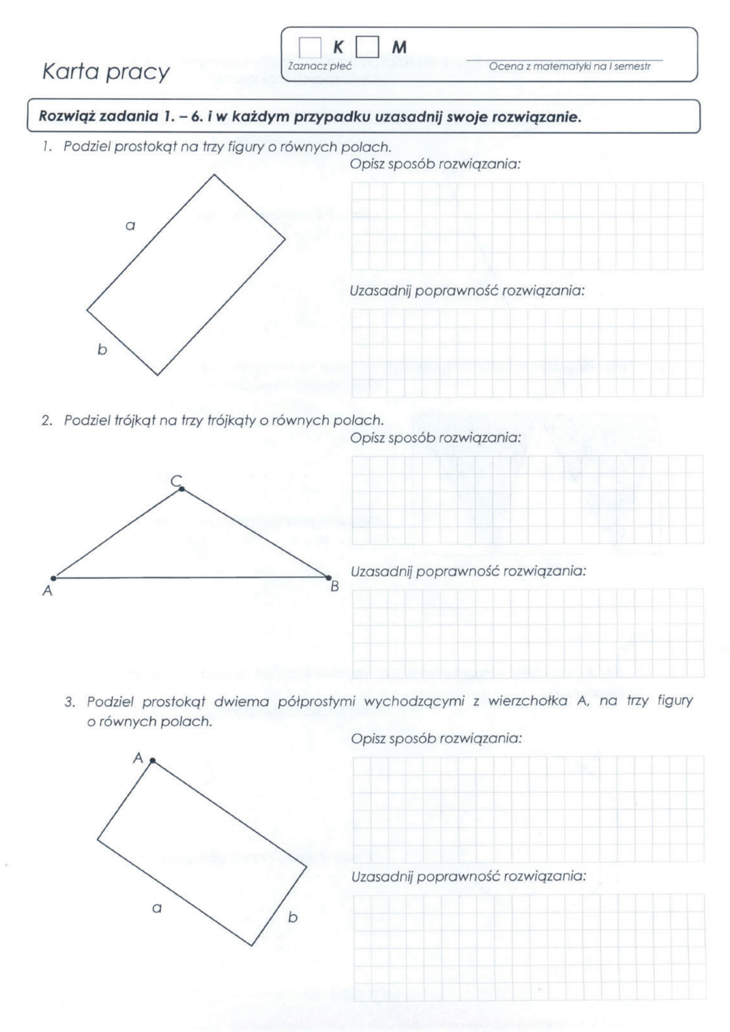
Appendix

The full worksheet used in the research.



Karta pracy

K M

Zaznacz płeć

Ocena z matematyki na I semestr

Rozwiąż zadania 1. – 6. i w każdym przypadku uzasadnij swoje rozwiązanie.

1. Podziel prostokąt na trzy figury o równych polach.

Opisz sposób rozwiązania:

Uzasadnij poprawność rozwiązania:

2. Podziel trójkąt na trzy trójkąty o równych polach.

Opisz sposób rozwiązania:

Uzasadnij poprawność rozwiązania:

3. Podziel prostokąt dwiema półprostymi wychodzącymi z wierzchołka A, na trzy figury o równych polach.

Opisz sposób rozwiązania:

Uzasadnij poprawność rozwiązania:

Text translation

1. Divide the rectangle into three figures with equal areas.

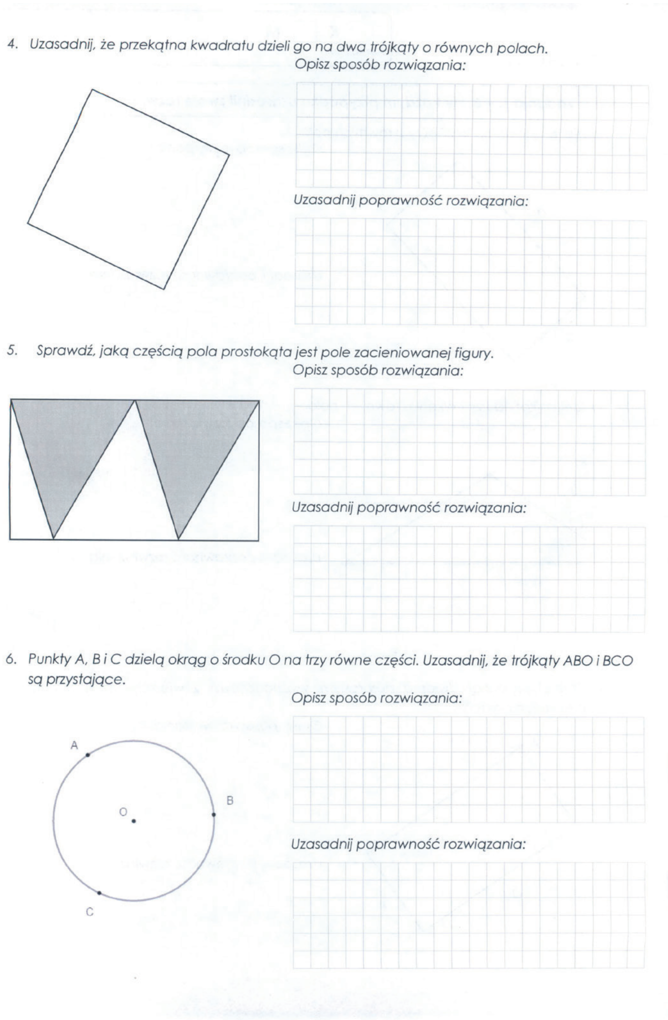
Describe the solution of the task:
Justify the correctness of the solution:

2. Divide the triangle into three triangles with equal areas.

Describe the solution of the task:
Justify the correctness of the solution:

3. Divide the rectangle into three figures with equal areas using two rays coming from the apex A.

Describe the solution of the task:
Justify the correctness of the solution:



4. Uzasadnij, że przekątna kwadratu dzieli go na dwa trójkąty o równych polach.

Opisz sposób rozwiązania:

Uzasadnij poprawność rozwiązania:

5. Sprawdź, jaką częścią pola prostokąta jest pole zacieniowanej figury.

Opisz sposób rozwiązania:

Uzasadnij poprawność rozwiązania:

6. Punkty A, B i C dzielą okrąg o środku O na trzy równe części. Uzasadnij, że trójkąty ABO i BCO są przystające.

A

B

O

C

Opisz sposób rozwiązania:

Uzasadnij poprawność rozwiązania:

Text translation

4. Justify that the diagonal of a square divides it into triangles with equal areas.

Describe the solution of the task:
Justify the correctness of the solution:

5. Check what fraction of the rectangle's area is occupied by the shaded figure.

Describe the solution of the task:
Justify the correctness of the solution:

6. Points A, B, and C divide the circle with center O into three equal parts. Justify that triangles ABO and BCO are congruent.

Describe the solution of the task:
Justify the correctness of the solution:

Edyta Nowińska & Elena Kok
Osnabrück University

Chapter 8

CHALLENGING ASPECTS OF METACOGNITIVE SUPPORT IN THE CLASSROOM AND HOW TO PREPARE TEACHERS FOR THEM

Summary: Supporting students' metacognition in teaching is both an important goal of teaching and a means for improving learning achievement. So far, however, there is a lack of studies on how to best prepare teachers for supporting metacognition in the classroom. This chapter explains the complexity of metacognitive support and describes an analytical tool that has proven useful in improving teachers' skills to foster students' metacognition. It also discusses the findings from a preliminary study in which prospective teachers learned how to support metacognition. Based on the theory-based considerations concerning the complexity of metacognitive support and on findings from this study, the chapter underscores the need for further research to help teachers improve their skills for metacognitive support.

Keywords: metacognitive support, metacognition, teacher noticing.

1. Introduction

Metacognition refers to a person's own cognition about cognition and regulation of cognition (Brown, 1978; Flavell, 1979). Since Flavell (1976; 1979), metacognition has been argued to be a critical factor in improving students' learning and enhancing their learning achievement (Brown, 1978; Glaser, 1990; Lingel et al., 2014; National Research Council, 2005; Schunk & Greene, 2018; Veenman & Elshout, 1995; Young & Fry, 2008).

Students, who behave in a metacognitive way by taking control over their learning process, plan, control and evaluate their cognitive activities. This, in turn, enables them to recognise the need for changing their current cognitive activities or reorganising their knowledge (Gunstone, 1991; Lee, 2005; Tsai, 2001). This kind of behavior becomes increasingly important not only in school education but also outside of it. The rapid growth of the amount of knowledge, information and technology requires a constant learning process, which has to be controlled by the learner to successfully master new challenges in private, professional, and social life. Therefore, acquiring skills for an adequate regulation of one's own cognitive activities becomes a necessity for students to succeed in the life-long learning in the 21st century.

In the context of school education, systematic reviews and meta-analyses of intervention studies have provided evidence that promoting students' metacognition increases their cognitive learning achievement (De Boer et al., 2018; Dignath & Büttner, 2008; Verschaffel et al., 2019). Furthermore, a meta-review of studies by Wang et al. (1990) showed that metacognition is the most important predictor of learning performance. This finding is consistent with the results from a meta-analysis by Hattie (2009). Supporting students' metacognition is therefore regarded both as an important goal of teaching and as a means of enhancing teaching effectiveness (Bransford & Donovan, 2005; De Boer et al., 2018; Hasselhorn, 1992). Consequently, it is not surprising that metacognition is gaining more and more attention in various disciplines related to school education (Dignath & Mevarech, 2021).

In his recent reflection on the field of research on metacognition, Azevedo (2020) concluded that there is an urgent need for exploring two questions: How can teachers foster metacognition in their students within their everyday teaching and how can the teachers be best prepared for this task? Azevedo's conclusion is based on research findings indicating that teachers lack knowledge about metacognition and rarely support it in their students (e.g., Dignath & Büttner, 2018; Dignath & Veenmann, 2020). Hence, more research is needed to overcome the gap between research on metacognition and practice in schools (Dignath & Mevarech, 2021).

To support students' metacognition, several instructional interventions have been developed and implemented in teaching (see Mevarech & Kramarski, 2003; Schneider & Artelt, 2010 for an overview of interventions implemented in teaching mathematics). Positive effects of these interventions on students' cognitive learning achievement led to the assumption that it is possible to train teachers in supporting students' metacognition. However, analysing the effects of the training offered to the teachers on the metacognitive support provided by these teachers in their classrooms was not the focus in these interventions. Therefore, no direct implications for a future teacher training can be derived. Furthermore, different approaches to support metacognition were implemented in different interventions. This makes it difficult to analyse and compare intervention studies with regard to their effects on the teachers' support.

Inspired by findings from metacognitive interventions, many researchers put forward recommendations for teachers of how to support students' metacognition in natural classroom settings. They stress, for instance, the need for engaging students in metacognitive activities, particularly in class discussions (Veenman et al., 2006; Zepeda et al., 2019; Zion et al., 2005). Supporting metacognition (hereafter: metacognitive support) is a complex process, in which teachers have to orchestrate their own instructions and students' reactions to them. This process involves an ongoing and focused observation of interactions in class discussions and cannot be reduced to a collection of easy instructions (Hasselhorn, 1992; Zepeda et al., 2019). Understanding its complexity is necessary to help teachers support metacognition in their teaching.

The purpose of this chapter is twofold. First, it explains a variety of aspects of class discussions teachers have to orchestrate to support metacognition in their students. In doing so, the paper sheds light on the complexity of metacognitive support. Second, it discusses the question of how teachers can learn to support metacognition in their students. Since the extent to which teachers support students' metacognition depends on what they notice in their classrooms, this discussion is framed by research on *teacher noting* (e.g., Jacobs et al., 2010; van Es et al., 2017).

In the following, first the meaning of metacognition and researchers' recommendations on how to promote metacognition in class discussions are explained. Next, the chapter addresses the question of how teachers can learn to support metacognition in their classrooms. In this discussion, the construct "teacher noticing" is explained. Afterwards, the paper describes the design of a course aimed at enhancing prospective teachers' sensitivity to various aspects of metacognitive support. It briefly illustrates the results of a pilot study, which explored the effects of this course. Finally, implications for further research are given.

2. Metacognition

Metacognition is traditionally defined as a person's own cognition about cognition and regulation of cognition (Brown, 1978; Flavell, 1976; 1979). It is generally accepted that a distinction can be made between "metacognitive knowledge" and "metacognitive skills" (Veenman, 2005). Metacognitive knowledge refers to the knowledge one has about the interplay between task, strategy characteristics and person. Metacognitive skills, in contrast, refer to the actual regulation of and control over one's learning (Flavell, 1979; Veenman, 2005).

Metacognitive knowledge can enable an individual to engage in a particular learning situation in a particular way, based on their current knowledge about this situation. The component *task* consists of knowledge about task difficulty, resources necessary to complete the task, knowledge about the goals of a particular task and possible outcomes (Flavell, 1979). The component *strategy* refers to knowledge about approaches that can be used to achieve a particular goal. It can inform decisions on how to proceed in a certain situation (Flavell, 1979). Finally, the component *person* includes, among other things, the individual's knowledge and beliefs about themself as a thinker or learner, particularly about their abilities to learn in a particular domain or to solve a particular kind of task (Flavell, 1979; Mevarech & Fridkin, 2006). According to Flavell (1979), all components of metacognitive knowledge overlap and are not clearly distinguishable from each other. An individual who activates this knowledge usually works with all three components of it, as far as these are available to them in the moment (Hasselhorn, 1992). Similarly to other kinds of knowledge, metacognitive knowledge can be inadequate or fragmentary, and does not automatically lead to an adequate strategic behaviour or to a better performance.

Interestingly, a longitudinal study conducted by Lingel et al. (2014) has generated empirical evidence that metacognitive knowledge related to learning mathematics is of substantial importance for students' mathematics achievement and for the development of this achievement at the beginning of secondary school. These findings suggest that providing students with opportunities to generate and use their metacognitive knowledge in classes can improve their learning achievement.

Metacognitive skills, in contrast, refer to the active monitoring, regulation and orchestration of cognitive processes, usually in order to achieve a concrete goal or objective (Brown, 1987; Flavell, 1976). Learners manifest their metacognitive activities by planning and controlling their thought processes and reflecting on them and their results. For instance, when proving a mathematical theorem or solving an equation, students who metacognitively regulate their learning, look for information given in their assignment, specify what they are asked for, plan how to best proceed in order to achieve their goals or sub-goals, and plan which strategies and previous learning experiences can be useful. While executing their plan, they control their argumentation, calculations, and their use of representations. They also control their progress and reflect if there is a need for adaptations. Finally, they reflect on their approach, the assignment, and their understanding of what they have achieved and learned in the particular situation. This reflection enables them to make their learning experience and learning achievement conscious for them and, consequently, to reorganise their knowledge, if necessary. Of course, metacognitive regulation does not only refer to solving concrete problems or tasks. It also manifests in controlling the correctness of an argument and reflecting about one's own knowledge, properties of mathematical objects, or one's own conceptions and misconceptions related to them (see examples in Cohors-Fresenborg & Kaune, 2007; Nowińska, 2016).

Intervention studies have generated empirical evidence that supporting students' metacognitive skills can improve student's learning achievement (e.g., Mevarech & Kramarski, 1997; Mevarech & Kramarski, 2014). This support is, however, a very complex task for teachers (Depaepe et al., 2010).

3. Metacognitive Support in Class Discussions

One challenging aspect of supporting students' metacognition in natural classroom settings is to help students develop a metacognitive habit toward their learning process, therefore enabling them to spontaneously and adequately regulate their own cognitive activities and comprehension (Hasselhorn, 1992; Schunk & Greene, 2018). Undoubtedly, such a habit develops with practice. Based on instructional interventions, researchers argue that teachers' actions and interactions with students in class discussions have the potential to facilitate this practice. They also claim that providing learning materials with integrated metacognitive guidance might support students' metacognition, but – as shown by an intervention study in science education (Eggert et al., 2013) – this is not sufficient to ensure that students use this guidance in an appropriate way. Thus, researchers' recommendations related to metacognitive support in teaching stress the crucial role of engaging students in metacognitive activities, particularly in class discussions (Veenman et al, 2006; Zepeda et al., 2019; Zion et al., 2005). There, students externalise their metacognitive thoughts and get feedback on them from their teacher or classmates. According to Zion et al. (2005), "metacognitive skills development is typically fostered by asking students to reflect on and explicitly monitor their learning performance" (p. 959). Veenman et al. (2006) also argue, that engaging learners in a metacognitive reflection on what to do, when to do it, and why it is done, and how to thereby achieve a particular cognitive goal, facilitates the development of students' metacognitive skills and metacognitive knowledge.

Teachers can support students' engagement in metacognition by integrating metacognitive activities into their own repertoire, for instance, by demonstrating metacognitive activities and explaining their significance for students' learning. This might, however, not be sufficient, because observing their teachers' behavior and metacognitive activities does not explicitly challenge the students to enact and internalise the demonstrated behavior (Depaepe et al., 2010; Schraw, 1998; Veenman et al., 2006; Zepeda et al., 2019). Teachers also need to explicitly instruct students to show a certain metacognitive behavior and, moreover, give them opportunities to regulate their learning activities in a self-determined way, i.e. autonomously and without encouragement from their teacher (Kramarski & Mevarech, 2003; Mevarech & Kramarski, 2003; Papleontiou-louca, 2003; Veenman et al., 2006; Zion et al., 2005). When providing metacognitive support, teachers not only need to enhance the quantity of students' metacognitive activities, but also the quality of them – the extent to which metacognitive activities are elaborate and combined with explanations. If the metacognitive activities are executed without an elaborate explanation, they could remain superficial. Superficial metacognitive activities might not result in the students identifying mistakes, misconceptions, and problems in comprehension and therefore recognising the need to reorganise their current course of action or knowledge.

Researchers also stress the necessity of helping students to precisely articulate their metacognitive thoughts by reflecting on questions like what is important in a certain learning situation, and why and how the important knowledge or skill can be used in other situations (Costa, 1984; Kramarski & Mevarech, 2003; Veenman et al., 2006). This reflection makes the learning process more conscious for students, which is important to extend their metacognitive knowledge. Teachers can make explicit references to these questions and stress the importance of asking them. Furthermore, they need to motivate students to make the articulation of their answers a habit of learning.

Addressing recommendations for metacognitive support, Bransford and Donovan (2005) further argue that students are better able to regulate their learning if their metacognitive knowledge includes knowledge about subject-specific lenses through which individuals in a particular discipline view the world and organise their knowledge. Therefore, they claim that metacognitive support provided by teachers should also include engaging students in discussions on intellectually challenging questions that make the subject-specific lenses visible to the learners. In mathematics, such questions can refer to epistemological and ontological aspects of knowledge or conceptions of proof or truth (Hiebert & Grouws, 2007; Hill et al., 2008; Lee, 2005).

The described recommendations on how teachers can support students' metacognition show the variety and complexity of factors involved in this support. Teachers are expected to invest an additional effort in class discussions to ensure adequate and prolonged connectivity between students' cognitive and metacognitive activities, quality of metacognitive activities and a clear articulation of what, when, why and how (Veenman et al., 2006).

Another important challenge that teachers have to overcome within their metacognitive support is to orchestrate all interactions in a class discussion. On the one hand, teachers must encourage individual students to externalise their metacognitive thinking. On the other hand, they must support students' interactions with each other and assure a focused class discussion. This is only possible if the students and the teacher frame their questions and contributions precisely, link them to what has been said or asked so far, elaborate and build on previous contributions by themselves or others, and justify why they agree or disagree with the statements made by others (Cohors-Fresenborg & Kaune, 2007; Kramarski & Mevarech, 2003; Michaels et al., 2008). Cohors-Fresenborg and Kaune (2007) describe this behavior as *discursive behavior*. They argue furthermore that discursive behavior also involves students and teachers making efforts to avoid actions and interactions that might impair deep thinking and cause misunderstandings, for instance talking at cross-purposes and providing confusing, superficial comments or inadequate summaries of what others have said. These are examples of what Cohors-Fresenborg and Kaune (2007) call *negative discursive activities*.

Despite many recommendations made by researchers so far, the question of how teachers can be prepared for implementing these recommendations in their teaching in a way that leads to sustainable improvements in students' metacognition and learning achievement remains open.

4. Learning to Support Metacognition

Supporting students' metacognition requires that teachers not only plan how to encourage students in metacognitive activities but also that they continuously observe the students' reactions and interactions in class discussions and interpret them from the perspective of metacognitive support. These classroom observation and interpretation activities form the basis for deciding how to respond to classroom events in the moment in order to promote students' metacognition. This important part of teaching must be considered to answer the question of how teachers can learn to consciously support metacognition in their students. In the following, this question will be discussed from the theoretical perspective of teacher noticing that is not only a common practice in teaching (Jacobs et al., 2010) but also an important skill for teaching (van Es et al., 2017).

Teacher noticing refers to teachers' in-the-moment decision making, which depends on what teachers attend to in their teaching and how they reason about what they have perceived to make decisions about how to proceed with their lesson (Jacobs et al., 2010). Over the last few years, several models of teacher noticing have evolved (see Santagata et al., 2021 for an overview). Their common feature is that they differentiate between the following interrelated components: *Attending* (to noteworthy features of classroom interaction), *Interpreting* (reason about what is observed) and *Decision making* in the classroom (decide how to respond) (e.g., Blömeke et al., 2015; Jacobs et al., 2010; van Es et al. 2017; van Es & Sherin, 2002; 2021).

Van Es and Sherin (2008) have shown that teachers can improve their noticing by analysing representations of practice – particularly move the focus of what they *attend to* (e.g., from a focus on teacher's actions to students' conceptions) and change their *interpretations* (e.g., from superficial evaluative comments to evidence based interpretative comments). This improvement, in turn, enables teachers to make well-grounded *in-the-moment decisions* on how to proceed with their own lessons (Sherin & van Es, 2009). To improve teachers' noticing, van Es et al. (2017) designed a framework for learning to notice. They tested it in the context of noticing "ambitious mathematics teaching" in representations of practice. Its first phase focuses on developing what candidates attend to in classroom interactions, the second phase emphasises the attention to the details and the reasoning about observed events, and the third phase brings together discrete observations and interpretations into a more integrated analysis. The authors argue that applying this framework can support teachers in their efforts to integrate ambitious instruction into their teaching (van Es et al., 2017).

Similarly, it seems plausible that improving teachers' noticing related to aspects of class discussions, that have the potential to support students' metacognition, can help teachers integrate metacognitive support into their teaching. To achieve a stronger focus of teachers' attention on those aspects, teachers need a conceptual understanding of metacognition and activities they are expected to promote in class discussions, thus of metacognitive and discursive activities. Their understanding must also include knowledge about aspects of class discussions that may hinder students' development of adequate metacognitive knowledge or an adequate use of metacognitive skills. As argued in the previous section, this refers particularly to negative discursive activities. Furthermore, teachers need to learn how to observe and interpret their teaching through a "metacognitive-discursive lens" or – in other words – to develop a "metacognitive-discursive vision" for their teaching that guides their attention and in-the-moment decisions in the classroom. Its guiding function manifests itselfs in teachers' sensitivity to details concerning their students' as well as their own metacognitive, discursive, and negative discursive activities, for instance, to the extent to which metacognitive activities are well-reasoned or superficial. Following the argumentation by van Es et al. (2017), we assume that teacher noticing related to these activities can be improved by involving teachers in a training course where they are supported in analysing and interpreting these activities in representations of practice. Due to the complexity of metacognitive support, as explained in the previous section, this is, however, a difficult learning process for teachers and a challenging task for teacher educators who have to design such a process. To make the learning process effective, it may be useful to provide an analytical tool (Levin et al., 2009; van Es et al., 2017) that the teachers can use and internalise as a metacognitive-discursive lens to guide their attention while analysing representations of practice.

The next section provides an example of an analytical tool that has proved useful in working with teachers to improve their noticing related to metacognitive support in class discussions (e.g., Cohors-Fresenborg et al., 2014; Kaune & Cohors-Fresenborg, 2010; Nowińska, 2018).

5. Analytical Tool for Analysing Metacognitive Support in Class Discussions

When observing and interpreting representations of practice in the form of videoed or transcribed lessons from the perspective of metacognitive support, teachers act as critical observers (hereafter: observer). To guide their attention and support interpretations of what they perceive, an analytical tool consisting of two elements has been developed.

Its first element is a *Category System* that decomposes metacognition into three categories of metacognitive activity: planning (P), monitoring (M), and reflection (R) (Table 3 in the appendix). For the reasons explained in the section on metacognitive support in class discussions, it also includes the categories "discursivity" (D) and "negative discursivity" (ND). Each category consists of subcategories, and some subcategories are further split into different aspects. The subcategories and aspects specify how activities of a particular category can manifest in class discussions. For example, the subcategory M5 states that metacognitive monitoring can manifest itself as checking the correctness or consistency of an argumentation or statement (see examples in Transcript *linear functions* in the appendix). The subcategory D1b, stating ones' own agreement or disagreement with answers or opinions proposed by others, is an example of a discursive activity (see examples in Transcript *linear functions* in the appendix).

When using the Category System, observers must pay attention to the details of metacognitive, discursive, and negative discursive activities, for instance, whether metacognitive activities are elaborate and combined with clear explanations and justifications. The Category System offers the possibility to label such activities by extending the code of the respective metacognitive or discursive activity with the prefix *r* (for reason). In addition, the prefix *d* (for demand) can be used to code instructions that explicitly require students to plan, control, reflect, or behave in a discursive way, by adding it to the code for the specific metacognitive or discursive activity that was demanded. The purpose of using this prefix is to analyse the extent to which learners engage in metacognitive activities in a self-determined way and the extent to which they do so only upon teacher instruction. When using the Category System to analyse class discussions, one must think of it as a network of concepts and codes for interpreting and describing metacognitive, discursive, and negative discursive activities rather than as a system of strictly distinct categories for labeling easily observable behaviors. As the coding is based on an interpretative process and most activities cannot be interpreted in an unambiguous way (see examples in Transcript *linear functions* in the appendix), the purpose of using this tool is *not* to find "the right" codes. The purpose is rather to consider different possible interpretations of an activity and choose a code that best describes its content and intention. In doing so, observers focus their attention on evidence allowing them to state if an activity can be interpreted in a particular way (e.g., as reflection) or if it is well explained or superficial. For each local interpretation of an activity, observers must also consider the consequences this activity may have for a shared understanding of issues discussed in the class for the other learners. This intensive interpretative process is not less important than its final product – the overview of metacognitive support offered by a teacher and of how it is utilised by students. On the one hand, considering different interpretations for each activity encourages observers to look for finest details. On the other hand, checking the consistency of the interpretations generated for every activity forces them to generalise their detailed observations. This, in turn, contributes to the observers understanding of metacognitive, discursive and negative discursive activities and enhances their sensitivity to these.

The second element of the analytical tool is a set of seven criteria to be answered by the observer based on their coded transcript or video. Each criterion focuses on one aspect of class discussions regarded as supporting students' metacognition. The criteria are formulated as questions to guide the observers' attention in their analysis of class discussions and are therefore referred to as *Guiding Questions*.

The purpose of using the guiding questions is to evaluate how well the observed class discussion supports students' metacognition and to underpin this evaluation with evidence from the coded class discussion. Thus, the Guiding Questions serve to draw connections between metacognitive, discursive, and negative discursive activities observed in a class and to analyse them *globally* from the perspective of metacognitive support (see section *Metacognitive support in class discussions*). The seven Guiding Questions focus on (1) engaging in metacognitive activities, (2) combining metacognitive activities with explanations and justifications, (3) effects of metacognitive activities on students' understanding of subject-specific content[1] discussed in class, (4) cultivating discursive activities, (5) dealing with negative discursive activities, (6) engaging in precise, focused discussions, and (7) discussions on intellectually challenging questions.

For each Guiding Question, the analytical tool offers a rating scale composed of 3-5 *answer categories* ordered according to the increasing quality of metacognitive support in class discussion. Higher quality refers to an increase in student engagement in metacognitive or discursive activities in a self-determined way. The answer categories include extensive descriptions of observable behavioral patterns in teachers' and students' activities and their interactions. Table 1 presents the guiding questions and a brief description of the key aspects of each answer category.

Table 1. Items (Guiding Questions, GQ) and scales (answer categories) for rating metacognitive support in class discussions

Engaging in metacognitive activities
GQ1: Do the learners and the teacher utilise metacognitive activities to elaborate on the subject-specific content of the class discussion and on their understanding of it?

Answer categories:
Metacognitive activities ...
1. ... hardly occur.
2. ... occur mainly in the teacher's contributions.
3. The teacher successfully motivates learners to utilise metacognitive activities.
4. The learners utilise metacognitive activities in a self-determined way.

Combination of metacognitive activities with explanations and justifications

[1] The term *subject-specific content* refers to subject-specific questions, problems, concepts, methods, strategies, representations, conceptions, or ways of reasoning and validating claims that occur in an observed class discussion in a certain school subject.

GQ2: Do the learners and the teacher combine their metacognitive activities with adequate, elaborate explanations and reasons?

Answer categories:
Metacognitive activities with adequate, elaborate explanations and reasons …
1. … hardly occur.
2. … occur mainly in the teacher's contributions.
3. The teacher successfully motivates learners to combine their metacognitive activities with adequate, elaborate explanations and reasons.
4. The learners combine their metacognitive activities with adequate, elaborate explanations and reasons in a self-determined way.

Effect of metacognitive activities and students' understanding of the discussed subject-specific content
GQ3: Do the teacher's and learners' metacognitive activities contribute to the learners' understanding of the subject-specific content discussed in class?

Answer categories:
Metacognitive activities that contribute to a deeper understanding of the subject-specific content …
1. … do not occur.
2. … occur only locally, i.e., in a few teacher's or student's contributions.
3. … occur systematically in a longer discourse involving several students interacting with each other and/or with the teacher.

Cultivating discursive activities
GQ4: Do the learners and the teacher engage in discursive activities and by doing so contribute to making the shared argumentation well "orchestrated", comprehensible, precise, and coherent with regard to the subject-specific argumentation and conclusions?

Answer categories:
Discursive activities …
1. … hardly occur.
2. … occur mainly in the teacher's contributions.
3. The teacher successfully motivates learners to engage in discursive activities.
4. The learners engage in discursive activities in a self-determined way.
5. Learners engage in discursive activities in a self-determined way and with a noticeably high degree of elaboration of different point of views.

Dealing with negative discursive activities
GQ5: Do the learners and the teacher make an effort to prevent negative discursive activities from hindering mutual understanding and understanding of the subject-specific content discussed in class?

Answer categories:
Negative discursive activities significantly hinder mutual understanding in the class discussion or understanding of the discussed subject-specific content …
1. … and no effort to change negative discursive behavior is made.
2. … the teacher unsuccessfully makes efforts to change the negative discursive behavior of learners.
Negative discursive activities occur, but do not directly affect learners' understanding of the discussed subject-specific content but …
3. … the class discussion loses its argument-based and structured character.
4. … some local individual contributions are difficult to understand.
5. Negative discursive activities have no significant effect on the learning process, or they hardly occur.

Engaging in precise, focused discussions
GQ6: Do the learners and the teacher practice metacognitive and discursive activities in such a way that coherent and focused discourses – called "discursive debates" – occur in class?

Answer categories:
Discursive debates …
1. … do not occur.
2. … are conducted by the learners, but are very short.
3. … are conducted by the teacher.
4. … are conducted by the learners and are quite long and elaborate.

Engaging in intellectually challenging discussions
GQ7: Does the teacher provide the learners with opportunities to discuss intellectually challenging questions?

Answer categories
Intellectually challenging questions …
1. … are not posed.
2. … are posed, but no metacognitive and discursive efforts are made to clarify them.
3. … are posed, but are not discussed in a clear and coherent way.
4. … are posed and discussed in a clear and coherent way.

The necessity to consider different answers for each Guiding Question forces observers to look for details in the whole class discussion and to generalise their observations. It should encourage observers to justify not only why they chose a particular answer for a certain Guiding Question, but also to justify why the remaining answers do not adequately describe the quality of metacognitive support. This also encourages observers to reflect on their decisions and how the observed teachers could have proceeded within the class discussions to improve the quality of their metacognitive support.

The guiding questions have originally been developed in the context of an interdisciplinary research study as a rating system to reliably measure metacognitive support in class discussions (Nowińska & Praetorius, 2017). In this context, it was important that the observer of videoed class discussions achieve a high level of agreement on their answers to the Guiding Questions. This agreement constitutes evidence that the answers do not substantially depend on the individual person using the Category System and the Guiding Questions. Locally, different observers may use different codes to capture the same activity, but since the answers to the Guiding Questions do not depend on the occurrence of the activities of a particular subcategory, local differences in coding do not automatically cause differences in global ratings provided by different observers.

Over the last few years, the analytical tool has been extensively used in the work with prospective teachers with the purpose to deepen their knowledge about metacognitive support and aspects of class discussions that support or hinder students' metacognition.

6. Explorative Pilot Study

The previous sections focused on theory-based implications for designing a training course for teachers, where they can learn how to support students' metacognition in class discussions. The core idea is that improving teachers' ability to notice and interpret features of class discussions that are relevant for metacognitive support can enable teachers to provide this support in their classes. Due to the complexity of metacognitive support in class discussions, it seems necessary that teachers learn to notice not only positive aspects of class discussions that can foster students' metacognition, but also negative aspects, meaning features of class discussions that may impair the effects of students' metacognition, for instance superficial metacognitive activities or negative discursive activities. To our knowledge, a systematic investigation of the effects of such a training course on teachers' noticing concerning metacognitive support is missing so far.

In the following, we share our first observations from an explorative pilot study with nine prospective teachers who took part in a course aimed at learning to support metacognition

(Kok, 2022). The main goal of this course was that the prospective mathematics teachers improve their noticing skills with regard to both functional (promoting) and dysfunctional (hindering) aspects of class discussions that are relevant for supporting metacognition. The course took place online due to the pandemic situation in summer 2021. There were meetings once a week for 90 minutes, 13 meetings in total.

The course consisted of two phases. The *first phase* provided the prospective teachers with information about metacognition and its role in learning mathematics, about metacognitive support, and recommendations for how to support metacognition in class discussion. The participants were introduced to examples of metacognitive, discursive, and negative discursive activities and were taught about the role these activities have in supporting students' metacognition. The goal of this part was to deepen the participants' conceptual understanding concerning metacognition and metacognitive support. The *second phase* consisted of analysing, interpreting and evaluating transcript-based examples of class discussions from the perspective of metacognitive support. The participants used the Category System and the Guiding Questions – the analytical tool described in the previous section – to guide their attention and interpret metacognitive, discursive and negative discursive activities. The decision to use transcript based representations of practice instead of videos was made for simple reasons. First, the participants could analyse a transcript at their own pace, without the dependence on the speed of a video. Second, it was possible to use anonymised transcripts from authentic class situations without violating data privacy. Third, little effort was needed to locally adapt the authentic transcripts to the needs of the seminar sessions. This was particularly important for transcripts that were intended to embody different positive aspects of metacognitive support that we sought to support the prospective teachers to notice. For this purpose, transcripts of authentic lessons were edited locally to make both functional and dysfunctional aspects of class discussions well observable within a shortened version.

In the course, three transcripts with positive and negative examples of metacognitive support were analysed. The prospective teachers had to analyse and interpret each transcript by conducting five reflection steps and documenting their individual work in writing (Table 2). To give each participant the opportunity to engage in a deep discussion, some of these steps had to be done as homework before a course session.

Table 2. Five reflection steps for analysing class discussions in transcripts

Step 1 – Homework assignment to be done in writing in the portfolio

- Read the transcript. What do you notice in the class discussion?
  - What features of the class discussion did you notice positively?
  - What features of the class discussion did you notice negatively?

Step 2 – Homework assignment to be done in writing in the portfolio

- *Analyse and code the transcript with the use of the Category System. Write a comment for each coded metacognitive, discursive and negative discursive activity to explain why you chose the particular code.*
- *What do you notice in the class discussion after coding the transcript? How did the result of coding the transcript influence what you are able to notice in the transcript?*

Step 3 – Discussion in a course session

- In the whole group discussion during the course session, the participants discussed the transcript which they had already analysed and coded. The goal of this discussion was to explore different interpretations for each teacher's and student's contribution in the transcript and to generate a shared interpretation.
- After the discussion on the local interpretation of each contribution, the participants were supported in interpreting the class discussion from a global perspective of metacognitive support. The Seven Guiding Questions served to guide this discussion.

Step 4 – Homework assignment to be done in writing in the portfolio

- *How has the discussion of the transcript in the course session influenced what you were able to perceive in the transcript?*
- *How has the discussion in the course session influenced your global interpretation of the class discussion?*

Step 5 – Homework assignment to be done in writing in the portfolio

- *Which functional examples concerning metacognitive support observed in the transcript would you like to integrate in your future work as a teacher?*
- *Which dysfunctional examples concerning metacognitive support observed in the transcript would you like to avoid in your future work as a teacher?*

The data collected in the pilot study consisted of two tests – one written test at the beginning of the training course and one 12 weeks later, at the end of the course. The goal of analysing the tests was to capture changes in the participants' ability to perceive and interpret features of class discussions that are relevant for metacognitive support. The participants were not informed about the purpose of the tests. The results of the pretests were not discussed in the course.

The tests consisted of two transcripts – A and B. After reading them, the participants had to write comments on which features they perceive in the transcripts and how they evaluate the quality of them. Transcript A embodied many features of class discussions that may hinder students' metacognition and very few that may support it. Transcript B, in contrast, embodied only the latter. The peculiarity of both transcripts is that they show smooth and clear teaching at first glance – but only the deep structure of transcript B stands up to this impression. Both tests were based on the same transcripts and tasks. The test-data were analysed by applying a qualitative content analysis (see Kok, 2022 for the discussion of the results).

With regard to Transcript B, the preliminary results indicated an increase in the number of relevant aspects concerning metacognitive support that each course participant was able to notice. The quality of the participants' interpretations for these details increased too. This was visible in the extent to which interpretations were supported with evidence from the transcript and written in a precise and elaborated way. A similar increase with regard to aspects of class discussions that can hinder students' metacognition was expected in the participants' comments to Transcript A. The preliminary results, however, did not indicate a clear tendency toward such a change. Compared to the first test, most of the participants commented on more activities that must be interpreted as negative discursive activities or superficial metacognitive activities in the second test. However, their interpretations of these activities were not more elaborate or precise than in the first test. Furthermore, some even misinterpreted these events as positive examples of metacognitive support. The results indicated that the participants' analysed the transcript in the second test with more details than in the first test, and their attention to noteworthy features of metacognitive support increased, but their interpretation of these features did not improve.

To sum up, the results of the explorative pilot study showed that the participants improved their noticing skills concerning features of class discussions that support students' metacognition, whereas recognising and interpreting features of class discussions that hinder students' metacognition was still a problem for most of them. This is a very important result, because teachers who are willing to support metacognition in their students also need sensitivity to negative discursive and superficial metacognitive activities. They are the starting point for changing students' learning behavior toward an adequate metacognitive regulation. Revealing these problems in the preliminary study was only possible because – in contrast to van Es et al. (2017) – we use both functional and dysfunctional examples of class discussions in our course and tests.

Our results show once more the complexity of metacognitive support. Further research is needed to find out how prospective teachers can be best prepared for functional *and* dysfunctional activities in class discussions in order to support metacognition.

7. Further Remarks

Despite the fact that metacognition improves students' learning, researchers' understanding of how to best implement findings from metacognitive interventions in classrooms' natural setting still need to be improved. Further research is necessary to help teachers support metacognition in their students. This paper explained the complexity of metacognitive support and shared our experience in preparing prospective teachers for promoting metacognition in class discussions. We see this as the first step to better understand how to improve students' metacognition. In a new study being in progress now, we have refined the design of the course described in the last section and put more efforts to develop the participants' sensibility to the important role of the discursive quality of class discussions when promoting metacognition. The analytical tool presented in this paper still plays a crucial role in this study. We hope that findings from this study will help us better understand how teachers learn to improve their teaching by putting more attention to metacognitive aspects of students' thinking.

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Appendix

Table 3. Category System for capturing metacognitive, discursive and negative discursive activities

	Planning
P1	indication of focus of attention, *e.g., with regard to tools / methods to be used or (intermediate) results or representations to be achieved*:
P1a	one-step planning activity
P1b	several-step planning activity or indication of an alternative approach
P2	planning metacognitive activities
	Monitoring
M1	controlling a subject-specific activity
M2	controlling terminology / vocabulary used for a description / explanation of a concept
M3	controlling notation / representation
M4	controlling the validity or adequacy of tools or methods used, *e.g., with regard to a planned approach or a modelling approach*
M5	controlling (consistency of an) argumentation / statement, *e.g., revealing mistakes or inconsistency or controlling an alternative argumentation (which has not been presented yet)*
M6	controlling whether the results answer the question, *e.g., with regard to the goal of a task or question and the answer given to it (controlling the factual and the intended situation) or with regard to the plausibility of the results*
M7	analysing a (mis)conception, revealing a misconception
M8	self-monitoring:
M8a	with regard to a subject-specific activity
M8b	with regard to terminology, description, explanation of a concept
M8c	with regard to notation
M8d	with regard to tools and methods
M8e	with regard to argumentation, statements
M8f	with regard to the correspondence between the achieved or intended results or answers and the discussed questions
M8g	with regard to metacognitive thoughts and activities
	Reflection
R1	analysing structure of a subject-specific expression:
R1a	without taking into consideration any additional rewriting or reorganisation
R1b	with additional rewriting or reorganisation of the given expression
R2	reflection on concepts, analogies, metaphors, conceptualisation
R2a	assignment of an object or an issue to a concept; classification of a concept within a concept hierarchy
R2b	thinking about the adequacy of a conceptualisation or about a subsumption, analogy, or metaphor related to a given concept
R3	a deliberate use of a (subject-specific) representation to express the results of a person's reflection:
R3a	identifying or marking some pieces of a given representation
R3b	creating a new representation
R3c	like a or b, but with a detailed explanation in order to promote understanding, uncover misconceptions, initiate a process of abstraction or metacognition

R4	analysing the contexts, goals, effects, and ways of using a subject-specific tool or method; indication of a tool or method needed to achieve an intended result
R5	analysing argumentation and reasoning with regard to content-specific or structural aspects
R6	reflection-based assessment or evaluation:
R6a	evaluation with regard to the content discussed, e.g., drawing an (interim) evaluation, elaboration on important or difficult aspects related to the content discussed in class
R6b	evaluation with regard to a person, e.g., (one's own) strengths, failings, mistakes, misconceptions, difficulties in understanding
R7	analysis of the interplay between external representations and internal conceptions
	Discursivity
D1	use of measures to improve the discursive character of a discussion:
D1a	naming of reference points or persons; asking for reference points or persons (in particular to ensure the basis of a conversation); indicating missing or wrong references
D1b	setting one's own contribution apart from others or stating agreement with another contribution
D1c	repetition of statements said before as a basis for further reasoning or to assure oneself of things meant or written by others
D1d	actions aimed at improving the structure of the class discussion and facilitating the discourse
D2	education fostering discursive behavior; clarifying, discussing, or deriving rules for fostering discursive behavior; request to respect the rules of a discourse; clarification of breaches of the rules of a discourse
	Negative Discursivity
ND1	superfluous contributions:
ND1a	asking a leading, or obvious question
ND1b	repetition of things already said without adding a new point of view to the discourse (also "teacher echoing")
ND1x	willful disturbance of the class discussion
ND2	use of inadequate, confusing vocabulary
ND3	violation of the rules for a well-orchestrated discourse:
ND3a	statements or questions do not recognizably refer to what was said or questioned or to what is to be discussed; the reference point is not explicit, or the argumentation is fragmentary and therefore unclear
ND3b	shortcomings with regard to grammar or sentence structure, broken sentences; at first glance, comprehensible sentences but it is not clear what is meant
ND3c	introducing alternative statements or proposals without setting them off against others; pretending to repeat or summarise a given contribution but making an essential change in the meaning without making this change "visible"
ND3d	uncommented change of the reference point or of the meaning of the issue discussed
ND3e	false logical structure of an argumentation
ND4	no intervention taken in the event of a severe disregard of discursivity rules, in particular when the discourse falls into different fragments and loses its argumentative character; ignoring a question or objection

Transcript Linear Functions

The transcript shows a class discussion in a 7th Grade class. It was developed based on an authentic class discussion in one class in the secondary school called "Gymnasium" in Germany (cf. Nowińska, 2016). Some changes in the original transcript were made to present the core part of the discussion with many examples of well-reasoned metacognitive activities.

The teaching unit deals with linear functions. To explain the use of linear functions in everyday situations, the teacher uses the following example:

> From a water pipe, water splashes into a cylindrical jar filled with water that is initially 2 cm high. The water level in the jar increases by 6 cm per minute. How does the height of water in the jar depend on the duration of the filling with water?

After one solution represented by the linear function $h(x) = 2 + 6x$ has already been discussed in class, the teacher points to another formal representation of a function provided by one student group ($h(x) = 6x + 2$). The subject of the discussion in the transcript is the question whether both formal representations of a function are correct.

T.	There was another equation in one group. Some in the group were of the opinion that the equation should not be h(x) = 2 + 6x, but h(x) = 6x + 2 *[T. writes down h(x)= 6x+2 on the blackboard]*	D1a	The teacher points to the formal representation h(x) = 6x + 2 of a function provided by one student group (D1a). In doing so, she clarifies the subject for the following discussion.
	They didn't quite agree in the group, though. What do you think? Alfred.	P2	The teacher gives an impulse that challenges students' metacognitive activities (P2) in order to evaluate the second formal representation of a linear function: h(x) = 6x + 2.
Alfred	Um, this is basically the same as the first equation. The term is just reversed.	R1a	Alfred compares the structure of both equations (R1a). Based on this reflection, he evaluates which one equation makes more sense for him (R6b).
	The equation just doesn't make as much sense as the first one.	R6b	Since he does not provide any explanation for his evaluation, it is unclear what 'making sense' means for him.
Lasse	I think they are saying that the water level is six centimeters at the beginning and that it always increases by two centimeters. But that's wrong, because in the text it's the other way around.	*r*M4	Lasse interprets the term 6x+2 and controls whether it adequately describes the increase of water; his control-activity includes an explanation (*r*M4). Unfortunately, the student makes a mistake, but his explanation enables the classmates to address this mistake as it can be seen in Julian's next contribution.
T.	Will you pick the next one?		The teacher manages the class discussions by asking Lasse to choose a classmate who should respond to his comment.

Lasse	Julian.		
Julian	No, they don't say that because the 6 represents the increase in water level per minute and they don't say 6 + 2x, they say 6x + 2. But according to the wording of the task, you would have to write down the function term as it is written above, that is 2+6x, because that would be exactly transferred. Because first there are the 2 centimeters of water, which are already in there, and then always the 6 centimeters per minute, which is 6x. And if there was written „It's increasing by six centimeters per minute and at the end two centimeters are added", then you could take the second equation. I would say: From the given task, you would have to take the first one.	*r*D1b *r*M5 D1a *r*M4	First, Julian clearly states and justifies his disagreement with Lasse's answer (*r*D1b). He verbalises his control regarding this answer and explains the mistake made by Lasse (*r*M5). Next, he clearly states the reference point of his following explanation (D1a) – "the wording of the task". At the end of this contribution, he refers to "the given task" again. In his further explanation, he controls and justifies, why the function with the term 2+6x does match the situation described in the text, whereas the other with the term 6x+2 does not (*r*M4). From mathematical point of view, his justification is not correct. His detailed explanation enables the classmates to follow his reasoning. This is visible, for instance, in a later contribution by Julian.
T.	Will you pick the next one?		
Julian	Josef.		
Josef	Well, I would say that, in general, you can swap two numbers during addition. That means that both function terms should be correct, because the result is always the same. Except, the first one makes a little more sense, because at the beginning there are already two centimeters of water in there and then it increases by six centimeters every minute. And it's not like that: At the beginning, there is nothing in it and it always increases by six centimeters per minute, and at a certain point, when you turn off the tap, it automatically increases by another two centimeters. The second term is perhaps not quite as reasonable, but in terms of the result it is also correct, I think.	*r*M4 D1a *r*R7 M4	First, Josef controls both solutions, and justifies why both terms are correct (*r*M4) He clearly states the reference point of his reasoning ("the results", "in terms of the results") (D1a). Next, Josef analyses both solutions. Since the purpose of this part of his contribution is not the controlling of the terms correctness, it is not coded as monitoring. He rather evaluates his understanding of both terms as descriptions for the increase in the water level. Thus, he engages in a reflection. This can be interpreted as his personal evaluation of which term is easier to understand for him (*r*R6a), or as a general reflection about interpretations and internal conceptions that can evolve in one's mind while thinking about these terms (*r*R7). Here the second interpretation has been chosen. Josef's reflection is combined with an elaborate explanation.
T.	Alfred.		
Alfred	Yes, indeed, what Josef says is true. Of course, it doesn't matter which term you take, but you'd better take the upper one because it makes more sense.	M5 *r*D1b	Alfred controls the answer given by Joseph (M5). He justifies his agreement with it (*r*D1b).

T.	What would you suggest if I find the lower solution in the exam? Should I subtract points? Yes or no? Maria.	*d*R6a	The teacher encourages the students to reflect and evaluate both solutions from the perspective of the teacher herself. Her question is not coded as a request for monitoring. Having heard different students' positions regarding the correctness and their understanding of both formal representations of the function, it seems plausible that the teacher's intention now is that the students reflect on both positions and combine them to a final consistent evaluative statement (*d*R6a).
Maria	I would say no, because in principle the second solution is almost the same as the first one, only that the numbers are swapped. Nothing changes in the result if you calculate 2+6x or 6x+2. The result is always the same.	*r*R6a	Maria evaluates the second solution from the perspective of the function value. Her evaluation is justified (*r*R6a).
T.	Julian.		
Julian	Yes, but um, Josef said that would mean that you increase by six centimeters per minute and then add two centimeters at the end when the tap is off. That is illogical and therefore one point should be lost.	*r*D1b *r*R6a D1c	Julian justifies his disagreement with Maria's position (*r*D1b). He evaluates the formal representation of the function from the perspective of Josef's interpretation (*r*R6a). By repeating parts of Josef's argumentations (D1c), he makes his contribution easy to follow.
T.	Josef.		
Josef	Yes, I would still say, although it doesn't make quite as much sense, that you don't subtract points from the rating.	*r*D1b R6a	Josef reacts to Julian's comments by specifying and justifying his own position in the discussion (*r*D1b). He states that his own interpretation of the second solution is not relevant for scoring this solution. His contribution can be interpreted as an evaluative reflection about the relevance of his previous argument for the final scoring of the solution (R6a).
T.	Mhm. You have mentioned different arguments. One relates to the result and the other to the fit of the term to the story in the wording of the task. We now need to clarify how we should take these arguments into account when evaluating the two functional equations. First, about the result: At the beginning of the school year, we talked about the question about when two terms are equivalent. And these two terms are equivalent. Why? Enno.	R5 P1 *dr*R2	The teacher clarifies the focus of the arguments provided by the students so far, thereby reflecting about the content of their argumentation (R5). Next, she clarifies what the students should focus on in the following discussion (P1) and asks the students to reflect and justify one mathematical property of both terms (*dr*R2); with other words: she *d*emands a *r*easoned answer.
Enno	We can apply the commutative law in this case.	*r*R2a	Enno answers the teacher's question (*r*R2a). His answer can be interpreted as a reflection about properties of both terms provided as solutions. However, it cannot be said for sure to which extent his answer is based on deep metacognitive reflection, and to which he only recalls his factual knowledge here.

T.	Sure, and with two terms that are equal in value, I can never subtract points, because the result is calculated correctly. Unless I had a very special task like for instance: Write the function equation in such a way that a student who has problems with the task can understand your function particularly well. And you are right, for a student who is reading the text and comparing it step by step with the term, the upper term 2+6x would be easier to follow than the lower one. Problem clarified?	*r*R6a	The teacher provides her final evaluation of the discussion provided by the students so far. To this end, she first refers to the argument concerning the result calculated by each term in the formal representation of the function for a certain value of x. Next, she clarifies the role of the second argument mentioned by the students. For this, she gives an example of a "special task" where the order of 6 and 2x in a solution would be important, and would lead to a lower scoring for the solution with the term 6x+2. The teacher's contribution is an example of an articulation of what was important, correct and relevant in the previous students' contribution and why (*r*R6a).

The seven guiding questions focus on (1) engaging in metacognitive activities, (2) combining metacognitive activities with explanations and justifications, (3) effects of metacognitive activities on students' understanding of subject-specific content[2] discussed in class, (4) cultivating discursive activities, (5) dealing with negative discursive activities, (6) engaging in precise, focused discussions, and (7) discussions on intellectually challenging questions.

From the perspective of the seven Guiding Questions, the class discussion presented in the transcript embodied positive aspects of metacognitive support. (1) The students engage in metacognitive activities in a self-determined way. The teacher provides a good opportunity for these activities. The students seem to be used to control their classmates' reasoning. (2) The students combine their metacognitive activities with adequate, elaborate explanations and seem to be used to do this in a self-determined way. (3) The metacognitive and discursive activities observed in this class have the potential to support the students' understanding concerning the use of functions to describe a functional relationship. Quite important are the final comments provided by the teacher. (4) The students behave in a discursive way when they react to their classmates' contribution. They do this without direct instructions from the teacher. (5) Furthermore, no negative discursive activities can be observed in the class, and (6) the discussion is led by the learners as they interact with each other in a focused way. (7) Although no challenging questions – in the sense of the seventh Guiding Question – can be observed in the transcript, the students' and teacher's metacognitive and discursive activities make the learning effects achieved in the short discussion well visible for the students.

2 The term *subject-specific content* refers to subject-specific questions, problems, concepts, methods, strategies, representations, conceptions, or ways of reasoning and validating claims that occur in an observed class discussion in a certain school subject.

PART III

Knowledge in the Context of Teaching Mathematics

Monika Grigaliūnienė
Vytautas Magnus University

Chapter 9

KNOWLEDGE OF MATHEMATICS TEACHERS FROM THE PERSPECTIVE OF THEIR STUDENTS

Summary: The aim of this research was to assess the knowledge of teachers (or lack thereof) from the perspective of students. A qualitative approach with a focus group interview was chosen as the data collection method. The focus group consisted of five Lithuanian ninth grade students. The participants mentioned a significant influence of previous experiences on their learning attitudes, defined what it means and how it feels to understand a topic, and noted that learning gaps result from superficial understanding, which is one of the reasons behind rational numbers being considered one of the most difficult topics. The participants mentioned both emotional/psychological and teacher-related aspects as reasons behind their learning challenges, as well as the need for teacher development. This article also discusses how these themes are related to mathematical knowledge for teaching and teacher competence.

Keywords: student perception, focus interview, teacher's knowledge, mathematics teachers.

1. Introduction

Good mathematical knowledge is associated with critical and analytical thinking, creative problem solving, effective information processing, reasoning, and argumentation skills (Jablonka & Niss, 2014). Unfortunately, despite the benefits, negative attitudes towards mathematics are widespread from early school age (Fiss, 2020). One of the reasons for this is, unsurprisingly, the teaching of mathematics (Lane, 2014; Askew & Venkat, 2019).

Despite the same foundations, school mathematics cannot be identified with a more general understanding of mathematics (Watson, 2008). School mathematics is taught based on curricula, formal requirements, and developmentally-appropriate abilities (e.g., Van Hiele, 1986). This means that the mathematics teacher needs to be able to adapt the concept that is being taught to the perceptions of the student at that age, selecting meaningful educational tools and responding to the student's existing knowledge, gaps, and level of understanding. It is obvious that pure mathematics knowledge (content knowledge), although essential (Lee, 2007; Hill et al., 2008), is not enough for quality mathematics teaching (Oonk et al., 2015; Loewenberg-Ball, 2000).

Mathematics teachers need to be able to teach even the most complex topics (e.g., Depaepe et al., 2018), but in order to do so successfully, they need to be knowledgeable about the different approaches to the topic at hand, the strategies of interpretation, and the use of didactic approaches (Ball et al., 2005). Such knowledge is famously defined as pedagogical content knowledge (Shulman, 1986). This knowledge is manifested not only in the teacher's ability to (re)organise their content knowledge, to apply it to the situation at hand, and to help students understand the topic at hand, but also in the teacher's identification of student's perceptions (errors) and the ability to respond to a challenge appropriately and select a solution (Cochran, 1997).

However, pedagogical content knowledge, while mostly tied to teaching mathematics, is just part of the overall knowledge that mathematics teachers must have to be able to work efficiently. According to Shulman, other types of necessary knowledge include: general pedagogical knowledge, knowledge of learners and their characteristics, and knowledge of educational ends, purposes and values, and others (Shulman, 1986). Another widely used description, mathematical knowledge for teaching [MKT], as defined by Ball et al. (2008), shares similar understanding. They define mathematical knowledge for teaching as a fusion of content knowledge and pedagogical content knowledge. Their definition of pedagogical content knowledge includes: knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum (Ball et al., 2008). For this study, pedagogical content knowledge is understood as defined by Ball et al. (2008).

This study focuses on students' perceptions of teachers' knowledge. Teaching is a demanding activity that involves interactions between teachers and students (Borko, 2004). Therefore, the teaching-learning experience needs to be examined not only from the teachers' (or observers') perspective, but also from the learners' perspective. The aim of the study was to assess the manifestations of teacher knowledge (or lack of knowledge) from the students' perspective and thereby evaluate important aspects that students are likely to notice during the learning process.

The research questions were as follows: (1) how do students explain the understanding of mathematical topics; (2) which topics do the students consider challenging; (3) how do the aspects that students relate to the quality of teaching relate to pedagogical content knowledge.

2. Methodology

A qualitative, descriptive approach was chosen for this research. A focus group interview was chosen as the method of data collection as it provides insight and understanding of the studied phenomena. The focus group interview encourages interaction and discussion between participants so that their shared experiences are highlighted. The focus group interview method also allows for differences between groups of individuals to be highlighted (Puchta & Potter, 2004; Wilkinson, 1998).

The focus group met on December 14, 2022 in a school in Kaunas, Lithuania. The duration of the interview was 40 minutes. The approach was to use a semi-structured interview structure, where the main questions were formulated in advance, but follow-up questions were also asked during the interview.

The main interview questions were as follows:

- How did you feel when you were succeeding in mathematics? E.g., you were studying, and the topic was clear to you.
- What made you realise that you really understood how to approach a topic?
- Were there difficult topics that you did not understand, that remained unclear for a long time (and maybe still do)? What was the topic? How did you find it unclear, difficult?
- Why, do you think, are these topics (still) unclear? How do you feel when you are dealing with them?

The interview was moderated and recorded by the researcher (two devices were used – the audio recording was done both on a mobile phone and on a computer). The focus group members were informed about the recording of the interview and the use of the interview data in compliance with all ethical and confidentiality requirements of the research. The participants consented to the recording of the interviews. The discussion was transcribed on the same day by attributing the statements to the group members. After the transcription, the data were pseudonymised.

Five ninth-grade students and a moderator took part in the study. The group was diverse in terms of gender and mathematics achievements – ranging from struggling students to high-achievers. All participants had attended more than one educational institution in the last four years.

3. Findings

The transcribed interview was analysed using the qualitative methodology approach. Five themes emerged from the analysis of the transcribed focus group participants' statements:

1. Previous experiences have a substantial effect on students' learning attitudes,
2. Understanding the topic includes knowing how and why the procedures work, and why you are learning the topic,
3. Challenging topics emerge from the shallow understanding,
4. Both emotional/psychological and teacher-related aspects were mentioned as the reasons behind the challenges,
5. Teacher development and the establishment of a connection between the teacher and pupils are necessary.

All of the mentioned themes are explained in-depth below.

3.1. Learning Attitudes and Previous Experiences

The focus group members noted that previous experiences have had a strong influence on their current attitudes towards learning and the likelihood of success. Some of the participants noted that the difficulty of learning mathematics is defined differently among them and is based on previous learning experiences – gaps from previous years and the need for more effort to achieve better results. Even students with more learning challenges noted that mathematics can be interesting and enjoy the feeling of understanding, but also noted that the success of learning can quickly become discouraging, and motivation disappears – "When I fail, I give up, I get nervous, I can't deal with that stuff anymore". The participants agree that mathematics is learnable ("When I have the determination to learn and to try, I understand everything"), but they differ in their assessment of the effort such a goal requires, "I need to put a lot of effort" vs. "I often succeed, I understand quickly".

3.2. Understanding

There is a consensus among the participants as to what it means to understand the topic being taught. In addition to being clear – "There is nothing to do, it's just clear" and not having any questions when solving the exercises – "When I can solve the exercises", the students emphasise a deeper understanding of the topic – "It is clear to me when I understand exactly what is going on. All the formulas and the topic itself, and why I am learning it" and the ability to explain "When I can explain what I am doing, why I am doing it", "Functions are a very good example, because there you really need to understand how the graph works, you need to understand it in depth". Also, the confidence – "I was not afraid to be wrong – then I understood what I was really doing in mathematics".

3.3. Challenging Topics

The participants agreed that one of the most challenging topics was rational numbers and mathematical operations with them, also relationships between fractions and decimal numbers. Equations and functions were also mentioned as the topics where achieving a deep understanding of the topic was more challenging. Looking back at previous classes, all agreed that the long division algorithm was challenging – "I decided a long time ago not to use the long division algorithm because I just don't understand how it works". Interviewees mention that those challenging topics are not as challenging when you really take the time to learn them and do not have gaps from the previous topics to deal with.

3.4. Reasons Behind Challenges in Learning Mathematics

The participants pointed to the emotional/psychological causes of learning challenges: 'When you don't understand, the topics get deeper, more difficult, <...> fear sets in'. Gaps of knowledge were the main focus of attention, with participants unanimously agreeing that gaps pose a major challenge for further learning: "It is very difficult when you don't have the basics" and pointed to the cyclical nature of the subject of mathematics – "Topics <...> kept coming back at a more complex level" – which makes the problem of gaps in mathematics particularly relevant.

The participants mentioned both the general negative emotions experienced in mathematics lessons – "Up to that point, all of my experiences with mathematics had been so negative, solely negative, that I didn't even think there could be anything positive" – and those stemming from the relationship with the teacher: "I was scared to ask her something", "My gaps were created when I was scared to ask the teacher".

All of the participants emphasised the role of the teacher in the teaching/learning process. When talking about their previous learning experiences, the participants pointed to the circumstances of their work resulting from teachers slowing down their development process. Some of the participants highlighted the lack of explanation: "My teacher did not explain anything at all", "She just read what was written in the book" or the lack of variety in explanation, "If you ask something, the teacher says 'I have explained it before and I will not explain it again, now find the answer to your question by yourself'", "My teacher used to say 'If something is unclear to you, ask', but if you ask, she would either say exactly the same thing or say 'I have explained this already, how could you not understand'", "I have just explained this very thing, how can you not understand".

3.5. Teacher Orientation and Focus in Mathematics Lessons

The participants in the discussion pointed to the need for teacher development – the ability to work in different contexts and to use tools to achieve the teaching/learning goal. One of the interviewees noted that during the pandemic, teachers were challenged by "Being quarantined – the teacher had very minimal knowledge of using all kinds of internet functions".

However, other members of the focus group concentrated on the teacher's work in the everyday environment: "The teacher who was teaching <...> didn't really understand what was going on with mathematics", "They just tell you to write the numbers, but why are you doing that, to understand the topic – often it just slips by and you get lost, you don't realise what's going on".

Almost all the participants did not blame the teachers: "It wasn't really the teacher's fault, she tried her best, as much as she could, and she didn't know much – she didn't know how to teach", but discussed how to work as a teacher in the current system: "I can't even think of a way to teach mathematics in this system – one person and many pupils, with a very limited time and difficult topics", "Teachers have to come to the school without being afraid to do as they please – because, most probably, they know better".

When discussing the aspects of the educational process that need improvement, the participants shared what helps them to learn: 'One-on-one time with the teacher', 'Solving problems at the blackboard', 'Quietness', but most of all, they highlighted a good learning atmosphere – 'Not being afraid to ask different questions' – and a classroom that is being managed: 'It disturbs me when everybody is talking loudly', 'It is very disturbing when others are shouting'.

3.6. Overview of the Results

The observations of the focus group interviewees are corroborated by researchers in mathematics education (e.g., Hill et al., 2008; Lee, 2007; Wu, 2010; Lortie-Forgues et al., 2015). The need for teacher development is noted, both in terms of mathematics teaching – didactic knowledge, the ability to answer students' questions, to plan tasks in a meaningful way, to engage in mathematical discussions – and classroom well-being – creating the right atmosphere, a good teacher-student relationship, classroom management.

4. Discussion

4.1. Interpreting the Results from the Perspective of Teacher Knowledge

The five themes that emerged from the analysis of the focus group interviews all relate to the knowledge of mathematics teachers, particularly pedagogical content knowledge (as defined by Ball et al., 2008).

Prior experiences affect students' learning abilities by influencing how capable they feel in learning mathematics. While prior experiences are something that has already happened and teachers have no way of correcting them, they could respond accordingly to fix the problem in its current state. Mathematical knowledge for teaching is the kind of knowledge that would be useful in these situations, specifically *knowledge of content and students* to identify which problems are most likely to occur, and *knowledge of content and teaching* to find a way to solve the problems.

The students mentioned understanding the subject as an important factor. This is directly related to the teacher's abilities to explain in a way that can be understood and to use mathematical knowledge for teaching. Being able to explain in a way that is easily understood is not only a skill, but also requires specific knowledge – how to address a misunderstanding, how to recognise and correct the mistakes of the pupils.

Participants cited the superficiality – lack of depth of the subject – as a gateway to not being able to fully understand the topic, and lacking the necessary understanding for when the same topic comes up again. For a teacher to successfully address this problem, they must have *specialised content knowledge* and *knowledge of content and students*; they must also have *knowledge of content and teaching* in order to choose the right teaching methods. To ensure that all efforts are in line with the objectives of the national/international curriculum, they must have *knowledge of content and curriculum*.

4.2. The Issue with Rational Numbers

The topic of rational numbers was mentioned by every single participant as being very difficult.

Consistent with the findings from previous studies, this research shows that fractions and fraction operations are one of the most difficult topics for students (Behr et al., 1993; Lamon, 2005; Ni & Zhou, 2005; Vamvakoussi et al., 2012; Lortie-Forgues et al., 2015; Wu, 2010), and it is therefore not surprising that this topic has been widely studied by researchers in mathematics education (McMullen et al., 2015; Vosniadou & Verschaffel, 2004). Of course, the question arises: if researchers, teachers, and the students themselves are aware of the problems the students face, why are they not being addressed?

Regarding this topic, researchers point out that many learning challenges begin with the concept of fractions (e.g., Lemonidis et al., 2017; Norton, 2019). Many researchers suggest that the right approach is to start with introducing young learners to fractions using real-world contexts, emphasising the relationship between each part and the original quantity, but also, at the same time, developing an understanding that the parts into which the whole is divided must be equal (Depaepe et al., 2018; Getenet & Callingham, 2019; Karakus, 2018; Sahin et al., 2016; Sahin & Korkmaz, 2019; Saran, 2018; Taylan & da Ponte, 2016; Zolfgari et al., 2021).

Rational numbers are a challenging and demanding subject for students. Good basic knowledge enables fluency and understanding of the individual steps, making it essential. Therefore, it is important that teachers not only have content knowledge (to know how to solve the tasks), but also mathematical knowledge for teaching – because the most important aspect of teaching is not knowing how to do it, but how to lead students to understanding.

4.3. Math Anxiety

Participants in the focus group interviews reported feeling discouraged, losing motivation, and being afraid of making mistakes and failing. All of these feelings can be traced back to the phenomenon of math anxiety (Ashcraft & Ridley, 2005).

This phenomenon has been shown time and again to be a significant cause of learning issues, which is inevitably followed by negative consequences such as low grades, shallow understanding, lack of motivation, etc. (Ramirez et al., 2018). Students who suffer from mathematics anxiety are more likely to perform worse, and those who perform worse are also more likely to develop more mathematics anxiety. The relationship is bidirectional (Carey et al., 2015) and can therefore be seen as even more problematic.

There is evidence of possible prevention-based interventions to reduce the level of mathematics anxiety related to the culture of the organisation (how students feel about their learning environment) (e.g., Hooper et al., 2016). Creating an accepting, safe culture is highly dependent on teachers, so it is not surprising that teachers play an important role in preventing mathematics anxiety. According to Lin-Siegler et al. (2016), normalising failure spurs students to push through the more difficult times. Experiencing positive feelings when dealing with mathematics leads to higher mathematics achievement (e.g., OECD, 2013) and can also be seen as a factor that strongly depends on the teacher. Even more, the enjoyment of mathematics represents a positive emotion that has a positive impact on academic performance – which in turn is something that the teacher can influence.

Cumulative scientific evidence supports the findings of this study, as it underlines the importance of the teacher in students' perceptions of mathematics and the power of teachers in changing students' perceptions.

4.4. Future Implications

The current study is a small study that provides insight into the students' perspective. The results are consistent with the findings of previous studies that addressed issues such as lack of pedagogical knowledge of mathematics teachers and the importance of a safe, accepting learning environment.

The idea of this study should be replicated with other sample groups in order to compare the results and explore the issues that have been raised in more depth.

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Matej Slabý & Ingrid Semanišinová
Pavol Jozef Šafárik University

Chapter 10

PRE-SERVICE TEACHERS' KNOWLEDGE OF STUDENTS' MISCONCEPTIONS ABOUT AND DIFFICULTIES WITH FUNCTIONS

Summary: This study presents a part of research focusing on the evaluation of a course for pre-service teachers (PSTs) aimed at developing mathematics teachers' specialised knowledge of functions. The course was developed as part of the FunThink project. In this paper, we focus mainly on pre-service teachers' knowledge of students' misconceptions about and difficulties with (M&Ds) functions. The research was carried out with 13 PSTs as part of a compulsory course at the Pavol Jozef Šafárik University in Košice. The PSTs solved a set of eight mathematical tasks and answered nine questions (some of them following the tasks) in pre-post test design. This paper describes the results of a qualitative content analysis of the answers to two of these questions. This analysis showed that the course partly helped to improve the PSTs' knowledge about students' M&Ds. On the other hand, it helped to identify some shortcomings in the course's design. The analysis of the PSTs' answers to selected questions will help us refine the design of the course to best help teachers to improve their knowledge about M&Ds in relation to functions.

Keywords: functions, misconceptions and difficulties, pre-service teachers, Mathematics Teacher's Specialised Knowledge model.

1. Introduction

The topic of function is one of the most important areas in mathematics education and in mathematics curricula in different countries. The development of functional thinking is important in both private and professional life (e.g., Vollrath, 1986; Leinhardt et al., 1990; Thompson & Carlson, 2017). On the other hand, several studies confirm the persistent difficulties that students have with functions. The reason for students' difficulties with functional thinking may be due to the abstract nature of "functions", which are accessible only through specific representations such as a graph, a formula, a table, or due to the need to transition between mathematics and the real world (e.g., Hadjidemetriou & Williams, 2002; Ostermann et al., 2018). It is very important for the mathematics teacher to be aware of these students' misconceptions about and difficulties with (M&Ds) functions without a temporal qualification. Based on this awareness, they can then manage the students' learning process in such a way that the students' M&Ds are avoided.

This study is part of the larger FunThink Erasmus+ project focused on the development of functional thinking. In the current study, we focused only on pre-service teachers' (PSTs') knowledge of students' M&Ds with functions. Within this study, we developed a tool to investigate the specialised knowledge of PSTs, focusing on functional thinking. The complexity and diversity of knowledge and information in mathematics makes it difficult to determine what a mathematics teacher who teaches at lower secondary and upper secondary level should know before entering the profession. Therefore, we focused only on the topic of functions (mainly linear functions) in Slovak, Polish, and German schools. We chose these countries because we prepared the same course for PSTs within the FunThink project. These countries are similar both in curricula as well as PSTs training at universities. Our tool consisted of eight specific mathematical tasks and nine questions (about functions, M&Ds, and the previous eight tasks) which were to be solved and answered by PSTs at the beginning and end of the Didactics of Mathematics course (the research tool can be requested from the authors of the article). In designing the tool, we assumed that teachers need to have a deep and broad understanding of school mathematics in order to be able to offer challenging mathematics to their students (e.g., Zakaryan & Leikin, 2004). In this article, we focus on two questions that address students' M&Ds from the perspective of Slovak PSTs. Therefore, our research question being addressed is: Did the course aimed at developing PSTs' functional thinking, scaffold the Slovak PSTs' knowledge of students' M&Ds about functions?

2. Theoretical Background

2.1. Functional Thinking

Dealing with mathematical functions encompasses the ability to manipulate the formulas representing them: it involves dealing with the notion of function in its versatility and developing a rich concept image which includes aspects such as: representation, generalisation, causality, regularity, and covariation. In recognition of this versatility, the concept of functional thinking has emerged (FunThink Team, 2021). Functional thinking is considered a way of thinking in terms of relationships, interdependencies, and change, and is the process of building, describing, and reasoning with and about functions (Blanton et al., 2015; Pittalis et al., 2020).

In the FunThink project, we focus on enhancing functional thinking in a comprehensive and transnational perspective, drawing on the specific and complementary expertise of the partners. Our professional partners are based in Germany, Poland, the Netherlands, and Cyprus, and we share a common vision that mathematics education can be significantly improved by enhancing functional thinking from primary to upper secondary school.

Following Pittalis et al. (2020) we distinguish four aspects of functional thinking which are related to different perspectives on functions:

1. Input-output aspect: This view on function as an input-output machine stresses the operational and computational character of the function concept. It includes exploring how a particular input value will lead to an output value (FunThink Team, 2021),
2. Covariation aspect: This aspect concerns the notion that "two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (Thompson & Carlson, 2017, p. 444),
3. Correspondence aspect: This view on function concerns understanding the relation between the independent and dependent variables (correlation between variables) and being able to represent it. This view helps answer questions on the global character of the relationship,
4. Function as a mathematical object: "A function is a mathematical object which can be represented in different ways, such as arrow chains, tables, graphs, formulas, and phrases, each providing a different view on the same object" (Doorman et al., 2012, p. 1246). This aspect is hierarchically the most difficult to understand, but this view of a function is important for the purpose of comparing a function with other functions or with other mathematical objects.

In Slovakia, the concept of a function is introduced in elementary school as an input-output system, followed by the covariation and correspondence aspects, which continue in secondary school and possibly reach the object aspect. In the latest Slovak mathematics textbooks (Kubáček, 2010) for secondary schools it is recommended to introduce the concept of function descriptively. Based on the analysis of textbooks focusing on aspects of the concept of function (Krišáková & Slabý, 2022), all four aspects of the function – 1) input-output, 2) covariation, 3) correspondence 4) object are adequately covered therein. However, the aspect of covariation is not covered sufficiently or is missing in the textbooks for middle schools. Moreover, our PSTs have been trained with older secondary school textbooks in which the correspondence and object aspects dominate.

2.2. Misconceptions and Difficulties Related to the Concept of Function

According to Hadjidemetriou and Williams (2002), misconceptions may be part of a faulty cognitive structure that causes, lies behind, explains, or justifies the error. "A misconception may develop as a result of overgeneralising an essentially correct conception or may be due to interference from everyday knowledge" (Leinhardt et al., 1990, p. 5). Students' misconceptions can lead to problems and learning difficulties. Several M&Ds arise in the context of functions for both PSTs and learners, and are the focus of this paper. In our research, we mainly focus on whether the course aimed at developing PSTs' functional thinking scaffolded their knowledge of students' M&Ds (errors) regarding functions. "Whereas science misconceptions often originate in children's observations and interpretations of real-world events, misconceptions of functions and graphs often are intertwined with previous formal learning" (Leinhardt et al., 1990, p. 31).

Difficulties and misconceptions about functions can be eliminated by promoting the development of functional thinking in students from an early age. Therefore, the development of functional thinking in students should start in the early grades and be improved gradually and extended over a long period of time (Warren, Cooper, & Lamb, 2006). It is therefore very important that these M&Ds are well managed, diagnosed, and then corrected by PSTs.

Several articles have described misconceptions about functions (e.g., Leinhardt et al., 1990; Hadjidemetriou & Williams, 2002; Ostermann et al., 2018). In this article, we will draw on the work of Leinhardt and colleagues (1990). They discuss difficulties together with misconceptions and describe them in more detail in the following eight categories:

1. What is and is not a function – including, for example, inaccurate ideas about what graphs of functions should look like; ideas that only patterned graphs represent functions, that functions given by more than one rule are not functions, that functions must be given by formula,
2. Linearity – the tendency to define a function as a relation that, when represented graphically, produces a linear pattern; the tendency to connect any two consecutive points by a straight line; overgeneralisation of the properties of linear functions to other functions,
3. Continuous versus discrete graphs – "representing or interpreting continuous data in a discrete manner and representing or interpreting discrete data in a continuous manner" (Leinhardt et. al., 1990, p. 34),
4. Representations of functions – M&Ds about transitions between ordered pairs, equations, graphs, tables, and verbal descriptions of relationships,

5. Concept of variable – "ideas that changing the symbol for the variable in a functional equation changes some critical aspects of the function; focusing on arbitrary symbol substitution but missing the central idea of a functional relationship between two variables; manipulating letters in equations without understanding the variable" (Leinhardt et al., 1990, pp. 42–43),
6. Notation – M&Ds "related to the unique notational systems inherent in both the graphical and algebraic symbols that are used to represent functions" (Leinhardt et al., 1990, p. 43); M&Ds in setting up two axes for a Cartesian coordinate system; scaling problems; confusing the two axes of a graph; thinking that graphs always go through the origin,
7. Correspondence – each y value must map to one and only one x value,
8. Relative reading and interpretation – M&Ds in constructing and interpreting graphs that represent real situations including slope-height confusion, graph as a picture misconception, and interval/point confusion.

2.3. Misconceptions and Difficulties in Pre-Service Teachers' Education

Different models of knowledge of mathematics teachers emphasise the need for teachers to be aware of their students' M&Ds. In the Mathematical Knowledge for Teaching model (Ball, Thames, & Phelps, 2008), this knowledge is part of Pedagogical content knowledge, particularly the Knowledge of Content and Students. Our course, which focused on developing PSTs' functional thinking, was designed within the Mathematics Teachers' Specialised Knowledge model (MTSK) by Carrillo and colleagues (2018). In this model, knowledge of students' M&Ds is part of the sub-domain of the Knowledge of Features of Learning Mathematics (KFLM). As noted by Carrillo and colleagues, the KFLM sub-domain includes knowledge about learning styles and, accordingly, includes theories about students' cognitive development. As such, this sub-domain considers teachers' knowledge of their students' ways of thinking and doing, particularly in mathematics, their errors, areas of difficulty, and misconceptions. In essence, this sub-domain involves an awareness of what students struggle with, an understanding of the process of learning different content, and what their strengths are, both in general and in relation to specific content.

One of the ways in which PSTs can develop their KFLM is by working with concrete student solutions, analysing videos of students presenting their thinking, or interviewing students. Research shows that if teachers are exposed to different aspects of students' thinking during their training, this can later influence how they understand their thinking, how they respond to students in class, and how they adapt their teaching to the current state of students' thinking (e.g., Wickstrom, & Langrall, 2020; Clements et al., 2011; Cobb et al., 1990). We have drawn on this research to design our course for PSTs, focusing primarily on the shift regarding M&Ds in connection with the concept of function.

3. Methodology

3.1. Participants and Context

The study involved 13 PSTs from the Pavol Jozef Šafárik University in Košice (four men and nine women) aged 22-23 years. They are first year master's students and all have a bachelor's degree in mathematics combined with another subject (4 PSTs – Mathematics and Geography, 2 PSTs – Mathematics and Biology / Physics / Slovak language, 1 PST – Mathematics and Chemistry / Informatics / Psychology). Note: Interdisciplinary studies are implemented as a mutual combination of two science disciplines.

The PSTs participated in the research as part of a compulsory course – Didactics of Mathematics – in the winter semester of the academic year 2022/2023. Informed consent was obtained from all participants involved in the study. It involved consent to the processing of personal data which will be used in accordance with the research objectives: dialogues and task solutions will be transcribed; participants' names will be changed; and all materials can be used for research purposes.

The course consisted of 26 lessons divided over 13 sessions (each lesson lasted 45 minutes, so one session lasted 90 minutes). The aim of the course was to develop mathematics teachers' specialised knowledge concerning functions according to the MTSK model.

3.2. Data Instrument and Collection

The data collection procedures used in this study included the PSTs' written solutions to eight tasks and nine questions focused on functional thinking and M&Ds regarding the concept of function and some properties of linear function. The research tool was developed in collaboration with colleagues from Poland and Germany who are participating in the FunThink project. The complete research tool is available on request from the authors.

At the beginning of the course (given to the PSTs during the first session) the PSTs solved the tasks and answered the questions in the research tool. These tasks and questions were completed again by the PSTs at the end of the course (during the last, thirteenth session). Neither the correctness of the solutions nor the PSTs' answers from the pre-test were discussed or published during the course, and none of the tests formed part of the final assessment.

In our research, we focused on two questions from the research tool, particularly questions 8 and 9 (Figures 1 and 2).

Firstly, we collected the data from each country and, consequently, we conducted a qualitative content analysis (Mayring, 2015). In order to create an appropriate code scheme, we used both a deductive and an inductive approach. During development, we considered both theoretical frameworks (Leinhardt's classification of M&Ds and the MTSK model) as well as the research question. We then coded the individual responses of the PSTs, modifying the coding scheme twice and changing some of the codes. We sent the third version of the coding scheme to our colleagues within the FunThink project (from Germany and Poland), who tried to use it to code the responses of the PSTs of the same research tool from their countries. Our Slovak team (both authors) re-coded the answers of the PSTs. The next step was a joint discussion with our aforementioned colleagues about the individual categories and problematic classifications of the PSTs' answers. The most important change was the division of the code concerning representations of function into three. On this basis, we modified the coding scheme again, which became its final form (Table 1). Both authors re-coded the responses of the Slovak PSTs. The final coding, as well as all previous encodings, was independently coded by both authors. Subsequently, small discrepancies in coding were discussed until unanimous agreement was reached. We used the program atlas.ti for coding. The PSTs were unaware of the coding of M&Ds (neither of the categories nor of the fact that we would be coding them).

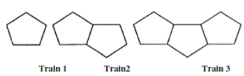
Figure 1. Question 8 from the research tool

8. What learning difficulties and misconceptions do you expect when teaching functions? Use as many examples as possible to depict your answer.

Figure 2. Question 9 from the research tool

9. Discuss mathematical problems which can be related to the following tasks: (continue pattern, table into the equation)

The first three trains in the pattern are shown below:



A) Determine the perimeter for the 4th train.
B) Determine the perimeter for the 100th train.
C) Write a description that could be used to find the perimeter of any train in the pattern.
Explain how you know.
How does your description relate to the visual representation of the trains?

From the following table, determine the prescription of the function:

x	0	1	2	3
f(x)	3	5	7	9

Possible problems:

Possible problems:

The coding scheme is described in Table 1. The short description of M&Ds is in the second column of the table. The code used for each M&D is in the third column. In the last column, the citation of the article shows the relation to the theoretical background. The categories 1-8 are taken from Leinhardt et al. (1990). The only difference is that we have divided the category Representations of Functions (REP) into three categories, namely Representations of Functions (1REP), Linking Representations (LREP), and Modelling (MOD). We proposed this division based on the PSTs' responses and a joint discussion with our colleagues from Germany and Poland. As the aspects of functional thinking are related to different perspectives on functions, we decided to create four categories according to the four aspects of the concept of function as described by the authors Doorman (2012) and Pittalis (2020). Moreover, the PSTs also mentioned M&Ds related to aspects of functions in the pre-test. The remaining categories arose from the need to classify all PST responses.

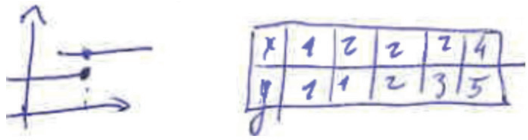
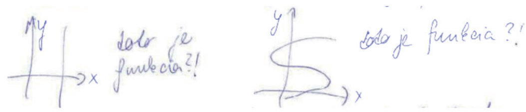
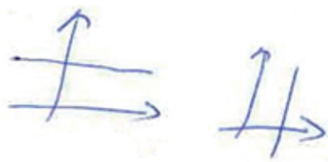
We coded all of the PSTs' observations of M&Ds from the research tool and then categorised them. We note that these are the M&Ds that the PSTs think will occur in the students' solutions, not the M&Ds of the PSTs themselves.

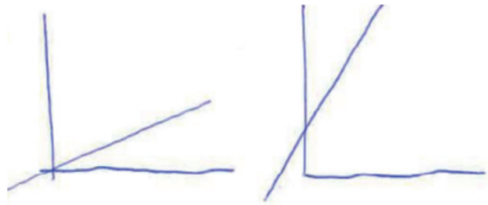
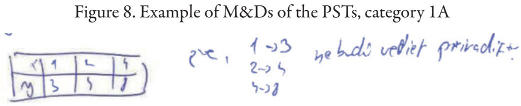
Table 1. Categories and codes of misconceptions and difficulties

1.	What is and what is not a function	**F/N**	**Leinhardt, Zaslavsky, Stein (1990)**
2.	Linearity	**LIN**	
3.	Representation of functions	**1REP**	
4.	Linking representations	**LREP**	
5.	Modelling	**MOD**	
6.	Concept of variable	**VAR**	
7.	Notation	**NT**	
8.	Continuous vs discrete graphs	**CvsD**	
9.	Input-output aspect	**1A**	**Doorman (2012), Pittalis et al. (2020)**
10.	Covariation aspect	**2A**	
11.	Correspondence aspect	**3A**	
12.	Function as a mathematical object	**4A**	
13.	General	**GEN**	
14.	Other meaningful answers	**Oth+**	
15.	Meaningless or incomprehensible answers	**Oth-**	
16.	No answer or I do not know	**BLK**	

Categories 1, 2, 6, 7, and 8 are taken directly from Leinhardt et al. (1990) without modification and we interpret them in the same way. Other categories are explained below. We will also describe each category we have created using a specific example from the PSTs' responses (Table 2).

Table 2. Examples of M&Ds of the PSTs for each category

Description of M&Ds	Examples of M&Ds of the PSTs
What is and what is not a function? (F/N)	(i) Misclassification of graph among functions – assigning two values to one – value. Figure 3. Example of M&Ds of the PSTs, category F/N_1  (ii) Students are not able to identify whether it is a function based on the graph. Figure 4. Example of M&Ds of the PSTs, category F/N_2  (iii) Function = graph.
Linearity (LIN)	(i) Students state that the graph of a linear function is any line. Figure 5. Example of M&Ds of the PSTs, category LIN  (ii) Direct proportion = linear function (but the function in the picture is not a direct proportion).
Continuous versus discrete graphs (CvsD)	(i) If the student draws a graph from the table and does not connect the resulting graph (e.g., pouring water into a glass of 100 ml – continuous graph).
Representations of functions (1REP) Representations of functions include different M&Ds with a single representation (it refers to only one of the representations: a graph, a formula, a table).	(i) Students are not able to draw a graph. (ii) Students do not understand what the formula (of the function) describes.

Linking representations (LREP) This category includes M&Ds in moving from one representation (graph, formula, table) to another representation (graph or table – transitions between representations only within mathematics). Note: We include transitions from graph, formula, and table to formula in the Correspondence aspect category (code 3A).	(i) Two formulas of functions and two graphs are given, and the students are not able to assign which graph belongs to which rule. Figure 6. Example of M&Ds of the PSTs, category LREP_1  (ii) Transitions between table, formula, and graph. Figure 7. Example of M&Ds of the PSTs, category LREP_2 
Modelling (MOD) This category includes M&Ds with transitions from verbal description to other representations and vice versa; also M&Ds with interpretations of an abstract mathematical solution. In this category, we also include misconceptions described in Leinhardt et al. (1990) as Relative reading and interpretation.	(i) Students are not able to interpret the slope of a line and that the slope of a line depends on the growth of the function – the greater the slope, the greater the speed (e.g., when we have a graph of path versus time). (ii) Students are not able to interpret graphs correctly (for example, if they were to relate the shape of the glass to a graph of the function of the height of the water in the glass versus time when the water is poured evenly into the glass).
Concept of variable (VAR)	(i) Confusion about the meaning of the coefficients in the formula.
Notation (NT)	(i) Thinking that graphs always cross both axes. (ii) The student is confused by the fact that the graph does not start at the coordinates [0,0]. (iii) Lack of awareness that $f(x)$ is another notation for y.
Input-output aspect (1A)	(i) Students are not able to fit numbers into a formula of the function to find the intercepts. (ii) Students will not be able to assign input-output. Figure 8. Example of M&Ds of the PSTs, category 1A 
Covariation aspect (2A)	(i) Not understanding why in the function x changes by 1 and $f(x)$ changes by 2. (ii) Students are not able to discern how x varies with $f(x)$ from the table and, if they do, they may not be able to write it down in a dependency.
Correspondence aspect (3A)	(i) Students do not know how to form a formula from the graph, they do not know what a formula is. (ii) Students are not able to discern from the table how x varies with $f(x)$ and, if they do, they may not be able to write it down in a dependency.
Function as a mathematical object (4A)	(i) Student are mistaken about the types of functions if they are not modified in their basic form.

General (GEN) This category arose from the need to include M&Ds reported by PSTs that were not related to the concept of function.	(i) The student does not know how to start solving the problem; misunderstanding of the task; the student calculates by using other data out of inattention.
Other meaningful answers (Oth+) Meaningful M&Ds related to functions that cannot be classified in any of the above categories are represented in this category.	(i) Pupils do not know the rules for modifying expressions with powers, logarithms.
Meaningless or incomprehensible answers (Oth-) M&Ds of PSTs that made no sense or were unclearly written.	(i) Misunderstanding of parents and their comments on student's learning.
No answer/ I do not know (BLK)	

4. Results

In Table 3, we can see which students' M&Ds were reported by the PSTs in the pre-test (blue star) and post-test (red star) sections (there is an indication of the occurrence of a category in the PSTs' response, not the frequency of M&Ds for each category).

If we look at the columns in Table 3, the important information that can be ascertained is that in all but one category (LREP) the number of occurrences in the PSTs' responses increased. We can also see a high frequency in categories 3A and 4A in both the pre-test and the post-test. On the other hand, the category CvsD is poorly covered. No PST mentioned it in the pre-test and only one in the post-test. The categories LIN, VAR, and 2A were rarely mentioned in the pre-test, although there is a substantial improvement in the post-test. The categories in grey in Table 3 are not particularly relevant for our research, they only serve as information about the completeness of the data we collected.

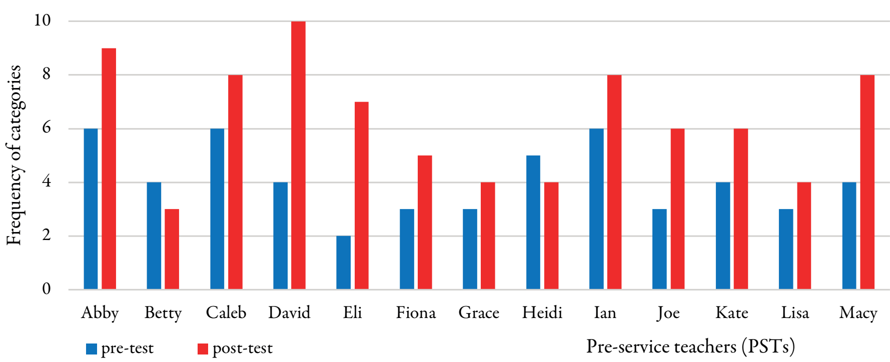
If we look horizontally at the individual PSTs, for almost all of them there was a shift in the number of M&Ds for each category, but some of the M&Ds mentioned in the pre-test were not mentioned in the post-test. For example, Betty identified an M&D in the 1REP category in the pre-test but did not identify any M&D that we could include in this category in the post-test. Eleven PSTs (all except Betty and Heidi) identified M&Ds in a greater number of categories in the post-test compared to the pre-test.

Table 3. Classification of pre-test and post-test M&Ds reported by PSTs

PST	Categories of misconceptions and difficulties																Σ
	F/N	LIN	1REP	LREP	MOD	VAR	NT	CvsD	1A	2A	3A	4A	GEN	Oth+	Oth-	BLK	Σ
Abby	**			*		**	*	*	**	**	**	**	*	*			6/9
Betty			*	* *			*				*	**	**	*			4/3
Caleb	*	*	*	*	*	*			**	**	**	**	*	*			6/8
David	**	*	*		*	*	**		*	*	**	**	**	*	*		4/10
Eli	*	*	*		*	*			*	*		**					2/7
Fiona			*	* *		*	*				**	*		**			3/5
Grace	*		* *				*				**	*	*	*			3/4
Heidi	**		*	*	*						**	**	*	*			5/4
Ian	**	*	*	* *	**	**			*	*	**		*	*			6/8
Joe	*		* *	*		*	*		*	*	*		*	**			3/6
Kate			* *	*	**		**				**	*	*				4/6
Lisa			*	*		*	**				*	*	**	*			3/4
Macy	**	**	*		*		*		*	*	**	*	*				4/8
Σ	6/8	1/5	7/8	6/6	4/5	2/8	4/8	0/1	3/6	2/7	11/10	7/10	8/6	9/3	0/1	0/0	

For ease of illustration, we have plotted the frequency of each category in the pre-test and post-test PSTs in a bar chart (Figure 9) which corresponds to the final column of Table 3. The most valuable progress can be seen in David's, Eli's, and Macy's answers. They mentioned students' M&Ds in 10 (6 are different than in the pre-test), 7 (6 are different than in the pre-test), and 8 (5 are different) more categories than in the pre-test.

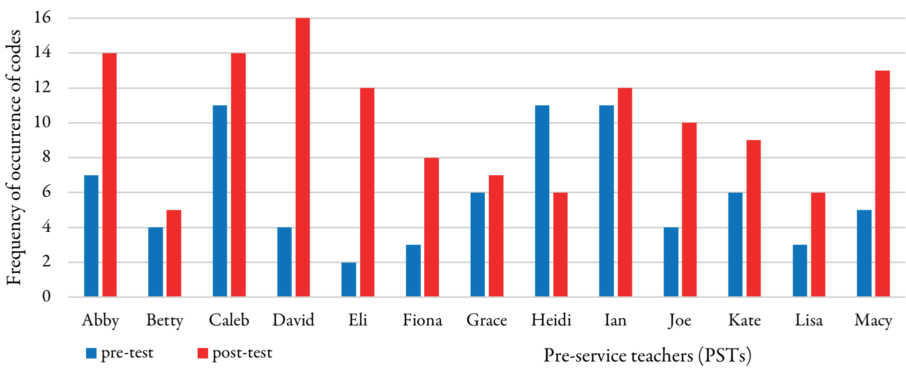
Figure 9. Number of categories identified by PSTs pre-test and post-test



The following graph (Figure 10) shows the number of all M&Ds identified by the PSTs before and after the test (total number of M&Ds identified by the PSTs). Again, we can see a great deal of improvement for David, Eli, and Macy in the post-test. They mentioned 16

(12 M&Ds more than in the pre-test), 12 (10 M&Ds more than in the pre-test) and 13 (8 M&Ds more than in the pre-test) M&Ds of the students. Contrary to the previous graph, we can also see progress for Betty, who reported more M&Ds in the post-test than in the pre-test, but only in the same category. Only Heidi did not make any progress in her answers, on the contrary. It is also worth noting that progress is observable for almost all PSTs (except Heidi and Betty) independent of coding categories.

Figure 10. Number of all M&Ds identified in PSTs pre-test and post-test



5. Conclusion

The results show that the course has partly helped to improve the PSTs' knowledge of students' M&Ds. All but one PST made at least some progress. There was also a shift in the number of categories of M&Ds reported by PSTs. On the other hand, many M&Ds that are commonly found in students and reported in research were also rarely (LIN, LREP, MOD) or hardly ever (CvsD) mentioned by PSTs in the post-test. Nor did the PSTs mention the misconception that each *y*-value must correspond to one and only one *x*-value (described by Leinhardt et al. (1990) as Correspondence). Another finding relates to PSTs' rather general descriptions of types of M&Ds. For example, they very often mention problems with reading from a graph but only one PST in the post-test specifically describes the misconception known as Slope/height confusion and only one mentions the misconception of Iconic interpretation (interpreting a graph of a situation as a literal picture of the situation). Both are categorised as MOD (Modelling). None of them mention interval/point confusion, where a student focuses on a single point instead of an interval. This is important information for us to keep in mind when we run the course again. An important aim for us is that each PST is aware of the known M&Ds before they start designing and teaching their lessons, so that they can prevent these M&Ds in their students.

The limitation of the study is its relatively small sample size. Results in other groups of PSTs might be different. Therefore, it would be interesting to include results from other partners within the FunThink project to obtain more generalisable data. Regardless of this limitation, the results help us see which kind of mathematics teachers' specialised knowledge concerning students' M&Ds about functions was less covered in our course. The analysis of PSTs' answers to the chosen questions will help us refine the course design in this aspect in order to best help teachers to improve their knowledge about M&Ds concerning functions.

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Veronika Hubeňáková, Monika Krišáková & Zuzana Gönciová
Pavol Jozef Šafárik University

Chapter 11

SLOVAK PRE-SERVICE MATHEMATICS TEACHERS' KNOWLEDGE ABOUT LINEAR FUNCTION DEFINITION AND THEIR BELIEFS ABOUT MATHEMATICS

Summary: This paper presents an investigation into pre-service mathematics teacher knowledge and beliefs in the context of a teacher course focused on the development of functional thinking. More specifically, it is focused on their Knowledge of Topic, Knowledge of Practices in Mathematics, and Beliefs about Mathematics as described in the Mathematics Teachers' Specialised Knowledge model. Such investigations can shed light on course development and inform research that still needs answers about connections between beliefs, knowledge, and practice. In this study, the following research questions are posed and answered: How do pre-service teachers define a linear function? Are pre-service teachers consistent with their own definition when discussing the linearity of a given function? What tendencies are visible in pre-service teachers' beliefs about mathematics? To answer these questions, qualitative methodology was used, where pre- and post-test answers of 13 course participants were analysed. On the one hand, the results revealed the prevalence of Platonist beliefs about mathematics. On the other hand, only a few of the pre-service teachers used the correct definition of a linear function (1 out of 13 in pre-test, 3 out of 13 in the post-test). Moreover, consistency of the definition and the argument used to decide about the (non-)linearity of the function was dependent on the context.

Keywords: Model MTSK, linear function, beliefs about mathematics, knowledge of topic.

1. Introduction

A quote from mathematician Georg Cantor states that the "essence of mathematics lies in its freedom". As with anything else, freedom does not equate to doing whatever comes to one's mind. It means making decisions and then accepting all consequences, in other words, being *responsible*. Often, we claim that the definition of a mathematical concept is a matter of convention, and, to some extent, it is arbitrary. For example, we can define a rectangle as a four-sided flat shape with straight sides where all interior angles are right angles. A consequence of this definition is that a square is a special case of the rectangle. This is a space in which teachers can practice mathematical freedom with students and teach them to anticipate consequences. Hopefully, not only in a mathematical context. A necessary condition, however, is that the teachers are *responsible* enough to make use of this freedom.

There are different models of (mathematics) teachers' knowledge which can help us conceptualise what teachers need to know to be able to enjoy the freedom of mathematics: starting from Shulman (1986), continuing, among others, with Mathematical Knowledge for Teaching (Ball et al., 2008), Knowledge Quartet (Turner & Rowland, 2011), and Mathematics Teachers' Specialised Knowledge (Carrilo et al., 2018), the last of which directly addresses the knowledge about practices of building mathematical knowledge in a logically correct way. For this reason, this paper will assume the terms of this model.

2. Theoretical Background

2.1. Mathematics Teachers' Specialised Knowledge Model

The model Mathematics Teachers' Specialised Knowledge (MTSK) is an analytical model which helps researchers gain insight into the teacher's knowledge, specifically the elements which this knowledge if made of and the interactions between them (Carrillo et al., 2013; Carrillo et al., 2018). The MTSK model (Figure 1) consists of three main parts: Mathematical Knowledge (on the left-hand side), Pedagogical Content Knowledge (on the right-hand side), and Beliefs (in the middle).

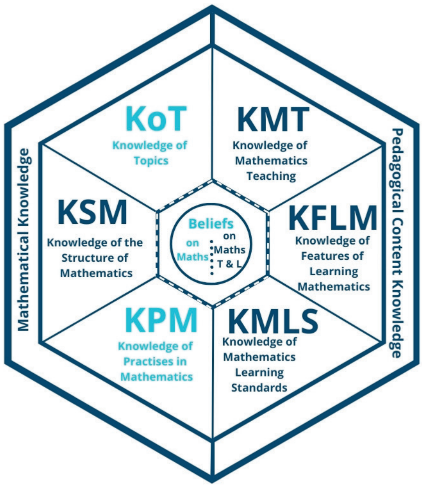
Mathematical Knowledge covers the "whole universe of mathematics, comprising concepts and procedures, structuring ideas, connections between concepts, the reason for, or origin of, procedures, means of testing and any form of proceeding in mathematics, along with mathematical language and its precision" (Carrillo et al., 2013, p. 2990). It is divided into the following subdomains: Knowledge of topics, Knowledge of the structure of mathematics, and Knowledge of practices in mathematics.

Pedagogical Content Knowledge "is comprised of the knowledge relating to mathematical content in terms of teaching-learning" (Carrillo et al., 2018, p. 240). It is also divided into three subdomains, which are: Knowledge of mathematics teaching, Knowledge of features of learning mathematics, and Knowledge of mathematics learning standards.

The third part of the model includes teachers' Beliefs about mathematics and their Beliefs about mathematics teaching and learning. "These are represented at the center of the figure to underline the reciprocity between beliefs and knowledge domains" (Carrillo et al., 2018, p. 240).

In this paper, we are only concerned with the subdomains: Knowledge of Topics, Knowledge of Practices in Mathematics, and Beliefs about Mathematics, so we only characterise these parts of the model in more detail.

Figure 1. Mathematics teacher's specialised knowledge model (Carrillo et al., 2018)



2.2. Knowledge of Topics (KoT)

This subdomain describes how mathematics teachers should know the topics of the mathematics they teach. It includes the knowledge they teach their students, but their understanding of it is deeper, more rigorous, and uses formal mathematical language. Naturally, the level varies across the school where the teacher is employed. As the topics can vary according to each country's curriculum, KoT is also specific to each country. However, a common similarity is that KoT consists of the following four categories:

1. **Procedures** – knowledge about how, under what conditions, and why something is done, and the key features which result in doing it.
   e.g., Knowing the mathematical apparatus for determining the missing coordinate of a point when a function is given by its formula.
2. **Definitions, properties, and foundations** – knowledge of descriptions and characterisations of a concept, and knowledge of the relationships between concepts and their properties within a given topic.
   e.g., Knowledge of the definition of a function, a linear function, a function of constant and direct proportionality.

3. **Registers of representation** – knowledge of the different ways a topic can be represented (graphic, algebraic, arithmetic, and so on).
   e.g., Knowledge of how a linear function can be represented (with a table, word, graph, equation, and nomogram), transitioning between representations.
4. **Phenomenology and applications** – knowledge of the applications of specific content, and the different contexts in which we may encounter that content.
   e.g., Recognition of a linear function in a real context.

Since KoT is specific to the mathematical and cultural context, let us focus on the Slovak curriculum concerning linear function. Moreover, due to the topic of the paper, we restrict ourselves to the definition of linear function – the second category. According to Slovak curriculum documents intended for lower secondary schools, pupils in Grade 9 are expected to know the concepts of linear dependence, linear function, and graph of a linear function. At the same time, they are meant to be able to work with these concepts and to determine the second coordinate of a point on the graph (Štátny pedagogický ústav, n.d.). In the target requirements for the mathematics matriculation examination, the standard specifies knowledge of the concept of a linear function and further specifies work with a linear function (Štátny pedagogický ústav, 2016).

When taught on the topic of functions, students also develop what is known as functional thinking. Functional thinking, simply put, considers a way of thinking in terms of relationships, interdependencies, and change. There are four aspects of functional thinking which are related to different perspectives on functions (Pittalis et al., 2020):

1. **Function as an input-output assignment:** Such a view of functions emphasises the operational and computational nature of the concept of function. It involves an examination of how a particular input value will lead to an output value. However, it does not require an awareness of the causal relationship between input and output (Pittalis et al., 2020). Suitable function representations are the input-calculation-output arrow chain or the input-output table.
   e.g., The total amount to pay as a function of the number of objects (candies, tickets) bought.
2. **Function as a dynamic process of covariation:** A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values having the property that, in the person's conception, every value of one quantity determines exactly one value of the other (Thompson & Carlson, 2017, p. 444). Suitable representations may be the function value table or the function graph, which can be scrolled through or traced.

e.g., In a linear function, the rate of change of the functional value per "unit" is the same.

3. **Function as a correspondence relation:** The correspondence relationship involves identifying the correlation between variables, using a function formula to predict distant values, and finding the value of one variable relative to the value of another variable (Pittalis et al., 2020). Such a view of function is an extension of the first aspect of the concept of function. For example, in an input-output table, the correspondence approach involves finding a numerical relationship between input and output values in addition to finding an explicit representation of the rule.
   e.g., Determine the coordinate of the intersection of the graph of the function $y = 3x + 5$ with the x-axis and the y-axis.
4. **Function as a mathematical object**, with its specific representations and properties which can be dealt with. Each representation provides a different view of the same object. This perspective is needed to compare a function with another function or with another mathematical object (FunThink Team, 2021).
   e.g., Observation of how the coefficient b affects the properties of the linear function.

These aspects highlight the key characteristics of function and can provide a foundation for teaching function and developing students' functional thinking.

From the given description of the MTSK, it is clear that one more subdomain is directly connected to the concepts' definitions. Or, to be more precise, linked to the process of defining. Namely, Knowledge of practices in mathematics (KPM).

2.3. Knowledge of Practices in Mathematics (KPM)

The term practice can be used in various ways by researchers. In this case, it means the workings of mathematics rather than the process of teaching it. KPM is defined as "any mathematical activity carried out systematically, which represents a pillar of mathematical creation, and which conforms to a logical basis from which rules can be extracted" (Carrillo et al., 2018, p. 243). It can be either general or topic-specific.

1. **General KPM** includes knowledge about how mathematics is developed beyond any particular concept.
   e.g., Understanding the meaning of necessary and sufficient conditions,
2. **Specific KPM** is a particular instance of general KPM. It is associated with the peculiarities of the topic in question.
   e.g., The use of proof by contradiction in proving the irrationality of some numbers.

Delgado-Reboledo and Zakaryan (2020) suggest that among other categories: “The KPM includes [...] the knowledge of how to define something in mathematics and the characteristics of a definition” (p. 571). Some of the characteristics of the definition are necessary due to the requirements of logic, while some are part of general culture. Van Dormolen and Zaslavsky (2003) suggest that the necessary features of definitions are:

1. **Criterion of hierarchy**: “Any new concept must be described as a special case of a more general concept. One or more properties must be used to describe this special case” (p. 94).
   e.g., Before we define a *linear function* as a function with a specific property, it is necessary to define a *function*.
2. **Criterion of existence**: A definition tells us what a concept is, but usually it does not say whether there exists an instance of such a concept within the current system [...] It must be proven that at least one instance of the newly defined concept exists in the current context (p. 94). The authors further state that the consequence of this criterion is that a well-defined concept needs to be followed by an example.
   e.g., We could possibly define a *quadrilinear function* as a function f in whose graph is a line and there exist $a, b, c \in R,\ a \neq 0;\ f:\ y = ax^2 + bx + c$. Yet, such a function does not exist. Its definition would lead to logical paradoxes.
3. **Criterion of equivalence**: When one gives more than one formulation for the same concept, one must prove that they are equivalent. In practice, this means that one must choose one of the formulations as the definition and consider the other formulations as theorems that have to be proved (p. 95).
   e.g., The linear function can be defined by its general equation or by its graph. Whatever we choose, we should be able to prove the respective feature as the consequence of the definition.
4. **Criterion of axiomatisation**: Some general concepts cannot be defined based on even more general concepts. Those concepts are implicitly defined in terms of axioms.
   e.g., In geometry, such concepts are point, line, and plane.
5. **Criterion of minimality** “demands that no more properties of the concept be mentioned than is required for its existence” (p. 96). If the definition does not meet this criterion, i.e., it also contains redundant properties, then the “definition” consists of the definition and at least one theorem.
   e.g., The definition of linear function where both features are stated – equation and graph – contradicts this criterion.
6. **Criterion of elegance** is the most subjective of all the criteria. It is a choice between two equivalent definitions: which one looks nicer, needs fewer words or fewer symbols, and so on.

7. **Criterion of degenerations**: "Degenerations are instances of a concept that we do not expect to be included when defining the concept. They are a logical outcome of the definition" (p. 99). We may change the terms of the definition so that such cases are not included. However, we must be careful not to disturb the development of the theory. Van Dormolen and Zaslavsky (2003) state these degeneracies are often accepted because it is difficult to predict the consequences of rejecting them. "Anyway, it should be shown that, if the definition allows for degenerations, properties that are proved for "normal" instances also apply to degenerations" (p. 99). e.g., Usually, when defining a quadratic function ($f: \ y = ax^2 + bx + c, x \in R$), we exclude $a = 0$, because it contradicts our perception of the quadratic function. On the contrary, when defining a linear function ($f: y = ax + b, x \in R$) it is arbitrary whether we include or exclude $a = 0$.

The last criterion is, from our perspective, the sub-criterion of the next requirement, which is missing from the list. Based on the definition, one has to clearly distinguish whether the object belongs to a set of defined objects. For instance, whether a square is or is not a rectangle or whether a constant function is or is not a linear function. Here, one more characteristic of definition becomes clear – definitions are arbitrary, and one can choose which definition one uses. The definition can be chosen, although its consequences are given.

As mathematicians, we often believe that each concept we use has a definition that fulfills the above-mentioned criteria. Yet, it is not true. Historically, many concepts were very well developed before the definition was established. Pinto and Tall (1996) recount two distinct purposes of the definition in mathematics:

> On the one hand, a concept which is already familiar to the student is given a definition to identify the concept. In this case, the concept determines the definition. On the other hand, in formal mathematics, the definition is used to construct the properties of the mathematical concept that it defines. In this case, the definition determines the concept (p. 140).

School mathematics usually heads from concept to definition. At some point, students need to know the "name" of the concept, and which image they construct. They study its properties and connections with similar concepts, and discuss whether the concept is a special case of some other concept, or whether those are disjunct sets of objects. However, students do not always need a formal definition. Similarly, mathematicians who are just discovering a new concept for which it takes time (sometimes centuries) until it has an established definition. Formal mathematics, conversely, starts with a definition and ends with the concept – its properties, examples, counter examples, and applications.

This dichotomy of approaches causes problems when providing textbook definitions. These "school" definitions sometimes only describe or characterise the concept and therefore do not fulfill some of the criteria stated by Van Dormolen and Zaslavsky (2003). We can see these difficulties in Slovak teaching materials when defining linear function.

2.4. Definition of Linear Function

Let us now analyse different definitions of linear functions stated in different textbooks and teaching materials used by Slovak teachers. It is worth noting that not all of them are labeled as "definitions" in textbooks. In those cases, it is out of the question whether the authors mean a definition or mathematical statement. This is exactly the point where we can see confusion between the concept image (Tall & Vinner, 1981) and its definition. From the perspective of what will come next, we spotlight the criteria of minimality and degeneration. In Table 1, we can see that three definitions (2, 3, and 4) do not satisfy the criterion of minimality. These definitions combine standard equation and graph of linear functions. Most of them accept a constant function as a special case of a linear function, although Bero and Berová (2015) exclude it. None of the definitions explicitly mention that the domain can be a subset of real numbers. However, Šedivý and colleagues (2004) admit that a graph of the linear function can be not only a straight line, but also part of a straight line. In addition, Hecht et al. (2001) use *time* as an example of the domain and therefore accept a subset of real numbers to be the domain of the linear function. Three definitions do not operate with the domain at all. Additionally, we can see that the standard definition of the linear function in Slovak context is based on the formula. This highlights the function as a correspondence relation. Only one definition (6) supports the dynamic process of covariation.

Table 1. Definition of Linear function in Slovak teaching materials

	Definition of Linear function	Resource	Does it deal with possible degenerations?		Criterion of minimality
			constant	$D(f) \subset R$	
1	The notation of the linear dependence of two quantities x and y in the form $y = k \cdot x + q$, where k and q are arbitrary real numbers, is called the equation of this linear dependence.	Kolbaská, 2014, p. 47 textbook recommended by the Ministry of Education	yes	not clear	Yes
2	A linear function is a function given by $y = k \cdot x + q$, where k, q, are arbitrary real numbers, $k \neq 0$. The graph of a linear function is a line.	Bero & Berová, 2015, p. 57 textbook	no	not clear	No

3	A linear function is any function given by the formula $y = k \cdot x + q$, where k, q, are arbitrary real numbers. The graph of a linear function is a line.	Kohanová et al., 2016, p. 45 textbook	yes	no (as a conclusion)	No
4	A function in the form: $y = k \cdot x + q$, where k and q are arbitrary real numbers and its defining domain is the set of all real numbers, is called a linear function. The graph of a linear function is a line or part of a line (if the domain is bounded).	Šedivý et al., 2004, p. 32 textbook	yes	no and yes (as a conclusion)	No
5	A linear function is any function given by the equation $y = a \cdot x + b$, where $a, b \in R$, the defining domain is the set R.	Vavrinčíková, n.d., p. 2 online material	yes	no	Yes
6	Linear functions are those functions where the rate of change of the functional value per "unit" (usually time) is the same.	Hecht et al., 2001, p. 23 textbook	not clear	not clear	Yes
7	The function $f\ :\ y = a \cdot x + b;\ x \in R$ is called a linear function. The numbers a, b, are denoted as the coefficients of the linear function.	Krynický, n.d., p. 1 online material	not clear	no	Yes

Our cursory observation implies that, in some textbooks, the concept of a linear function is not well defined in terms of the criteria of the definition and correctness of the content. Those could more or less work as descriptions of the concept image of linear functions. To some extent, the precision of the definition is related to beliefs about mathematics, which are part of MTSK.

By a completely correct definition of a linear function, we mean definition 5, although we could accept the following formulation: A linear function is any function given by the equation $y = a \cdot x + b$ where $a, b \in R$, the defining domain is the set R, <u>or its subset</u>. An important aspect is how the examples where, for instance, R^+ is a domain, are treated. Alternatively, the covariance approach could be used (definition 6) to define linear function. However, this textbook definition should be completed with information about possible rates of change (is "0 change" the change?) and also specify the domain to be considered fully correct.

2.5. Beliefs about Mathematics

To study pre-service teachers' beliefs, we drew on the classifications in Carrillo and Contreras (1994) who provided a categories system and descriptors for a more detailed characterisation of the teacher's beliefs of mathematics and its teaching. To identify teachers' beliefs about mathematics, there are three tendencies, the names of which were adapted from Ernest (1989, as cited in Carrillo & Contreras, 1994):

1. **Instrumentalist** – Mathematics is like a bag of tools, made up of an accumulation of facts, rules, and skills to be used in the pursuance of some external aims. Thus, mathematics is a set of unrelated but utilitarian rules and facts.
2. **Platonist** – Mathematics is a static but unified body of knowledge, a crystalline realm of interconnecting structures and truths, bound together by filaments of logic and meaning. Mathematics is discovered, not created. The aims of mathematical knowledge are internal.
3. **Problem solving** – Mathematics is a dynamic, continually expanding field of human creation. Mathematics is a process of inquiry, and its results remain open to revision. The aim of mathematical knowledge is intellectual development.

It would be expected that holders of different beliefs about mathematics will have a different approach to the role of definitions. It seems that the Platonist tendency should mostly support the direction going from definition to concept. However, this tendency will be especially strict about fulfilling the criteria of a good definition.

3. Methodology

3.1. Context of the Study and Data Collection

The presented study was conducted in the context of the FunThink Erasmus+ (Enhancing functional thinking from primary to upper secondary school) international project. Three main objectives were set. Firstly, designing learning environments through which it will be possible to develop functional thinking in primary and secondary school pupils. Secondly, creating a course for pre-service teachers that empowers them to develop the functional thinking of their future pupils. The third objective was to create an online platform where materials will be available to a wide professional audience (FunThink Team, 2021). This study is anchored in a pilot version of the course that was conducted in the Slovak Republic. The course lasted 14 weeks with a weekly allocation of 90 minutes. It was attended by 13 pre-service teachers in the first year of their master's study. For them, it was the first course directly related to the didactics of mathematics (pedagogical content knowledge). Pre-service teachers were aware of the conducted research based on the course and agreed to participate. The structure of the course was as follows (Table 2):

Table 2. Structure of the pilot version of the course in the Slovak Republic

Week	Module	Topic
1	Introduction	Course description Pre-assessment (test and lesson plan development)
2	KoT	Independent solutions of tasks focused on Knowledge of Topic
3		Discussion about tasks and their solutions
4	Functional thinking	Aspects of Functional thinking and representations of functions
5		Aspects of Functional thinking in tasks, solutions, and definitions
6	Design principles of FT project	Inquiry and Digital tools
7		Situatedness
8		Embodiment
9		Pre-service teachers' task design
10	Formative assessment	Functional thinking in students' solutions
11		Teacher's reaction to incorrect answer (video analyses)
12	Curriculum	Aspects, representations, and application of function in Slovak curriculum
13	Outro	Post-assessment (test and lesson plan development)
14		Feedback in terms of MTSK

The data on which this article is based was collected during the pre- and post-assessment tests (sessions 1 and 13). The pre-service teachers' solutions and answers from the pre-test were not discussed during the course; their accuracy was not disclosed. None of the tests were part of the final assessment, thus the research participants' motivation was not affected by the pressure of getting a better grade.

Before the course, as part of their undergraduate studies, they were to gain a deep mathematical understanding mainly of advanced mathematics. Most of the mathematics courses they have taken were constructed in the manner of Definition – Theorem – Proof. One of the more important courses deals with mathematical logic. Thus, we can say that from a mathematical point of view, these pre-service teachers were taught in a Platonist way.

In this context and concerning the stated theoretical background, we pose the following research questions:

RQ1: How do pre-service teachers define the linear function? Does it change after the course?

- How – in the sense of fulfilling criteria of definition,
- How – in the sense of the aspects of functional thinking used.

RQ2: Are pre-service teachers consistent with their own definition when discussing the linearity of a given function? Does it change after the course?

- Consistent – in the sense of accepting the consequences of their own definitions,
- Consistent – in the sense of using the same concepts in the definition and the argumentation.

RQ3: What tendencies are visible in pre-service teachers' beliefs about mathematics?

3.2. Research Tools and Data Analyses

The pre-test and post-test contained eight identical tasks and nine questions, from which we analysed those which helped us answer our research questions (Table 3). The data was analysed by two authors (MK and VH) who first coded the data independently, then compared, refined, and unified the coding. The code for each task is stated in the results section.

Table 3. Tasks and questions

No.	Task / Question	Purpose of the task
T2	Decide whether the functions below are linear. Justify your reasoning. $C: y = 14$ $D: y = 2x + 1$ $F: y = \frac{x^2}{x} + 1$	In these tasks, we wanted to see whether their decision and argumentation about the linearity of the given functions is consistent with their definition of linear function. We focused on the domain of the function (F – Is the function linear even though 0 is excluded from the domain?) and the range of the coefficients (C – Is a constant function a special case of the linear function?)
T8	Define the following concept. If you consider it important, state more definitions. Linear function:	
Q1	Write down your personal definition of mathematics.	Using these questions, we wanted to find out what perspectives pre-service teachers have on mathematics, as well as what they think is the importance of teaching functions.
Q2	What is the difference and what are the similarities between mathematics as a science and mathematics as the subject you will teach?	
Q6	What do you consider to be the goal of teaching about functions? Expand your answer.	

4. Results

4.1. Definition of Linear Function

Table 4 presents the codes we used in the data analysis of Task 8, where pre-service teachers were asked to define a linear function. Table 5 summarises the results of the coding. Green or blue cells represent pre-test results, and green or yellow cells represent post-test results.

Table 4. Coding for Task 8

No.	Codes
T8	**Correctness** • Domain ▪ Real numbers ▪ Real numbers or a subset of *R* ▪ Unspecified • Redundancy **Underlying concept** • Equation ▪ *a*, *b* from *R* ▪ *b* from *R*, *a* from *R* – {0} ▪ *a*, *b* unspecified • Graph ▪ Line ▪ Line or its part • Polynomial (exponent) • Constant rate of change

Table 5. Definition of Linear function in pre- and post-test

	Pre-service teachers nick name		Adam	Bejka	Cecília	Danka	Emil	Fany	Gabika	Heňa	Ivana	Jana	Karol	Maša	Noro	only pre-test [blue]	only post-test [yellow]	both tests [green]	total in pre-test	total in post-test
T8: Definition of Linear function	correctnes		[green]	[yellow]	[yellow]										[yellow]	0	3	1	1	4
	redundancy			[blue]		[green]		[yellow]		[blue]		[yellow]	[yellow]	[yellow]	[blue]	3	4	1	4	5
	domain discussed (pre-test - post-test)		R-ssR	not-R	not-ssR	not-R									not-R					
	equation	**any of the next**	[green]	[green]	[yellow]	[green]	[yellow]	[yellow]	[green]	[green]	[yellow]	[yellow]	[green]	[green]	[green]	0	5	8	8	13
		a, b from R	[green]	[yellow]	[yellow]	[yellow]	[yellow]	[yellow]	[blue]	[green]	[yellow]	[yellow]	[blue]		[green]	2	7	3	5	10
		b from R, *a* from R-{0}												[green]		0	0	1	1	1
		a, b unspecified		[blue]		[blue]			[yellow]				[yellow]			2	2	0	2	2
	graph	**any of the next**				[green]		[green]		[blue]	[blue]	[yellow]		[yellow]	[blue]	3	2	2	5	4
		line				[green]		[blue]		[blue]	[blue]			[yellow]	[blue]	4	1	1	5	2
		line or its part						[yellow]				[yellow]				0	2	0	0	2
	polynomial			[blue]	[blue]		[blue]									3	0	0	3	0
	constant rate of change												[yellow]			0	1	0	0	1

In the pre-test, only one of the participants gave a completely correct definition of the linear function. The difficulties included redundancy of the information given in the definition (n = 4). The criterion of minimality was mostly broken by using an equation and a straight line to define linear function. Except for one pre-service teacher, no one indicated the domain of the function within the definition. One participant, Jana, stated a definition which was not describing linear function at all. Her definition had more semblance to a one-to-one function. Pre-service teachers relied mostly on an equation (n = 8) or a graph (n = 5). Three of them perceived linear function as a special case of a polynomial function. Among those who rooted their definition in an equation, most (n = 10) stated that coefficients are real numbers, one of them expelled slope equal to 0, and two of them did not specify the range of the coefficients.

In the post-test, we identified a correct definition in the responses of four participants. Redundancy was again present (n = 5). Interestingly, this was a repeated error for only one participant. The other three made some progress. However, four others now made this error. In addition, five participants had already indicated the domain in this test: three as a set of real numbers, and two as an arbitrary subset of real numbers. Each participant used an equation as a basic concept for the definition and the polynomial approach disappeared, however, one pre-service teacher suggested a definition based on a constant rate of change. Four participants used the concept of the straight line for defining linear function. Two of them also accepted part of the straight line to be the graph of a linear function.

In both tests, we observe a clear tendency to define the linear function using a formula. Only one pre-service teacher in the post-test (Karol) used the covariational approach. These and the previous observations are in alignment with what we perceived in the Slovak teaching materials, even though the pre-service teachers were not systematically instructed to study them.

4.2. Consistency between the Definition and Decision

We mentioned in the theoretical introduction that definitions are arbitrary, however, the consequences of the choice of definition are given by logical rules. Therefore, we look closely at the pre-service teachers' consistency between their own definition of a linear function and their decisions about the linearity of three functions given by equations (tasks 2C, D, F). Table 6 presents the codes we used in the data analysis of Task 2. For this analysis, we presumed, if not stated otherwise, the domain for x and the range for a, b to be real numbers. The results are displayed in Table 7, in the rows "Is it LF?"

Table 6. Coding for Task 2

No.	Codes	
T2	**Is it a linear function in the terms of the definition given by the pre-service teacher in T8?**	
	- yes	(1)
	- no	empty cell
	- difficult to decide	(?)
	Is it a linear function (test answer)?	
	- yes	(1)
	- no	empty cell
	- it depends on the definition	(?)
	Argument	
	- equation *(range of coefficients)*	
	- graph *(line or line or its parts)*	
	- polynomial	
	- rate of change	
	- constant function *(implies linear)*	

Firstly, we notice that the number of consistent answers tended to increase from pre-test to post-test: from 8(+1) to 10, from 9 to 11, and from 2 to 7(+1) for the given functions. Next, the most difficulties appeared with the task 2F, where the domain of the function was $R - \{0\}$. Even though some of the pre-service teachers discussed the domain of the given function in the pre-test, their decision ended up inconsistent (Adam, Emil, Fany, Heňa, Ivana). In the post-test, the discussion about the domain always led to consistency with one's own definition.

Moreover, we analysed their argumentation and its match to the one used in their definition (see rows Arguments, Table 7). The meaning of the codes (blue, yellow, and green colors) is explained next the table.

Again, here we can see a minor shift towards more consistent argumentation. If we count the amount of the same concepts used in the definition and the argumentation, we can see that for task 2C, the number of matches increased from 3 to 8, for task 2D, from 8 to 9, for task 2F, from 5 to 11. Using a different concept in the argument and the definition is not necessarily an error. We perceive using the "line argument" to explain the decision about the linearity of the function while defining a linear function using only the concept of the equation as being a bit problematic from the perspective of logic. In this case, pre-service teachers should be asked how they know the graph is a line. We assume a circular argument ("because it is a linear function") could possibly occur. Similarly, using the argument "it is a constant function" without mentioning it in the definition. In the given context, we were not able to discern more information. It is a possible enhancement for our future research.

Table 7. Consistency between the definition and the decisions in pre- and post-test

			Pre-service teachers nick name	Adam	Bejka	Cecília	Danka	Emil	Fany	Gabika	Heňa	Ivana	Jana	Karol	Maša	Noro	consistency	included in the definitio	included in the argumer	included in both
T2C: decide (constant)	Pre-test	is it LF?	in terms of definitic	1	1		1		1	1	1	1	1		1	1	8			
			test answer	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**		**?**		**1**	+1			
		arguments	equation															7	0	1
			graph															3	2	2
			polynomial															3	0	0
			rate of change															0	0	0
			constant function															0	8	0
	Post-test	is it LF?	in terms of definitic	1	1	1	1		1	1	1	1	1	1		1	10			
			test answer	**1**	**1**	**1**		**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**				
		arguments	equation															7	0	6
			graph															3	0	1
			polynomial															0	0	0
			rate of change															0	0	0
			constant function															0	7	1
T2D: decide (y=2x+1)	Pre-test	is it LF?	in terms of definitic	1	1		1	1	1	1		1	1			1	9			
			test answer	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**				
		arguments	equation															4	0	4
			graph															2	3	3
			polynomial															2	2	1
			rate of change															0	0	0
	Post-test	is it LF?	in terms of definitic	1	1	1	1		1	1	1		1	1	1	1	11			
			test answer	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**				
		arguments	equation															5	0	8
			graph															3	1	1
			polynomial															0	1	0
			rate of change															1	0	0
T2F: decide (R-{0})	Pre-test	is it LF?	in terms of definitic	1				1			1				1	1	2			
			test answer		**1**	**1**	**1**		**1**			**1**	**1**		**1**	**1**				
		arguments	equation															4	3	4
			graph															4	1	1
			polynomial															3	0	0
			rate of change															0	0	0
			domain discussion	1				1	1		1	1		1				6	0	0
	Post-test	is it LF?	in terms of definitic	1		1		1		1	1		1	1	1		7			
			test answer	**1**		**1**	**1**	**?**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	+1			
		arguments	equation															4	0	9
			graph															2	2	2
			polynomial															0	0	0
			rate of change															1	0	0
			domain discussion	1		1					1			1				4	0	0

Furthermore, the pre-service teachers used more than the one necessary argument to correctly argue their decision only several times (2C pre-test: 0, post-test: 2, 2D: pre-test and post-test: 0, 2F: pre-test: 1, post-test: 2). This redundancy of arguments appeared only in formulas which could possibly be interpreted as a linear function. It is likely that the pre-service teachers felt the need to provide a "stronger" argument to justify their decision.

4.3. Beliefs

Table 8 presents the codes we used in the data analysis of Question 1, 2, and 6. Table 9 displays the results of coding for Questions 1, 2, and 6 in both tests. The color green stands for a code present in both tests. The color blue is used when the code was visible only in the pre-test, and yellow is used when the code was observed only in the post-test.

Table 8. Coding for Question 1, 2 and 6

No.	Codes

Q1
Q2
Q6
	Beliefs tendencies about mathematics:
	• Instrumentalist
	  ▪ External aims of mathematical knowledge
	• Platonists
	  ▪ Internal aims of mathematical knowledge
	• Problem solving
	  ▪ Intellectual development as an aim of mathematical knowledge

Table 9. Beliefs tendencies about mathematics in pre- and post-test

Pre-service teachers nick name			Adam	Bejka	Cecília	Danka	Emil	Fany	Gabika	Heňa	Ivana	Jana	Karol	Maša	Noro	pre and post test (green)	only post-test (yellow)	only pre-test (blue)	total in pre-test	total in post-test
Beliefs tendencies about mathematics	Instumentalist	Aims	green	green	yellow	blue	blue		yellow	blue		green	blue	green	blue	4	2	5	9	6
		Other				green					green	yellow	blue	green		3	1	1	4	4
	Platonist	Aims	blue	yellow	green	yellow	blue	green	green	blue	green	green	green	green	blue	7	2	4	11	9
		Other	green	green	yellow	blue	green	green	green	green		blue	blue	green	blue	7	1	4	11	8
	Problem solving	Aims	green	green	green		green			green			yellow		green	6	1	0	6	7
		Other			blue		green						yellow		blue	1	1	2	3	2

Important information derived from the data is the following: Firstly, the beliefs of individual pre-service teachers did not show a clear tendency in most cases. Seven of them showed all three tendencies, five of them showed two tendencies, and one pre-service teacher showed beliefs from only one tendency (when including both pre- and post-test). We do consider differences between the tests as the change of beliefs, because the tool was not adjusted for identifying "non-beliefs". Therefore, we cannot say whether the pre-service teacher did not hold a particular belief, or it simply did not appear. Secondly, this variability was primarily due to beliefs from the "Aims" category. When omitting this category, one pre-service teacher revealed all three tendencies, six proclaimed two tendencies, and five showed only one tendency. Thirdly, "Platonistic" emerged as the strongest tendency in this group of pre-service teachers. Their experiences in mathematics courses during their undergraduate studies could easily justify this result.

5. Conclusions and Discussion

Thompson (1992, p. 131) stated that "to look at research on mathematics teachers' beliefs and conceptions in isolation from research on mathematics teachers' knowledge will necessarily result in an incomplete picture". Our research demonstrated this statement once again. If we observed only the beliefs of our pre-service teachers, we would expect exact definitions and precise argumentations. Instead, we found a small number of correct definitions and many inconsistent decisions. Somehow, we are in a paradoxical situation, in which we have pre-service teachers whose beliefs claim that mathematics must have a system and respect clear rules while simultaneously not following this requirement. The reasons behind this dichotomy can differ from one subject to another and require deeper research including interviews with the participants.

Firstly, we can challenge the information about the Platonist beliefs of pre-service teachers. Even though these were clearly present in their answers and supported by university courses, they are not necessarily central in the belief system of pre-service teachers. Moreover, the connection between beliefs and behavior is not straightforward. Even salient belief is filtered several times and it takes several steps to transform a belief into intention, and intention into behavior (e.g., Bosnjak, Ajzen, & Schmidt, 2020). In this context, one of the "filters" is pre-service teacher knowledge.

In general, knowing the definition of a linear function is part of KoT, because the Slovak curriculum requires this piece of knowledge to be taught. Therefore, at first sight, we could say that problems in definition formulation and argumentation of decisions were caused due to a missing piece in this subdomain. However, the majority of the participants were able to formulate definitions of a linear function (except one pre-service teacher in the pre-test) that described their concept image of a linear function. This concept image was correct. Most of them used plausible arguments when deciding whether the given equations represent linear functions or not. However, the definitions and arguments lacked precision and clarity, and did not respect the criteria of a good definition.

This leads us to the conclusion that the more challenging subdomain might be KPM, although we cannot be sure whether it is a problem of personal or enacted knowledge (as suggested by Carlson et al., 2019). From the perspective of course design, it matters if pre-service teachers do not have the knowledge about the criteria of definitions, or they do not enact this knowledge when defining a concept. One way or another, our research states that the unclear borders between concept image and concept definition of a linear function, which is present also in Slovak teaching materials, need to be addressed in a specific way. In further research, we will uncover what is behind the problems of pre-service teachers and re-design the course, and/or provide feedback to our colleagues, informing them to do so in their courses.

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Begüm Özmusul & Ali Bozkurt
University of Gaziantep

Chapter 12

AN EXPERIMENTAL STUDY ON MIDDLE SCHOOL PRE-SERVICE MATHEMATICS TEACHERS' ALGEBRAIC KNOWLEDGE FOR TEACHING

Summary: In this study, our primary objective was to investigate the impact of algebraic instruction centered on problem-solving and emphasising the cultivation of algebraic habits of mind. The aim was to evaluate the effectiveness of this instructional approach in enhancing the proficiency of pre-service elementary mathematics teachers in algebraic teaching. The research employed a quasi-experimental method and involved 66 pre-service teachers enrolled in the elementary mathematics teachers' education department at a state university, all of whom were taking the Algebra teaching course. Participants were randomly assigned to either the experimental group (31 participants) or the control group (35 participants) in a randomised pretest-posttest control group design. The data collection instrument utilised was the Elementary Patterns, Functions, and Algebra-Content Knowledge (PFA) test. Mean scores and ANCOVA values were computed for the data analysis. The study revealed that the pre-test mean of the pre-service teachers in the control group exceeded that of the pre-service teachers in the experimental group. However, upon examining the post-test averages, it became evident that the pre-service teachers in the experimental group outperformed those in the control group. The ANCOVA results indicated a statistically significant difference between the post-test average scores, adjusting for the pre-test scores of the pre-service teachers in the experimental and control groups. These findings suggest that the training provided to the experimental group significantly contributed to the enhancement of the pre-service teachers' algebraic knowledge for teaching.

Keywords: algebra knowledge for teaching, Algebraic Habits of Mind (AHoM), problem-based algebra teaching, middle school pre-service mathematics teachers.

1. Introduction

Algebra is a fundamental branch of mathematics that plays a crucial role in developing students' abstract thinking skills. Research suggests that for pre-service teachers to teach algebra effectively, they must have a strong understanding of algebraic concepts (Magiera et al., 2013). To help students develop algebraic thinking, teachers need to support their understanding and connections between mathematical ideas (NCTM, 1997; Kieran, 2007).

However, there is still a lack of research on effective strategies to enhance pre-service teachers' proficiency in this area (Magiera et al., 2013).

Teachers' knowledge and skills are key factors influencing the success of instructional practices (Borko & Putnam, 1996; Mewborn, 2003). Therefore, well-designed teacher education programs that focus on both subject knowledge and pedagogical knowledge are essential for improving algebra instruction (Ball & Bass, 2000). Effective teacher training programs should be structured to integrate both mathematical content and pedagogical strategies (Hill, 2010).

This study explores the impact of problem-based algebra instruction on the algebra teaching knowledge of pre-service elementary mathematics teachers. Specifically, it investigates how an instructional approach based on Algebraic Habits of Mind (AHoM) influences pre-service teachers' ability to teach algebra. By comparing a traditional teaching approach with problem-based instruction, this research aims to assess the contribution of AHoM-based teaching to the development of pre-service teachers' algebra knowledge for teaching.

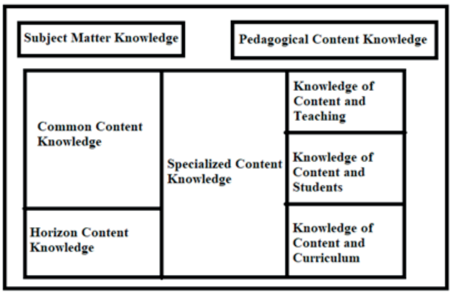
2. Theoretical Framework

Within the purview of this study, we aim to investigate the transformation in the algebra teaching proficiency of pre-service elementary mathematics educators, specifically focusing on problem-based algebra teaching within the conceptual framework of Algebraic Habits of Mind (AHoM). The study is theoretically grounded in the frameworks of Mathematical Knowledge for Teaching (MKT) and AHoM.

2.1. Mathematical Knowledge for Teaching

There has been a noteworthy surge in research examining teachers' pedagogical knowledge and the requisite competencies within this domain (Darling-Hammond, 2006; Fernández-Soria, 2013; Morris et al., 2009). Within this scholarly discourse, Ball et al. (2008) focused on Shulman's (1987) delineation of subject knowledge and pedagogical content knowledge, specifically advancing the conceptualisation of mathematical knowledge for teaching. This concept has transitioned from a mere depiction of what teachers ought to know and execute in the mathematics teaching domain (Ball & Bass, 2003) to an intricate model that classifies various facets of teacher knowledge (Ball et al., 2008). The researchers delineate six distinct categories, while acknowledging the inherent challenge of unequivocally demarcating the boundaries between them (Ball et al., 2008).

Figure 1. Domains of Mathematical Knowledge for Teaching (Ball et al., 2008)

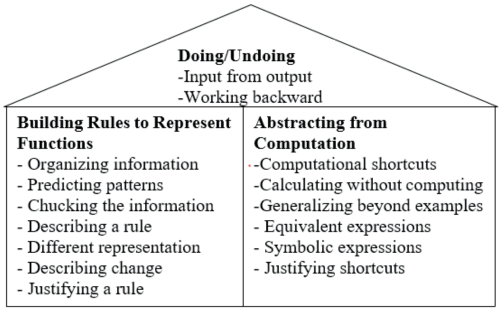


The conceptualisation of teacher knowledge, encompassing the distinctive attributes of knowledge essential for effective teaching and the processes through which this knowledge is acquired and reinforced during teacher preparation, has undergone significant evolution over the past three decades. Predominantly, paradigm shifts have occurred within the domains of teacher knowledge, specifically content knowledge and general pedagogical knowledge (Magiera et al., 2017). While Shulman's seminal theory (1986) delineates a foundational knowledge base requisite for proficient teaching, the present study by Hill et al. (2008) is grounded in the evolved practice-based theory of Mathematical Knowledge for Teaching (MKT). MKT not only aids educators and pre-service teachers in cultivating the decision-making acumen vital for effective classroom instruction (Johnson, 2009), but it also furnishes a robust theoretical framework and practical applications for teacher education programs (Hill et al., 2005).

2.2. Algebraic Habits of Mind

Algebra, within the field of mathematics, is concerned with symbols and generalised numerical entities extending beyond basic arithmetic operations to address equation-solving, analyse functional relationships, and elucidate the structure of a representation system comprising expressions and relations (Lew, 2004). Practitioners of algebraic discourse commonly opt to initially emphasise specific properties before delving into the implications of these properties, for instance, some direct attention towards abstract features that distinguish algebra from arithmetic. From this standpoint, algebraic thinking is characterised as the ability to operate on an unknown quantity as if the quantity is known, as opposed to arithmetic reasoning, which involves operations on known quantities' (Langrall & Swafford, 2000). Others emphasise the role of problems in algebra, defining algebraic thinking as the capability to represent quantitative situations in a manner that renders relationships between variables visible. Driscoll (1999) posits that possessing algebraic thinking involves contemplating functions and their workings, along with considering the impact of system structure on computations. These dual facets of algebraic thinking are nurtured by specific habits of mind.

Figure 2. Algebraic Habits of Mind (Driscoll et al, 2001)



Driscoll (1999) conceptualised the habits of mind of building rules to represent functions and abstracting from computation under the Doing-Undoing habit of algebraic thinking (Figure 2). It includes a set of sub-habits that encourage algebraic thinking about algebraic concepts and making sense of algebraic problems in every habit.

Doing-Undoing: This algebraic habit of mind serves as a framework for the other two habits. Students should be able to conclude an algebraic operation as well as work backwards from the result of an algebraic operation to reach its starting point. With this habit of mind, students do not only focus on reaching the result, but also think about the process. For example, if $x^2 - 1 = 0$, they should be able to find the solution of this equation as well as the equation with roots $x = 1$ and $x = -1$.

Building Rules to Represent Functions: This habit of mind involves recognising and analysing patterns, investigating and representing relationships, generalising beyond specific examples, analysing how processes or relationships change, and looking for evidence of how and why rules and procedures work. Driscoll (1999; 2001) describes the processes in Figure 2 that characterise the algebraic habit of building rules to represent functions in a mathematical situation. More specifically, Driscoll identifies the features that provide insights into the specific processes underlying the act of analysing patterns and relationships in a mathematical situation and describing them using a functional rule.

Abstracting from Computation: It is the habit of thinking about computation independently of the numbers used. Abstraction is important for this habit of mind. Abstraction is the process of extracting mathematical objects and relationships based on generalisation. For example, when calculating the sum of the numbers $1 + 2 + 3 + ... + 50$, students can regroup the numbers to get 51 and reach the result as 50 + 1 = 51; 49 + 2 = 51; 48 + 3 = 51, ... In this process, it is important to allow students to think in different ways and to find different solutions.

In the literature pertaining to algebraic thinking, numerous studies have investigated aspects such as assessing students' algebraic thinking skills (Chimanoi et al.; Kaput, 1999), examining its developmental trajectory (Driscoll, 1999), and determining levels of proficiency in algebraic thinking (Hart et al., 1998). Drawing inspiration from Cuoco et al.'s (1996) exploration of beneficial cognitive approaches to mathematical content, denoted as 'habits of mind', Driscoll (1999) construed algebraic thinking as the cognitive process involved in contemplating quantitative situations that facilitate the clarification of relationships between variables. Presenting a theoretical framework, Driscoll (1999) delineated the requisite habits that students must cultivate to develop proficiency in algebraic thinking. He argued that as students acquire the ability to interpret symbols, they take a pivotal step in expressing generalisations, unveiling algebraic structures, establishing relationships, and formulating mathematical situations. However, the inquiry arises as to whether this framework exerts an influence on the development of algebra teaching knowledge among pre-service elementary mathematics teachers.

This study endeavors to investigate the impact of problem-based algebra teaching on the advancement of pre-service elementary mathematics teachers' knowledge of algebra teaching within the conceptual framework of habits of mind related to algebraic thinking. Magiera et al. (2013) assert the importance of prioritising the design of robust teacher education programs that effectively foster the development of algebraic thinking in primary school students. In this context, the research aims to scrutinise the effect of problem-solving-oriented algebra teaching grounded in the acquisition of AHoM on the enhancement of pre-service elementary mathematics teachers' knowledge of algebra teaching. Aligned with the research objective, the investigation seeks to answer the following question:

– Is there a statistically significant difference in the post-test achievements related to algebraic teaching knowledge between pre-service elementary mathematics teachers who received problem-based algebra teaching within the framework of acquiring algebraic habits of mind and those who did not?

3. Method

In this investigation, a quasi-experimental model involving both experimental and control groups was employed. Experimental research is designed to explore causal relationships between variables (Evans, 2005). However, educational research, as outlined by Campell and Stanley (1963), often necessitates the use of a quasi-experimental model due to the absence of random group selection in this context (Campell & Stanley, 1963). In light of this consideration, the utilisation of a quasi-experimental design is deemed appropriate, allowing for the examination of potential differentiation in the algebra teaching knowledge of pre-service elementary mathematics teachers by applying distinct educational processes to the experimental and control groups. Additionally, the research design, to be implemented within the scope of this study, is succinctly outlined in Table 1.

Table 1. Research design

<table>
<tr><th>Groups</th><th>Pre-test</th><th>Education process</th><th>Post-test</th></tr>
<tr><td>Experimental group (n=31)</td><td rowspan="2">MKT-PFA
(A form)</td><td>Algebra teaching based on problem solving within the framework of AHoM</td><td rowspan="2">MKT-PFA
(B form)</td></tr>
<tr><td>Control group
(n=35)</td><td>Algebra teaching within the scope of the curriculum of Council of Higher Education (CoHE)</td></tr>
</table>

The "Mathematical Knowledge for Teaching-Elementary Patterns Functions and Algebra-Content" (MKT-PFA) was developed in English by Ball and Hill (2009) as part of the "Learning Mathematics for Teaching Project" conducted at the University of Michigan. This assessment tool aims to evaluate the algebra teaching knowledge of pre-service elementary mathematics teachers. The test comprises two equivalent forms, namely A and B forms, with 29 items in Form A and 27 items in Form B, three of which are common to both forms. The MKT-PFA test was adapted into Turkish for the purpose of this study. Pre-service elementary mathematics teachers participating in the experimental and control groups were administered the MKT-PFA (Form A) prior to the commencement of a 12-week algebra teaching practice, and the MKT-PFA (Form B) following the completion of the same teaching practice.

In this study, the instructional intervention applied to the experimental group was designed based on the Algebraic Habits of Mind (AHoM) framework. As outlined in the theoretical framework, AHoM emphasises problem-solving approaches that promote algebraic reasoning, such as Doing-Undoing, Building Rules to Represent Functions, and Abstracting from Computation (Driscoll, 1999). These habits were explicitly embedded in the instructional activities throughout the 12-week intervention. Pre-service teachers in the experimental group engaged in structured problem-solving tasks that required them to analyse patterns, generalise relationships, and think flexibly about algebraic structures. In contrast, the control group received traditional instruction aligned with the Council of Higher Education (CoHE) curriculum, which primarily focused on procedural fluency rather than conceptual exploration. This quasi-experimental design allows for an investigation into whether problem-based algebra instruction within the AHoM framework leads to a significant improvement in algebra teaching knowledge compared to conventional methods.

3.1. Participants

The research was conducted during the spring semester of the 2021/2022 academic year with pre-service elementary mathematics teachers enrolled in the "Algebra Teaching" course within the elementary mathematics teaching program at the education faculty of a state university in Turkey. The course, "Algebra Teaching", is bifurcated into morning and afternoon sessions. The researcher designated the morning session as the experimental group and the afternoon session as the control group. Consequently, the study comprised a total of 66 pre-service elementary mathematics teachers, with 31 assigned to the experimental group and 35 to the control group. The assignment of students to their respective groups was executed through a random allocation process.

3.2. Data Collection Tools

In the process of adaptation, the initial step involved the translation of items in the A and B forms into Turkish. Subsequently, four meetings were conducted with a group comprising two academics specialising in mathematics education and five mathematics teachers pursuing doctoral studies, to finalise the translation. Adaptations were made with considerations for general culture, school culture, mathematical language, and other contextual nuances as recommended by Delaney et al. (2008). Following the adaptation, algebra teaching knowledge tests, totaling 53 items, were administered to 183 pre-service elementary mathematics teachers enrolled in the elementary mathematics teaching department of a state university in Turkey.

The reliability of the adapted data collection tool to Turkish was assessed using the Kuder Richardson 20 and 21 formulas (KR-20, KR-21), which serve as indicators of the test's internal consistency (Wallen & Fraenkel, 2013). The KR-20 values for the adapted A and B forms were .784 and .799, respectively. According to Wallen & Fraenkel (2013), a KR20 reliability coefficient of .70 and above is recommended for attaining a reliable score. Additionally, point-biserial correlation estimates in both the USA and Turkey yielded highly similar results for Form A ($r = .658$; $t = 4.540$; $p = 0.0001$) and Form B ($r = .721$; $t = 5.215$; $p = 0.0000$). Consequently, it can be affirmed that the Turkish version of the algebra teaching knowledge tests exhibits reliability and validity, rendering it suitable for assessing the knowledge of teachers and pre-service elementary mathematics teachers in the domain of algebra teaching.

3.3. Data Collection Procedure

For the purposes of this study, conducted within the framework of the Algebra Teaching course, a total of 66 pre-service elementary mathematics teachers were randomly assigned to either the experimental or control group. Subsequently, the participants were administered the Algebra Teaching Knowledge Test (Form A). The pre-service elementary mathematics teachers comprising the experimental group underwent a 12-week instructional intervention, where algebra was taught through a problem-solving approach within the conceptual framework of Algebraic Habits of Mind (AHoM). Throughout this period, the pre-service teachers engaged in weekly problem-solving activities designed to apply AHoM principles. As an illustrative example, the provided problem in Figure 3 necessitated generalising the operations utilised in arithmetic, facilitating a seamless transition from mathematical concepts to algebraic reasoning.

Figure 3. Something Nu Problem (Driscoll, 1999, p. 55)

Something Nu

Consider the operation of counting the factors of a whole number. This function is usually called "ν" (the lowercase Greek letter for "nu"). For example, the number 6 has the factor 1, 2, 3, and 6, so $\nu(6) = 4$. Here's some practice:

1. If the input to ν is 5, what is the output? What if the input is 12?
2. What is $\nu(24)$? $\nu(288)$? $\nu(2^3 \times 3^2 \times 5^4)$?
3. Find some numbers that ν takes to 6.
4. Classify all numbers n so that $\nu(n) = 3$. Classify all numbers n so that $\nu(n) = 2$.
5. What can you say about a number m if $\nu(m) = 12$?
6. Find two numbers n and m so that $\nu(nm) = \nu(n)\nu(m)$. Find two more. Compare with the findings of others.

This study involves a deliberate emphasis on fostering Algebraic Habits of Mind (AHoM) among pre-service elementary mathematics teachers, with an explicit encouragement for them to think in terms of constructing rules for representing functions. The instructional approach includes framing the 'number of factors' as a functional relation and delving into the concepts of input and output within the context of the Something Nu problem. Challenges such as "Find some numbers that takes to 6" prompt engagement with the notions of doing-undoing. The overarching goal is to enhance the pre-service teachers' proficiency in AHoM. Additionally, the participants received guidance on developing AHoM alongside algebraic teaching by addressing guiding questions during problem-solving activities.

In contrast, the pre-service elementary mathematics teachers in the control group underwent a 12-week algebra instruction, following the framework established by the Council of Higher Education (CoHE, 2018). This curriculum covered various aspects of algebraic thinking, emphasising its importance in mathematics teaching. The content included the pre-algebraic period, the arithmetic-algebra relationship, generalised arithmetic and functional thinking, basic algebraic concepts, different representations in algebra teaching, and the teaching of variables, algebraic expressions, equations, and inequalities. Furthermore, the instruction involved organising course content, utilising appropriate teaching materials and strategies, and assessing student knowledge of these topics, including understanding and interpreting students' thinking about concepts, identifying difficulties, errors, misconceptions, and exploring the practical connections of these topics with daily life and other courses.

3.4. Data Analysis

Tabachnick et al. (2007) proposed that a group is considered normally distributed when the skewness and kurtosis values for the normality distribution fall within the range of -1.5 to +1.5. Therefore, both the experimental and control group pre-service teachers in this study are deemed independent, and the MKT-PFA post-test scores for both groups exhibit a normal distribution (experimental group skewness=-1.036, kurtosis=.601; control group skewness=-.394, kurtosis=-.956). Following the analysis, it was determined that the variances between the groups with respect to the pretest scores of the students in the experimental and control groups were homogeneous ($F(1, 66)=2.379$, $p=.128>.05$). This implies that the slopes of the regression lines, calculated for predicting post-test scores based on pre-test scores for both groups, are equal. Consequently, it is inferred that the necessary assumptions for the Analysis of Covariance (ANCOVA) are satisfied. Given the fulfillment of these assumptions, descriptive statistics were computed for both groups' post-test scores, and subsequently, ANCOVA was conducted to ascertain whether there existed a statistically significant difference in student achievements between the groups.

4. Results

In this study, descriptive statistics were computed for both groups, and the Analysis of Covariance (ANCOVA) was conducted for the post-tests. The aim was to investigate whether there was a statistically significant difference in the performance of pre-service elementary mathematics teachers in the Algebra Teaching course, as assessed by the MKT-PFA test, between the experimental and control groups (Table 2).

Table 2. Averages of MKT-PFA Pre-Test and Post-Test Results

	Groups	N		%	sd
Pre-test	Experimental	31	21.81	.80	3.45
	Control	35	23.47	.77	1.91
Post-test	Experimental	31	25.34	.93	1.88
	Control	35	24.47	.91	1.68

When Table 2 is examined, it is seen that the pre-test averages of the pre-service elementary mathematics teachers in the control group are higher than the pre-test averages of the pre-service elementary mathematics teachers in the experimental group. When the post-test averages of the algebra teaching knowledge test are examined, the post-test average score of the pre-service elementary mathematics teachers in the experimental group is =25.34 and the post-test average score of the pre-service elementary mathematics teachers in the control group is =24.47. However, when the pre-test scores are checked, it can be said that there is a change in the post-test scores of the pre-service elementary mathematics teachers in the experimental and control groups.

The ANCOVA results for the significance of the difference between the post-test scores of the pre-service elementary mathematics teachers in the experimental and control groups according to their average scores are given in Table 3.

Table 3. ANCOVA Results of MKT-PFA Post-Test Scores

Source	Sum of squares	Df	Mean square	F value	p	Eta square
Model	277.576	1	7.167	100.194	.000	.809
Pre-test	57.807	1	3.204	24.964	.000	.284
Group	31.379	1	2.781	13.551	.000	.177
Error	145.882	63				
Total	235.068	66				
p<0,05						

When Table 3 is examined, it is seen that the difference between the post-test mean scores of the pre-test scores of the pre-service elementary mathematics teachers in the experimental and control groups according to the ANCOVA results is statistically significant ($F(1, 63) = 13.551$, $p<.05$). Therefore, the increase in the pretest-posttest achievement scores of the pre-service elementary mathematics teachers in the experimental group is statistically significantly higher than the increase in the achievement scores of the pre-service elementary mathematics teachers in the control group. On the other hand, the fact that the eta square value, which is an indicator of practical significance, is 1.77, indicates that the effect is very close to high (Cohen, 1988).

5. Discussion and Conclusion

This study examined the impact of problem-based algebra instruction on the algebra teaching knowledge of pre-service elementary mathematics teachers within the framework of Algebraic Habits of Mind (AHoM). The findings indicate a significant improvement in the algebra teaching knowledge of pre-service teachers in the experimental group after completing the instructional process. This suggests that problem-based algebra teaching, structured around AHoM, positively contributes to pre-service teachers' understanding and instructional skills in algebra.

The positive outcomes observed in the experimental group can be attributed to the instructional approach, which encouraged pre-service teachers to analyse and explore algebraic thinking through guided questions. This aligns with previous research emphasising the importance of providing teachers with opportunities to interpret and evaluate students' mathematical reasoning (Stump, 1999; Asquith et al., 2007; Lynch & Star, 2014). Additionally, studies have shown that mathematical knowledge plays a critical role in shaping effective teaching practices and improving student learning (McCrory et al., 2012; Hill & Ball, 2004).

While the results highlight the benefits of AHoM-based instruction, there are some limitations to consider. The study was conducted with pre-service teachers from a single university, which may limit the generalizability of the findings. Furthermore, this research focused specifically on problem-based instruction within the AHoM framework, and its effects in other instructional contexts remain an open question. Future studies could explore AHoM's impact at different levels of teacher knowledge or examine how teachers' algebraic habits align with those of their students.

Despite these limitations, this study provides valuable insights into how problem-based algebra teaching can enhance pre-service teachers' instructional knowledge. The findings support the integration of AHoM into teacher education programs as a means to develop stronger algebra teaching practices.

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Barbara Nawolska
University of the National Education Commission in Krakow

Chapter 13

REDUCTIVE REASONING OF PEDAGOGY STUDENTS IN THE PROCESS OF SOLVING A TEXT TASK ENTITLED: HOW MANY PEARLS WERE IN THE CASKET?

Summary: Everyone needs the ability to solve problems. In school education, the formation of this skill can and should be implemented by solving text tasks. The teacher, in order to teach task solving, should have such a skill himself. The article presents the results from a study of the ability to solve a certain task by students of pedagogy – future teachers of early childhood education.

Keywords: text task, solving a task, reductive reasoning, pre-service teachers of early childhood education.

1. Introduction

Everyone, regardless of their role in life, needs a broad-based ability to see and solve problems in a variety of life and work situations. And at the same time, the ability to think logically and critically is important. This is a lifelong learning skill and the earlier we start learning it, the better the results we will achieve. It is therefore not without reason that the core curriculum of 14 February 2017 states that 'Primary education is the foundation of education' (Podstawa Programowa [Core Curriculum], 2017). The purpose of this education is:

> the development of competences such as creativity, innovation and entrepreneurship; the development of skills of critical and logical thinking, reasoning, argumentation and inference (...) equipping students with a body of knowledge and the formation of skills that enable them to understand the world in a more mature and structured way (Podstawa Programowa [Core Curriculum], 2017, p. 11).

In view of this, the most important skills developed as part of general education in primary school include: proficient use of the tools of mathematics in everyday life, the ability to think mathematically, the ability to search for, organise and use information from various sources responsibly and the ability to critically analyse and evaluate it, the ability to solve problems in various fields with the conscious use of methods and tools derived from computer science and programming (ibid., pp. 12–13).

The mentioned goals can be achieved, among other things, in mathematics education through solving text tasks, as tasks of this type constitute a special case of a problem situation. The ability to solve them therefore, in addition to its educational value, has a great practical dimension being a paradigm of action in any problem situation.

While solving tasks and problems[1], students acquire the ability to analyse facts, synthesise events, estimate risks, make rational decisions, improve abstract thinking, learn to conduct correct deductive and reductive reasoning, make inferences not only in familiar but also new situations, both simple and complex, typical and atypical (ibid. p. 26). Thus, learning mathematics serves to develop logical and critical thinking. Logical and critical thinking is what we need most nowadays and what will be useful in the as yet unknown future in both our private and professional lives. Therefore, the greatest challenge facing schools and teachers is to educate pupils to be people who think logically and critically. It is about thinking understood as a process of "modifying the uncertainty of judgements (judgements) under the influence of information obtained both through logical and experimental analyses" (Nosal, 1988, p. 16).

2. Deductive and Reductive Reasoning

Reasoning is the resolution of issues by means of inference or the derivation of one sentence from another on the basis of a logical result relation (Dictionary of the Polish Language, 1981, p. 126). Thus, it is a process in which certain beliefs are followed by further beliefs, linked to the previous ones by logical inferential relations (Czyżewski, 1993, p. 399). The process of reasoning can be presented orally or in writing and takes the form of a sequence of sentences in which premises and conclusions are distinguished. The premise is the sentence that is the beginning of the reasoning, i.e., it is the basis for the recognition of its results, while the result of the reasoning is the conclusion.

A special role in reasoning is played by conditional sentences, called implications, which have the form *if p then q*, where *p* and *q* are sentences. A sentence *p is* called the predecessor and a sentence *q* the consequent of an implication. There are differences between inter-

[1] Word problems are a good tool for developing pupils' competences. Solving them contributes to developing the habit of critical thinking, which facilitates the evaluation of different situations, verifying whether there are grounds for a thesis or not, and thus makes it possible to arrive at the truth.

preting an implication in natural language and its formal (mathematical) sense. These differences are exhaustively discussed by Helena Siwek (2005, pp. 255–259).

In logic and mathematics, the relation which exists between the sentence *p* and the sentence *q* when it is excluded that the predecessor of *p is* true, while the successor of *q* is false, is called the relation of the **resultant**. We say then, that from the predecessor of an implication its corollary follows, and the sentence *p is* called the predicate of the sentence *q,* while the sentence *q* the **corollary** of the sentence *p. The* relation itself is called the relation of the predicate to the corollary. The premise and the conclusion are related to each other by the relation of the result, i.e., the relation of the rationale to the corollary. There are two possibilities of opposing these parts of the reasoning: premise – conclusion, reason – consequence. This leads to two types of reasoning: deductive and reductive. Deduction is reasoning in which the premise is the rationale (the antecedent of the true implication) and the conclusion is the consequent (the successor of the implication). Reduction, on the other hand, is reasoning in which the premise is the consequent, while the conclusion is the rationale. Thus, **deduction** is *direct* reasoning, i.e., based on known **reasons** (causes), **consequences** (effects) are inferred. **Reduction**, on the other hand, is *reverse* reasoning, i.e., from the consequences (effects) one deduces their rationale (causes), and not from the rationale about the consequences; in this reasoning one goes backwards from the facts to their causes.

Deductive reasoning is very natural and is often called cause-and-effect thinking. It takes the form of a true implication, e.g., *If there is no electricity on the grid, then the light bulb does not light up.* The sentence *there is no electricity on the grid,* is the cause of the *bulb not being lit,* while the sentence the bulb is *not lit,* is the effect of *there being no electricity on the grid*. The first of these sentences is the premise of deductive reasoning and at the same time *the* rationale for the corollary, which is the sentence bulb is not lit, while the sentence bulb is *not lit* is the conclusion and corollary of the fact that *there is no electricity in the grid*. This is because we conclude that the absence of electricity in the grid is always the cause of the bulb not being lit. Reasoning in the opposite direction: i.e. reasoning in which, on the basis of the established fact that the light bulb is not lit, we believe that this condition was caused by the lack of current, is reductive reasoning.

In solving tasks in grades I-III of primary school, we mostly use deductive reasoning. We use it in all typical calculation tasks: e.g., *components are given, find their sum; determine the difference of a given minuend and a given subtrahend; factors are given, determine their product; determine the quotient of a known dividend and divisor*. The same is true of most text tasks. E.g., in solving the tasks: *Adaś had 5 zlotys, he bought juice for 3 zlotys. How many zlotys does he have left?* and *Staś has 4 lorries and 5 more cars. How many cars does he have?* deductive reasoning is used. In all such tasks, the premises (data from the task) are at the same time the causes and the conclusions are the effects of the known causes; the conclusions are the calculated quantities that were unknown and about which the task asks. Most pupils are good at solving tasks of this type, as logical thinking emerges in children in late childhood. This makes it possible to make cause-and-effect inferences. A child at this age is able to explain phenomena and predict theoretical consequences of various events (Stefańska-Klar, 2006, p. 135).

Tasks requiring reductive reasoning (backwards) are somewhat more difficult, as solving them involves reversing the actions (activities) referred to in the task. In order to solve tasks of this type, the child must develop the ability to internally reverse the given actions or a certain state of affairs. That is, the child must reason operationally.

Consider an example of a task: *There were a few zlotys in the piggy bank. When I put 3 zlotys into the piggy bank and counted all the money, it turned out that there were 10 zlotys. How many zlotys were in the piggy bank at the beginning?*

Solving this task reductively, we start from the final state: there were 10 zlotys at the end. Since there was that much after the previous addition of 3 zlotys, so going backwards, we have to reverse the addition action, i.e., take away 3 zlotys from the 10 zlotys. This leaves 7 zlotys. So there must have been 7 zlotys at the beginning. Not all early childhood education pupils are able to reason this way straight away. Some guess the unknown, others find it using memorised facts (7 + 3 = 10), others use trial and error. However, despite a certain degree of difficulty, there can be no lack of such tasks in early childhood education, as they are essential for developing the skills of reductive reasoning so important both in mathematics and in many practical situations, e.g., medical diagnosis[2].

3. Skills of Students – Future Early Childhood Education Teachers – In Solving a Pearl Reduction Task (Quantitative Analysis)

The ability to solve text tasks, is a key competence. Teachers and also pedagogy students, as future teachers, should be able to solve such tasks in order to use them with children. In doing so, it is important that they not only know how to solve them somehow, but also that this solution is accessible to pupils in the younger grades.

In March 2023, during my first classes in the subject "strategies for mathematics education in grades I-III", I asked fourth-year students of pre-school and early childhood pedagogy at University of the National Education Commission in Krakow to solve the following task in writing:

[2] Based on the effects of the patient's illness, the doctor must determine the cause of the illness (determines what caused the symptoms of the illness).

> The Maharaja gifted 3 of his daughters with pearls stored in a casket. To the eldest he gave half the contents of the casket and one pearl, to the younger half the rest and one pearl, and to the youngest half the remaining pearls and 3 more pearls and then the casket remained empty. How many pearls did the Maharaja have in the casket and how many pearls did each daughter get?

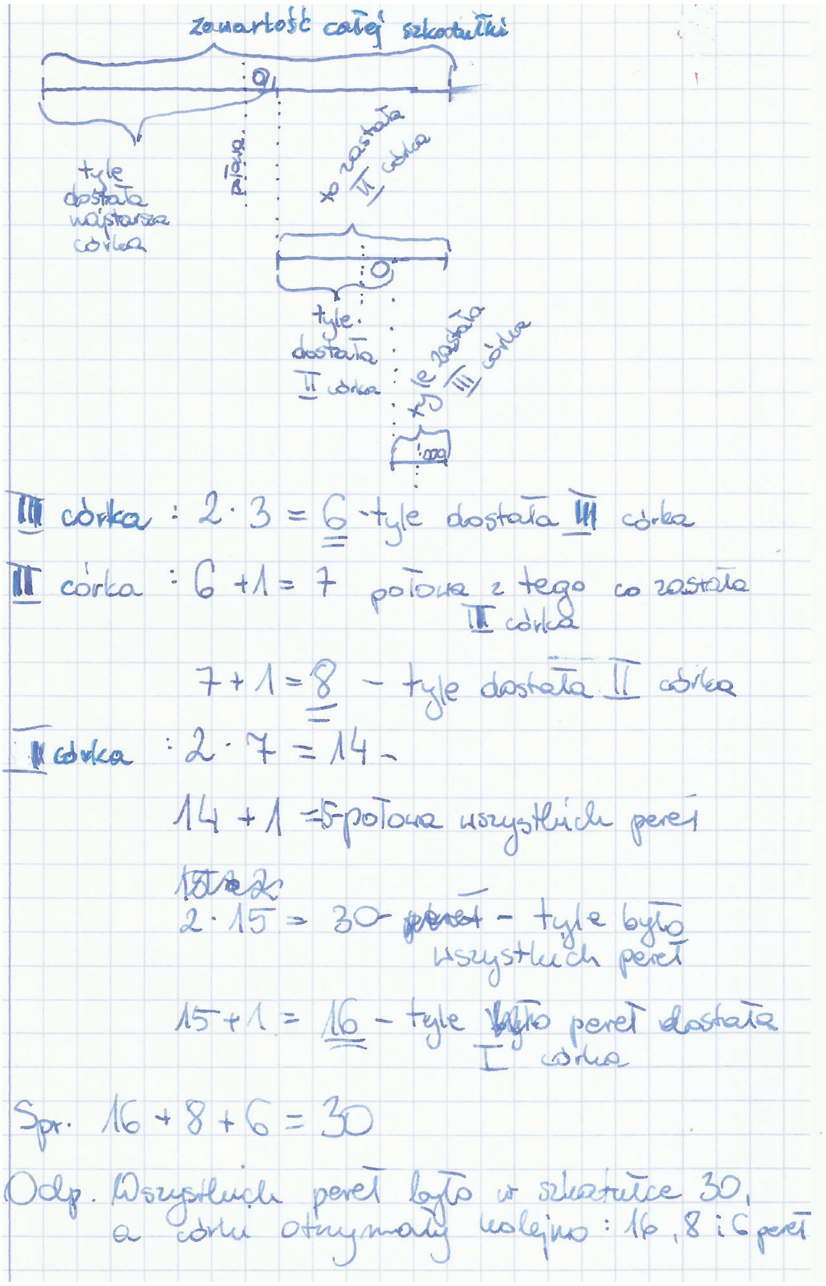
A prerequisite for correctly solving a task is understanding its content. This understanding consists of the verbal (linguistic) layer of the task text (the ability to read the text with comprehension is important here) as well as the conceptual layer connected with understanding the mathematical concepts contained in it. In this case, it is necessary to understand what "half" means (the result of dividing into 2 equal parts). You also need to understand that the size of the half depends on the size of the whole being divided. In the task, each time (in the case of each daughter) the maharaja divides a different size and, in addition, reduces the remainder by 1 (in the case of the first two daughters) or by 3 (in the case of the last daughter). However, the ability to read with understanding alone is not sufficient. What is also needed is the ability to mathematise the situations presented in the task, which represents the greatest difficulty for a large group of solvers. The difficulty is related to the fact that often the same words from the mother tongue can be translated differently into the language of mathematics and vice versa, many different words can be translated into the language of mathematics in the same way. The ability to make such a translation should grow with experience and therefore with the age of the solver. The longer we study, the more experience we have and the more skill we have in this area. Do students, who have a lot of experience (after all, they have almost 15 years of schooling behind them), know how to read with understanding and know how to mathematise the situation described in the problem?

Furthermore, reductive reasoning should be used in solving this task. And a graphical (drawing) representation of the situation described therein can be a great help. Is reductive reasoning within the students' grasp? And will a drawn representation make it easier for them to solve it?

4. Examples of Correct Solutions to the Pearl Task (Qualitative Analysis)

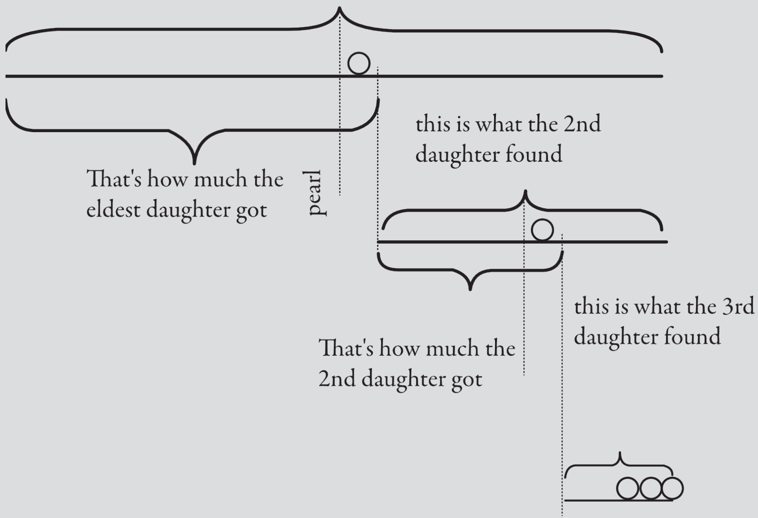
Examples of work using apt illustrations are shown in Figures 1, 2, and 3.

Figure 1. Correct solution supported by drawing – work by Anna Z.



Text translation

Contents of the entire casket (above the top bracket)



3rd daughter 2 · 3 = 6 this is how much 3rd daughter got

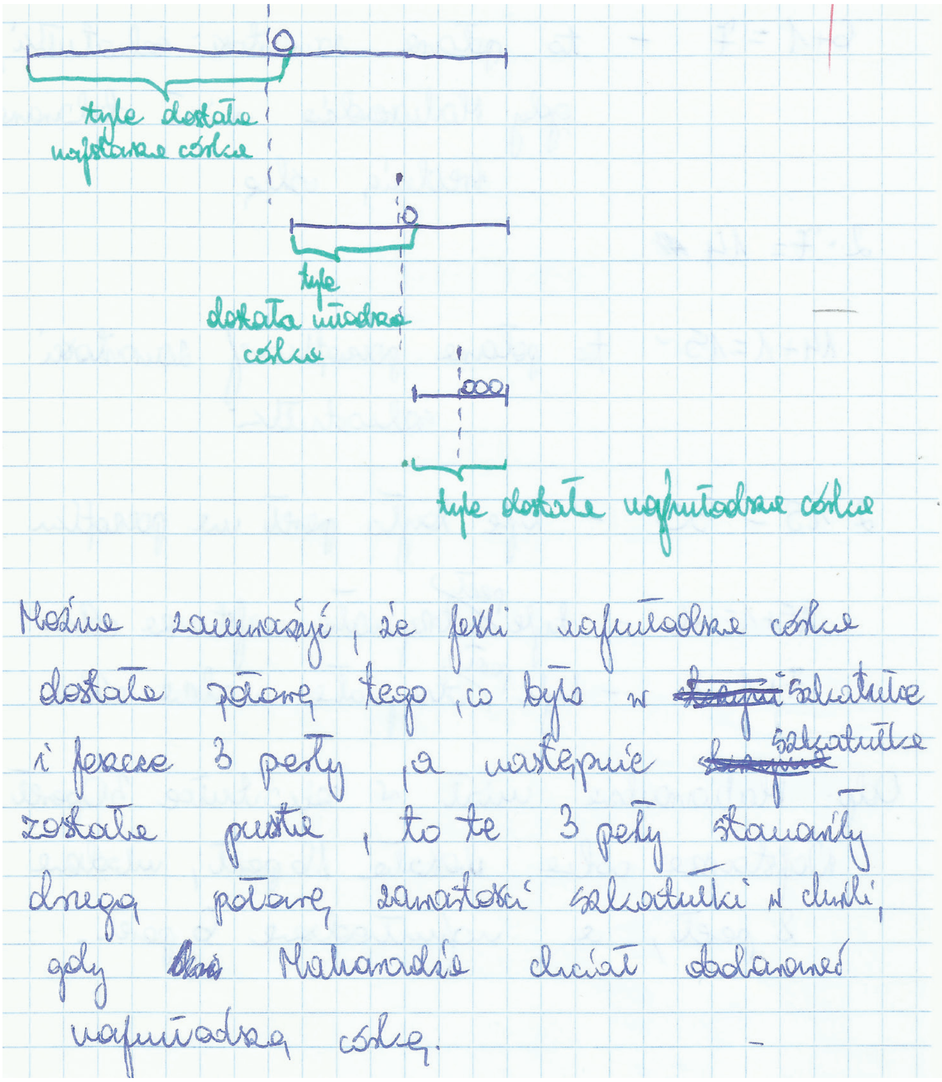
2nd daughter 6 + 1 = 7 half of what 2nd daughter found
7 + 1 = 8 – that's how much the 2nd daughter got

3rd daughter 2 · 7 = 14
14 + 1 = 15 half of all pearls
2 · 15 = 30 that is how many pearls there were
15 + 1 = 16 that's how many pearls I daughter got
Spr. 16 + 8 + 6 = 30
Answer: There were 30 pearls in the casket.
And the daughters received 16, 8, and 6 pearls, respectively.

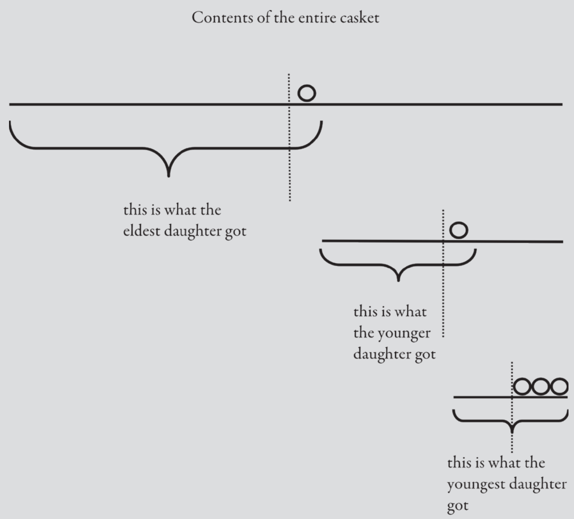
Anna Z. (Fig. 1.) presented the pool of pearls by means of a segment, which she divided into 2 equal parts and assigned one of these parts (left part) together with an additional pearl drawn with a circle in the second (right) part to the 1st daughter (*this is how much the eldest daughter received).* She redrew the rest of the pearls by means of a segment and proceeded in the same way as for the first division, assigning half of the new segment and 1 more pearl to the 2nd daughter (this is how much the *2nd daughter got).* She redrew the remaining pool again using another section, which she divided into 2 equal parts. In the last (second) part of this new division she drew 3 pearls. She assigned this entire pool (section) to the 3rd daughter. Using the drawing from bottom to top, she saw that the 3rd daughter got $2 \cdot 3$, i.e., 6 pearls. She also saw in the drawing that the half of the pearls found by the 2nd daughter was 6 + 1, i.e., 7, and calculated that, therefore, the 2nd daughter got 7 + 1, i.e., 8 pearls, while she found 14 (because $2 \cdot 7 = 14$). When she added 1 to 14, she found that half of all the pearls were 15, so there must have been $2 \cdot 15$, i.e., 30 pearls. Calculating how many pearls the eldest daughter got was very easy: 15 + 1 = 16. She then checked that the sum of the pearls distributed was 30 (16 + 8 + 6) and answered all the questions in the task.

A similar drawing and the reasoning beneath it can be seen in the work of Barbara K. (Fig. 2.).

Figure 2. Correct solution supported by drawing – work by Barbara K.



Text translation



It can be seen that if the youngest daughter was given half of what was in the casket and 3 more pearls and then the casket was left empty, then these 3 pearls constituted the other half of the contents of the casket at the time the Maharaja wanted to bestow the youngest daughter.

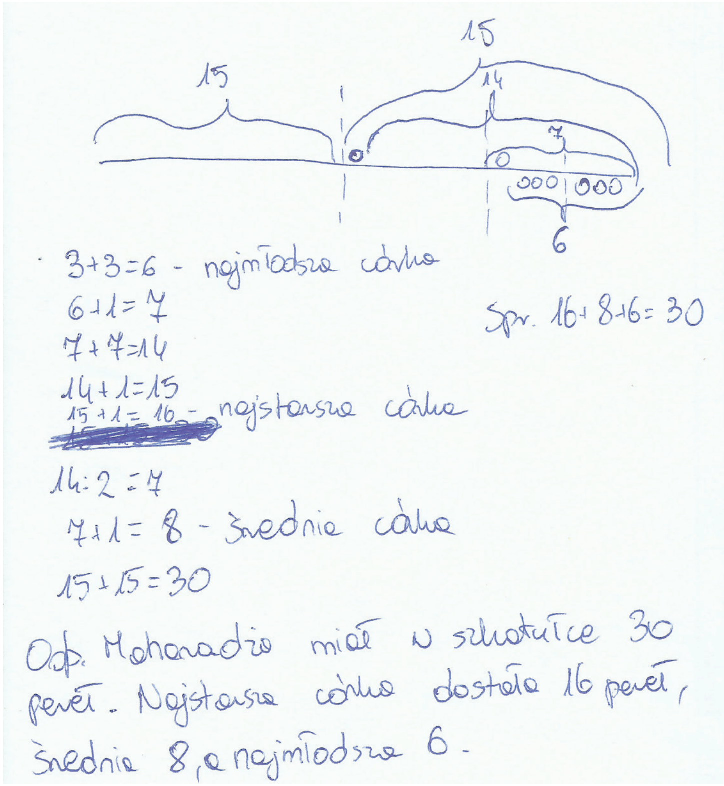
On the back of the sheet of paper Barbara K. wrote down the following further reasoning:

2 · 3 = 6 this is how much the youngest daughter got
6 + 1 = 7 is half of the contents of the casket when the Maharaja was to bestow the middle daughter,
2 · 7 = 14, 14 + 1 = 15 is half of the initial contents of the casket,
2 · 15 = 30 that is how many pearls there were at the beginning.
15 + 1 = 16, 7 + 1 = 8.
Ans. The Maharaja had 30 pearls. The eldest daughter got 16 pearls, the younger daughter got 8 and the youngest daughter got 6.

Neither Anna Z. nor Barbara K. wrote any numbers on the drawing. They made calculations based on the illustrations and wrote them under the illustrations. Furthermore, they depicted each state of the pearls before the next distribution in a separate drawing.

In contrast, Kinga G. (Fig. 3.) illustrated all the stages of pearl distribution in a single drawing and immediately used this drawing for her calculations by writing the numbers of pearls found in the "from the end" calculations onto it.

Figure 3. Correct solution supported by drawing – work by Kinga G.



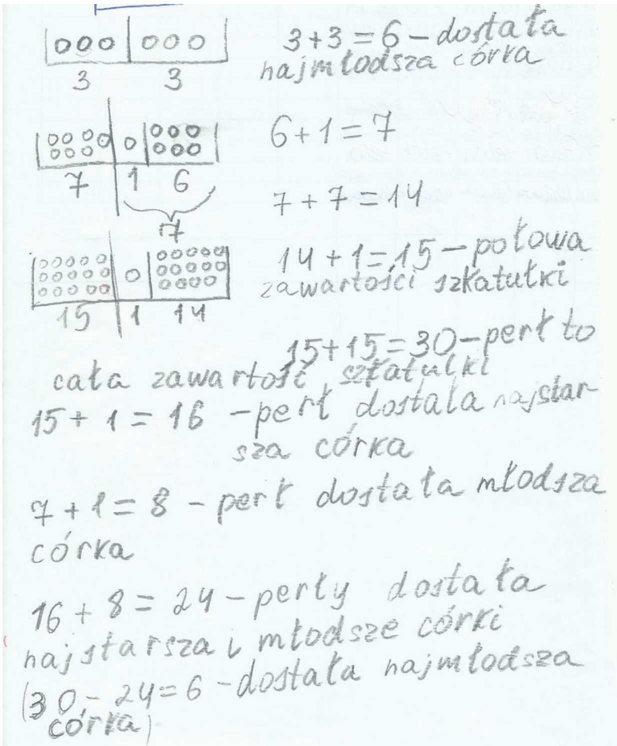
Text translation

3 + 3 = 6 youngest daughter
...
15 + 1 = 16 eldest daughter
...
7 + 1 = 8 middle daughter
15 + 15 = 30. Verification: 16 + 8 + 6 = 30
Ans. The Maharaja had 30 pearls in his casket. The eldest daughter got 16 pearls, the middle daughter got 8 and the youngest daughter got 6.

In all the works presented in Figures 1, 2, and 3. the illustrations were created from the beginning, i.e. from the first hand of pearls, while the calculations were created from the end, i.e. from the last hand of pearls.

It is different in the work presented in Figure 4. The author starts both the calculations and the illustration with the presentation of the final situation. She carries out the calculations in the same way as in the works in Figure 1 and Figure 2. Only at the end, although she has already calculated at the very beginning how many pearls the youngest daughter got, she again, but in a different way, determines the same number. First, using addition 16 + 8, she determines that the two older daughters received 24 pearls. Then, using subtraction 30 – 24, she calculates how many the youngest daughter got.

Figure 4. Correct solution supported by drawing – work by Iryna S.

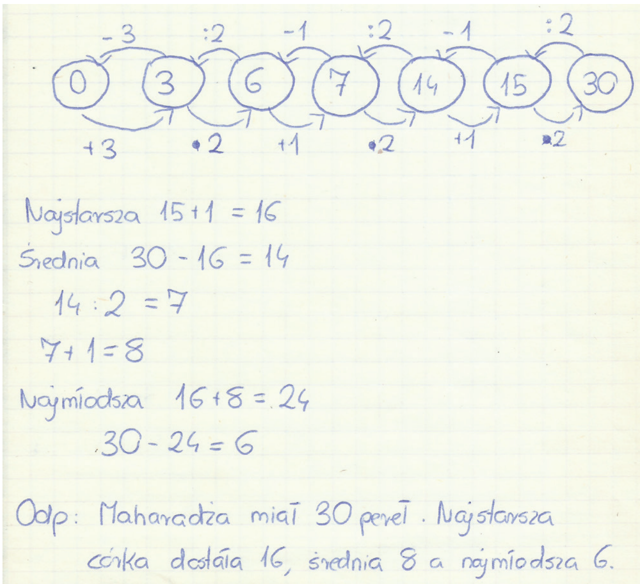


Text translation

3 + 3 = 6 – got the youngest daughter
6 + 1 = 7
7 + 7 = 14
14 + 1 = 15 – half of the contents of the casket
15 + 15 = 30 – is the content of the entire casket
15 + 1 = 16 – the pearls were given to the eldest daughter
7 + 1 = 8 – the pearls were given to the younger daughter
16 + 8 = 24 – pearls were given to the eldest and younger daughter
30 – 24 = 6 – got the youngest daughter

Another tool used in solving the task was the arrow graph. The way in which the graph was used in the solution is presented in Figures 5, 6, and 7. In the work in Figures 5 and 6, the graph was drawn starting from the left, indicating the final state, and successive arrows from left to right marked the operations opposite to those performed in the description of the task, while the results of the calculations were entered in the boxes of the graph, which led to the initial number of pearls. This was enough to later (under the graph) calculate how many pearls each daughter received. Furthermore, in the graph in Fig. 5, the arrows in the right-to-left direction were also completed, whereas in the work in Fig. 6, this was no longer done. Apparently, the author of this work did not feel the need to do so.

Figure 5. Correct solution supported by a full graph – work by Iwona M.



Text translation

Eldest 15 + 1 = 16

Average 30 – 16 = 14

14 : 2 = 7

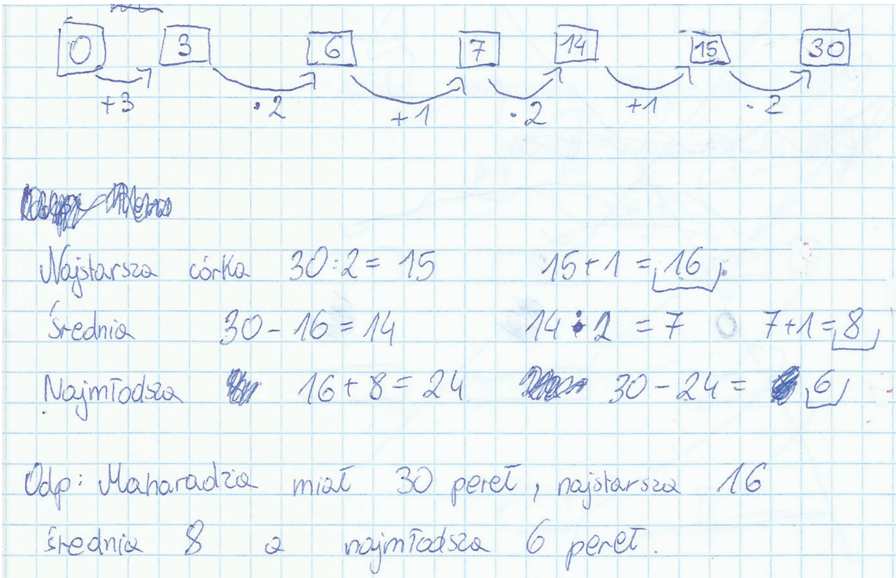
7 + 1 = 8

Youngest 16 + 8 = 24

30 – 24 = 6

Ans. The Maharaja had 30 pearls. The eldest daughter got 16, the middle daughter got 8 and the youngest daughter got 6.

Figure 6. Correct solution supported by an incomplete graph – work by Klaudia O.

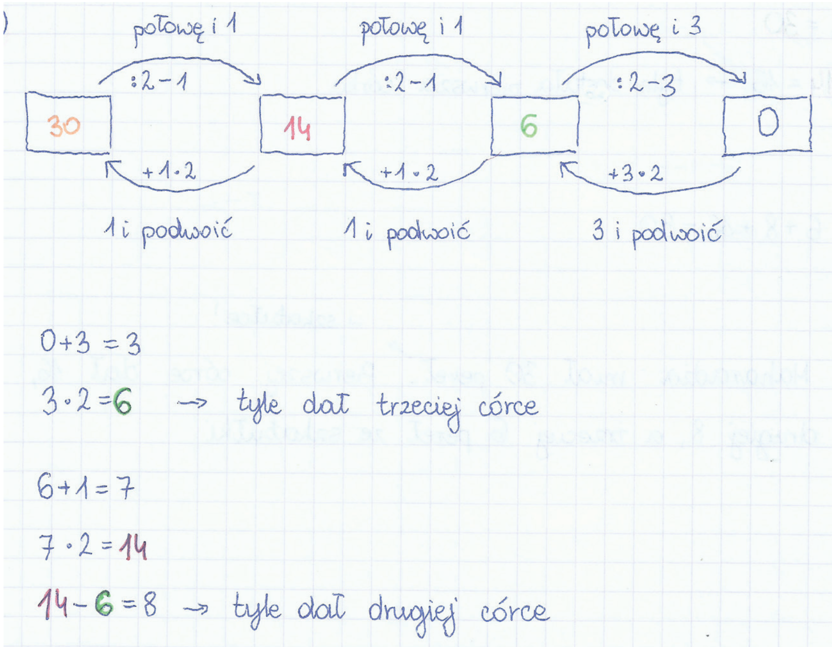


Text translation

Eldest daughter 30 : 2 = 15 15 + 1 = 16
Average 30 – 16 = 14 14 : 2 = 7 7 + 1 = 8
Youngest 16 + 8 = 24 30 – 24 = 6
Ans. The Maharaja had 30 pearls, the eldest 16, the middle 8 and the youngest 6 pearls.

The graph shown in the work in Figure 7 is like a shortened version of the full graph (one arrow represents two actions: catching and subtracting). Moreover, in this graph, the first box from the left presents the initial state and, when the graph was drawn, this box and the subsequent boxes were empty. It was only in the calculations carried out from right to left and presented by the arrows leading down, that the numbers that were written into these, initially empty, boxes were determined.

Figure 7. Correct solution supported by a reduced graph – work by Catherine J.



Text translation

half and 1 half and 1 half and 3

1 and double 1 and double 3 and double

3 · 2 = 6 -> That's how much he gave to his third daughter

14 – 6 = 8 -> That's how much he gave to his second daughter

In 14 papers, the correct solution was only obtained by reductive reasoning without drawing or graphing. In all these works, the reasoning was as follows:

To the youngest he gave half and another 3, so the half he gave to his daughter was 3, and before he gave her there were 6 pearls (3 + 3 = 6).
After giving the middle daughter half of the pearls and one, there were 6 left. If he hadn't given the one, there would have been 7 left in the casket and that was half of all, so by the time he gifted the middle daughter, there were 2 · 7 = 14 pearls.
When he gave the eldest half and one, there were 14 left. If he had not given the one, there would have been 15 (14 + 1 = 15), half of all.
Therefore, at the beginning there were 2 · 15 = 30.
The eldest got 30 : 2 + 1 = 16, there were 30 – 16 = 14 left.
The average got 14 : 2 + 1 = 8, left 14 – 8 = 6,
The youngest got 6 : 2 + 3 = 6 was 6 – 6 = 0.

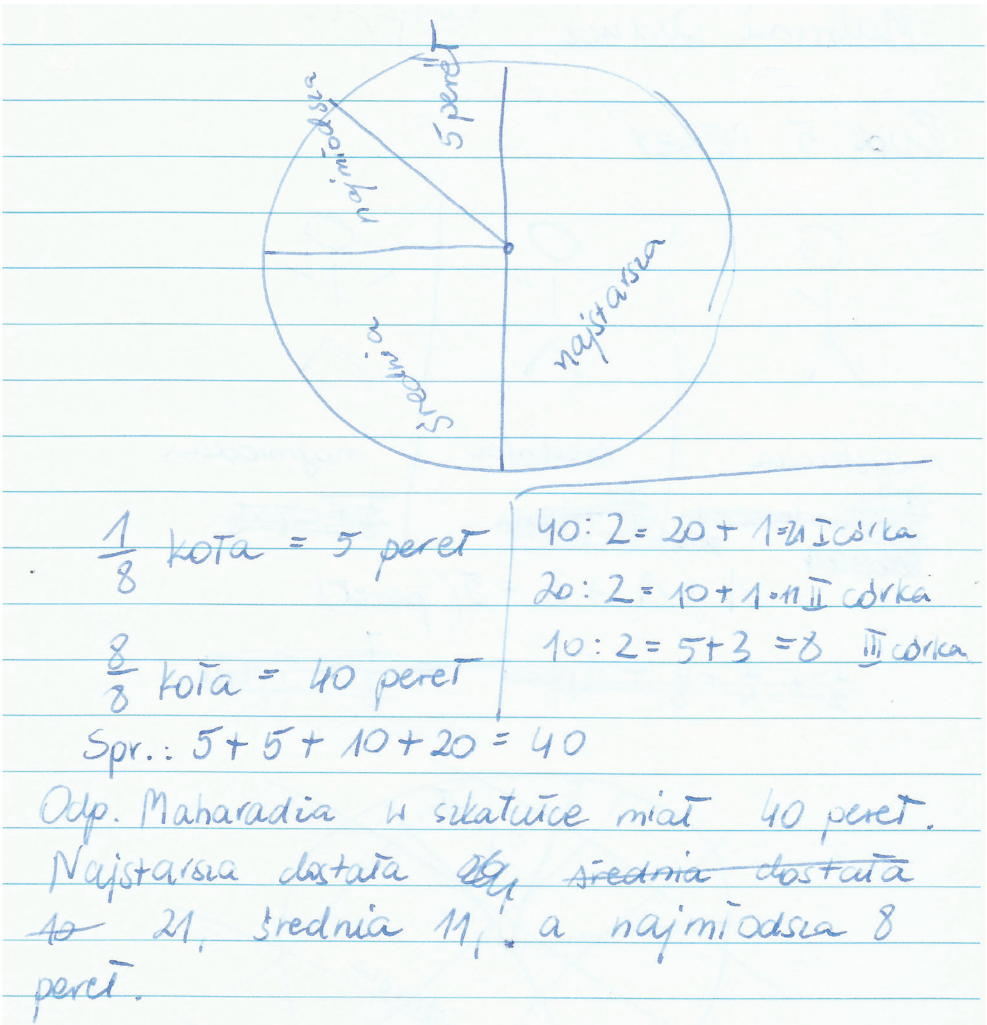
Students in the discussed group demonstrated the ability to plan concrete, imaginary and abstract operations in accordance with Z. Krygowska's theory of active-based teaching of mathematics (1977, pp. 81–128). Numerous examples of such operations from different areas of mathematics and for different levels of education are presented by H. Siwek (1998). In some of the cited student solutions there is clearly a combination of two operations, namely – imaginary and abstract. Students represent the pearl numbers in the task by drawing beads (individually or in a casket) or draw a segment (strip) of length according to the data in the task (measurement aspect) – which can be interpreted as a reference to concrete operations, to then move on to operations on numbers, sometimes using an arrow graph. Such solutions using imaginary and abstract operations are more numerous than those containing only abstract operations – i.e., based on the verbal description of the activity, the notation and the execution of actions on numbers. This indicates that in the adult group studied, there are few people who used only abstract operations in solving the task – the highest from the point of view of the action method (14 out of 142). It can be hypothesised that the availability of formal mathematical thinking, in students of the humanities, is at a very low level. There are very serious tasks ahead of college classes, in the basics of mathematics education and the methodology of mathematics education in grades I-III, to prepare good teachers for pre-school children and students in grades I-III. Many valuable tips on how to do this can be found in the book by M. Cackowska (1993), in which the author provides detailed scenarios for solving tasks taking into account the principles of the activity method and the principle of graded difficulty. The use of the methodical solutions proposed by the author in mathematics lessons can significantly contribute to the development of students' ability to solve text tasks, so important in the process of learning mathematics.

Interesting research on the application of the activity method in the process of solving mathematical tasks was presented by Z. Zamorska. In order to find out how different types of tasks are solved in lessons within a specific curriculum slogan and how they are implemented, the author observed lessons conducted by qualified early childhood education teachers (cf. Zamorska, 1996, p. 109). In her conclusions, the author found that in the process of solving tasks "Concrete activities predominated, especially when the instructor was informed about the purpose of the observation [...] the studied teachers implement activity-based teaching only to the extent limited to the etymological sense of the name. Often the schematisation and pre-mathematisation did not proceed correctly, leading to erroneous intuitions of the concepts considered" (Zamorska, 1996, p. 116).

5. Examples of Incorrect Solutions to the Pearl Task (Qualitative Analysis)

Unfortunately, not all students solved the task correctly. The prerequisite for solving the task correctly, as already mentioned, is understanding the content and being able to mathematise it. Not all students understood the content or understood it but were unable to mathematise it. They also did not know how to verify whether their ideas were good or whether the numbers they obtained met the conditions of the task. Below I present 10 examples with their analysis.

Figure 8. Incorrect solution, answer 40 – Work by Malwina D.



Text translation

eldest [right half of the circle] / middle [bottom-left] / youngest / 5 pearls [other parts]

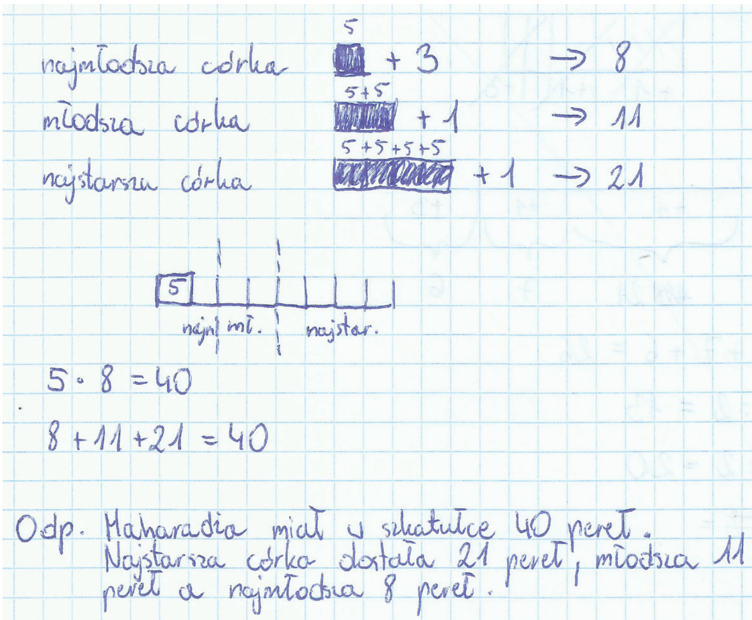
1/8 circle = 5 pearls 8/8 circle = 40 pearls	40 : 2 = 20 + 1 = 21 I daughter 20 : 2 = 10 + 1 = 11 II daughter 10 : 2 = 5 + 3 = 8 III daughter

Verification: 5 + 5 + 10 + 20 = 40
Ans. The Maharaja had 40 pearls in his casket. The eldest got 21, the middle got 11 and the youngest got 8 pearls.

In the illustration in Fig. 8, the author presents the entire pearl pool by means of a circle and in it she distinguishes half, then half of half and half of what remains. She does not take into account at all the fact that the half remaining from the first division was reduced by 1 pearl and, similarly, the half of what remains was also reduced by 1 pearl. On the basis of this erroneous drawing, the author considers that the smallest parts in the drawing are the eighth parts and additionally, completely unexpectedly and without any justification, considers that there are 5 pearls in such an eighth part. One can only guess that she arrived at this number by counting the pearls added to the daughters: 1 and 1 and 3, i.e., 5. With such an assumption, there are 40 pearls in all. But then there are already completely incomprehensible and incorrectly presented conclusions, that the 1st daughter got 21 pearls, because "40 : 2 = 20 + 1 = 21". This incorrect notation shows that the student clearly does not understand what the equals sign means. Moreover, since the 1st daughter got 21 pearls and there were 40 pearls in all, there are 19 pearls left to be further divided and it is impossible to divide them into 2 equal parts. However, this does not bother the author at all and she further divides not 19 but 20 pearls, only that she cannot have so many, which she does not see at all. In her opinion the 2nd daughter got: "20 : 2 = 10 + 1 = 11" (again, incorrect notation). Similarly, the 3rd daughter got: "10 : 2 = 5 + 3 = 8" (also incorrect notation). If we wanted to check how many pearls the Maharaja distributed to his daughters in such a case, the total: 21 + 11 + 8 is different from the sum of: 5 + 5 + 10 + 20, by which the Author makes the check. This check by the author has nothing to do with actually checking whether the resulting numbers 21, 11, and 8 meet the conditions of the task, i.e. she fails to verify her ideas.

A solution with a similar error to that in Malwina's work is also found in the work in Figure 9. It is evident from both the images and the calculations that the added pearls 3 and 1 and 1 are not initially treated as part of the divisible whole (they are as if from a different pool / are not pearls from the casket). In contrast, the pearls from the casket are presented using squares: the pearls for the youngest daughter are presented as one square, the middle daughter as 2 squares and the eldest daughter as 4 squares. Thus, starting with the youngest daughter, each successive daughter has (in squares) 2 times as many pearls as the previous one. Hence, all the pearls from the casket form 7 equal portions (squares).

Figure 9. Incorrect solution, answer 40 – work of Patricia O.



Text translation

5

Youngest daughter .... + 3 → 8

5 + 5

Younger daughter .... + 1 → 11

5 + 5 + 5 + 5

Eldest daughter .... + 1 → 21

| 5 | | | | | | | |

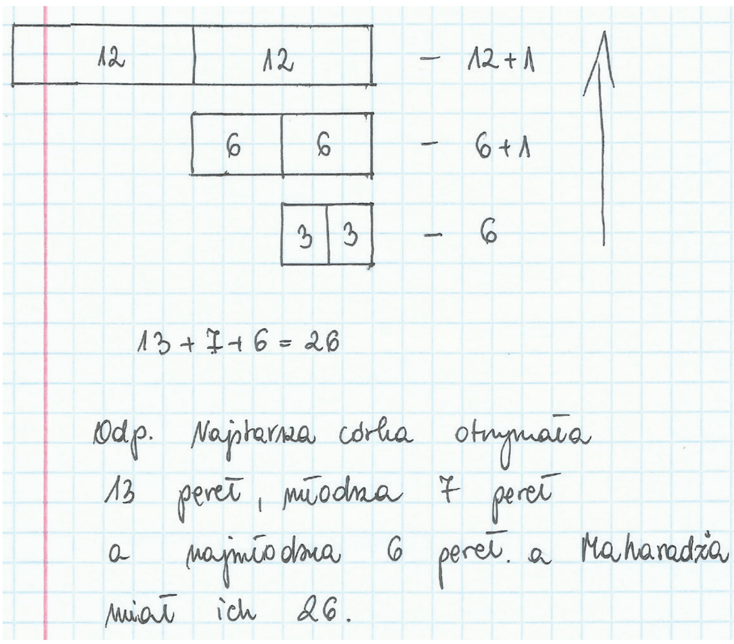
youngest | younger | eldest

Ans. The Maharaja had 40 pearls in his cascet. The eldest daughter received 21 pearls, the younger one 11, and the youngest 8 pearls.

In the next figure, a further eighth portion of pearls is added to these 7 portions shown on one line, presumably formed from these originally separately treated five pearls. The solver then recognises (it is not clear why) that there are as many pearls in each portion as in this added portion. This is evidenced by the product of $5 \cdot 8 = 40$ (although $8 \cdot 5 = 40$ would be more accurate). The result of this multiplication, i.e. 40 is considered to be the number of all pearls, and the daughters got 8 (because $5 + 3 = 8$), 11 (because $2 \cdot 5 + 1 = 11$) and 21 (because 4 · 5 + 1 = 21) respectively. In calculating how many pearls the daughters got, again the exact pearls are as if from a different pool and are not part of the divisible total. When determining the total number of pearls received by the daughters, again the "added pearls" are included in the pearls in the casket and thus the author of the work has a concordance of the sum of the distributed pearls: 8 + 11 + 21 = 40 with the number of total pearls she calculated. Although the presentation of the pearls in Figures 8 and 9 is different, the reasoning in both papers apparently follows a similar pattern and thus the results obtained are identical. And what is more interesting is that as many as 39 people reasoned analogously.

Another error in reasoning – for as many as 13 people – led to the result: there are 26 pearls in all, with the daughters receiving 13, seven and six respectively.

Figure 10. Incorrect solution, answer 26 – work by Carol A.



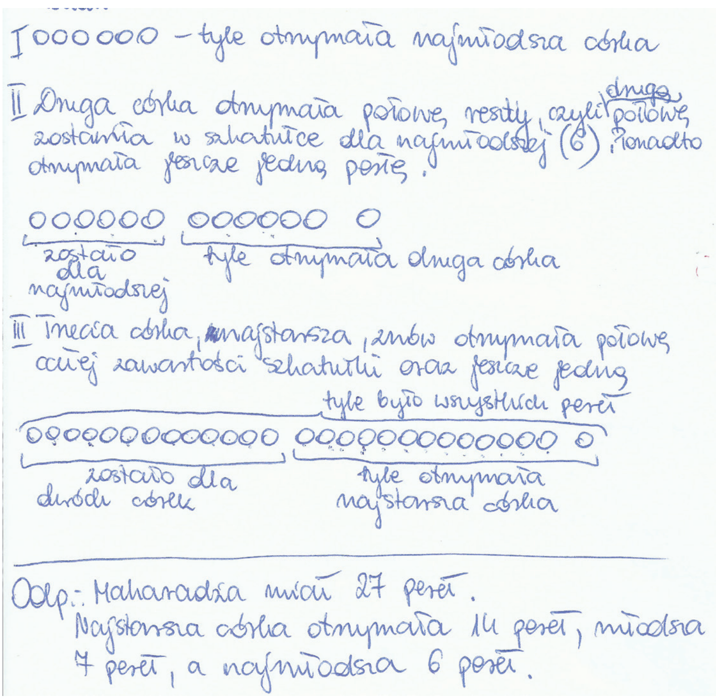
Text translation

Ans. The eldest daughter received 13 pearls, the younger daughter 7 pearls and the youngest daughter 6 pearls. The Maharaja had 26 of them.

From Figure 10 it is easy to read how the author reasoned. First she represented the whole pool of pearls with a rectangle and then divided it into 2 equal parts. Then she redrew one part and also divided it into 2 equal parts, only to redraw one part again and again divide it into 2 equal parts. She correctly concluded that the latter part (received by the youngest daughter) contained 6 pearls. She then performed the calculations according to the direction of the arrow pointing upwards. She carried out two types of calculations in parallel. In the drawing, she calculated how many pearls she thought were in the casket at successive stages of their distribution starting with the last distribution, while next to the drawing she calculated how many pearls each daughter received, starting with the youngest. Since the drawing was not a model of the situation described in the task, such a procedure could in no way be successful. The author formulated the answer and did not notice at all that, since there would be 26 pearls in all, the eldest daughter would have to get half of the 26, i.e., 13 and one more pearl, so ultimately 14 pearls, so the numbers given do not meet the conditions of the task.

Another erroneous result appearing in as many as 7 papers is 27. I show an example of such a paper in Figure 11. Here the reasoning for determining the number of pearls of the youngest and middle daughter of the maharaja is analogous to the reasoning in Figure 10.

Figure 11. Incorrect solution, answer 27 – work by Alexandra M.



Text translation

I: o o o o o o – that's how much the youngest daughter received
II: The second daughter received half the change, i.e. left the other half in the casket for the youngest (6). In addition, she received another pearl.

|o o o o o o| |o o o o o o| o|

was left for the youngest || so much the second daughter received

III: The 3rd daughter, the eldest, again received half of the entire contents of the casket and another

so many were all the pearls

|o o o o o o o o o o o o o| |o o o o o o o o o o o o o| |o|

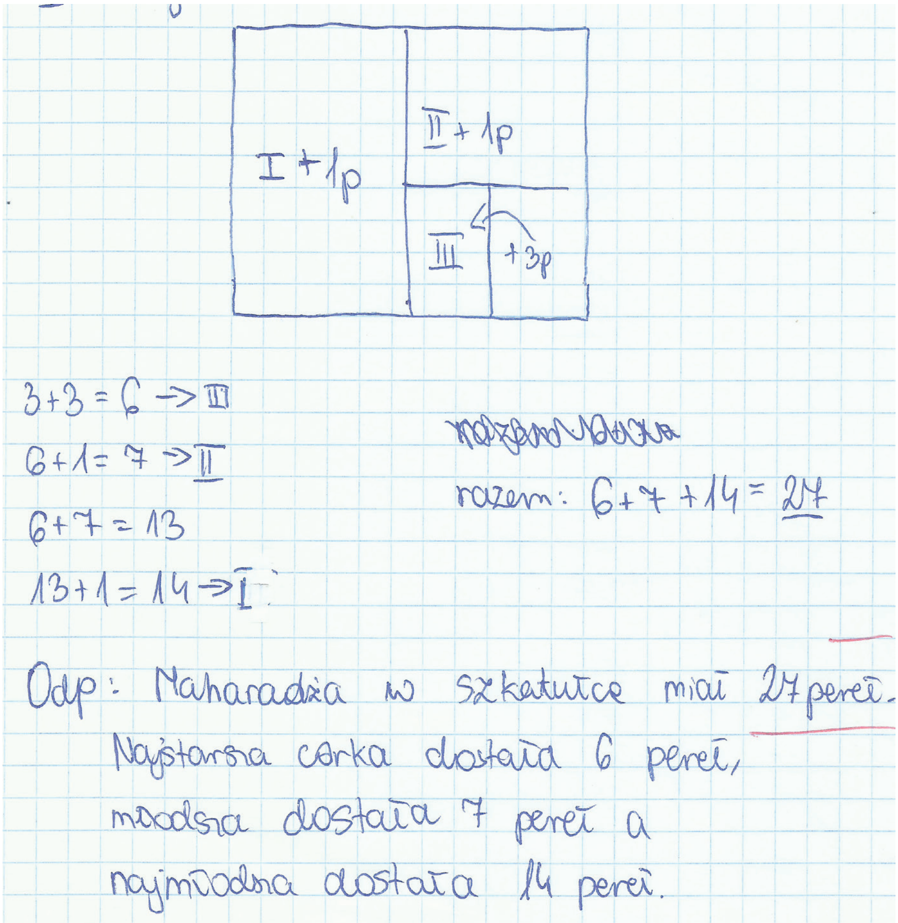
was left for two daughters || as much as the eldest daughter received

Ans. The Maharaja had 27 pearls.
The eldest daughter received 14 pearls, the younger daughter seven pearls and the youngest daughter six pearls.

The difference is in how to calculate how many pearls the eldest daughter got. The author first illustrated the pearls of the youngest (6) and middle (7) by drawing 13 circles. To the eldest, she allocated 13 and one more, i.e., 14. The total number of pearls was, in her opinion, 27, and it did not bother her at all that the number was not divisible by 2.

An identical solution, further supported by a graphical diagram in the form of a split square, is at work in Figure 12.

Figure 12. Incorrect solution, answer 27 – work by Hanna K.



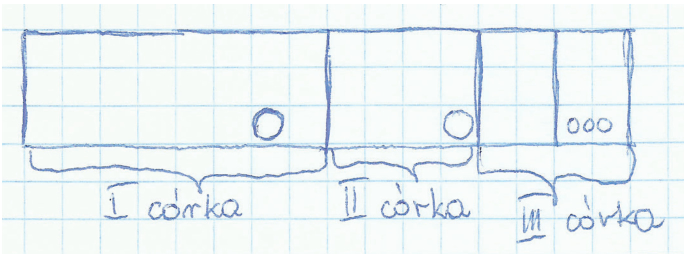
Text translation

Total 6 + 7 + 14 = 27
Ans. The Maharaja had 27 pearls in his casket.
The eldest daughter received six pearls,
the younger one got 7 pearls and the youngest one got 14 pearls.[3]

A score of 44 is repeated in 4 papers and is obtained in a manner analogous to that in the paper in Figure 13, which includes an illustration and commentary.

[3] She got the order of the daughters wrong in her answer

Figure 13. Incorrect solution, answer 44 – work by Alexandra K.



Text translation

I daughter II daughter III daughter

3 pearls are half of the third daughter's pearls
$2 \cdot 3 = 6$ number of pearls of the third daughter
$2 \cdot 6 + 1 = 13$ number of pearls of the second daughter
$2 \cdot 12 + 1 = 25$ number of pearls of the first daughter[4]
$25 + 13 + 6 = 44$ number of pearls in the casket.

In the solution method of Figure 13, the pearls received by the daughters are presented as rectangles in such a way that the width of the next one is 2 times smaller than the previous one (the younger daughter has 2 times less than the older one). The additional single pearls of the eldest and middle daughter are drawn in the first and second rectangle. In the last, fourth rectangle, 3 pearls are drawn and the number of pearls of the youngest is correctly determined ($2 \cdot 3 = 6$). Unfortunately, further reasoning contains similar errors to the reasoning already presented and in this case leads to the conclusion that there were 44 pearls in all.

A similar understanding of catching (doubling) pearls, albeit with a different calculation, is found in the work of Alexandra J. shown in Figure 14. The author correctly determined the number of pearls of the youngest daughter ($3 + 3 = 6$) and, using this, doubled 6, obtaining 12, and doubled 12, obtaining 24.

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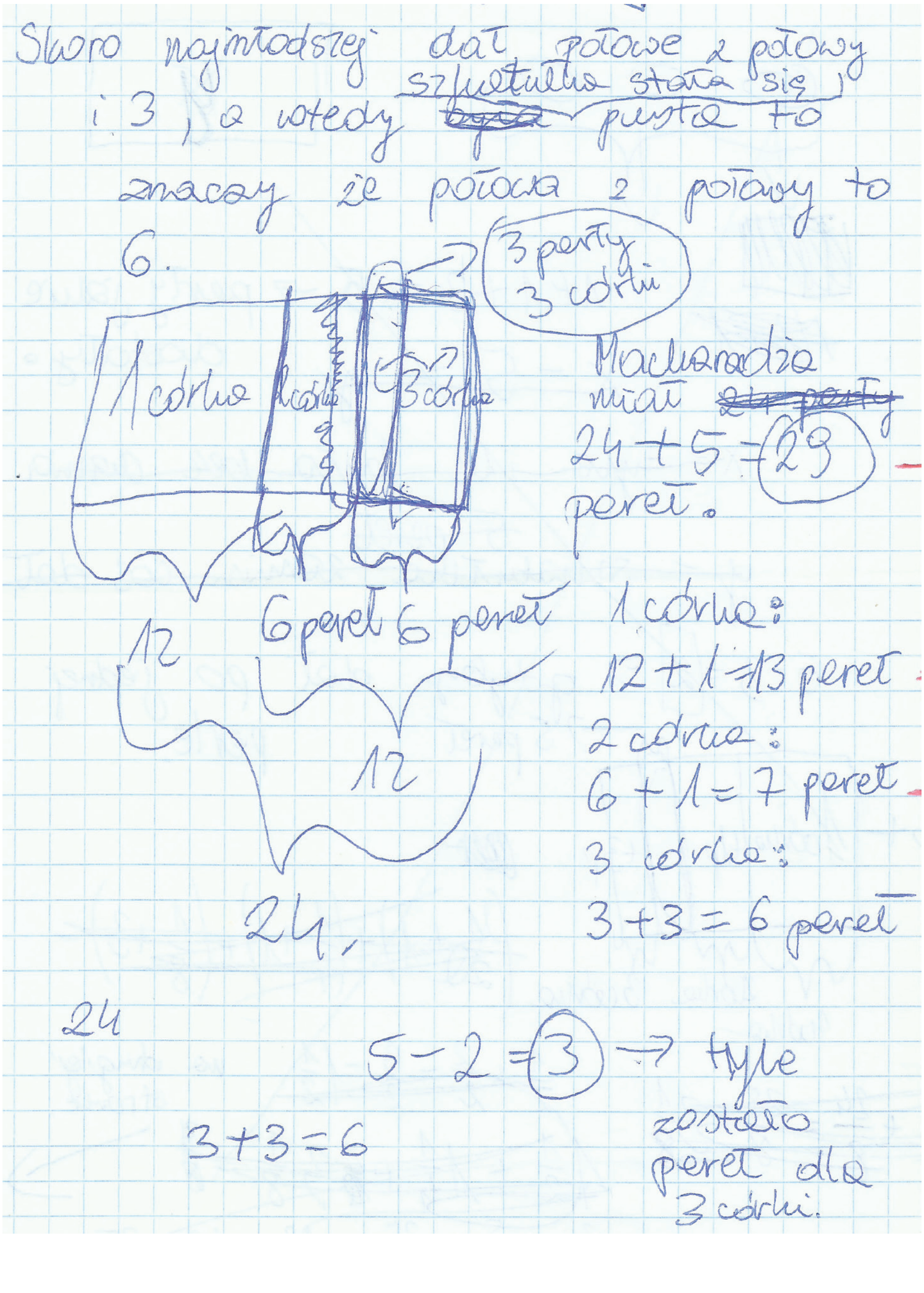
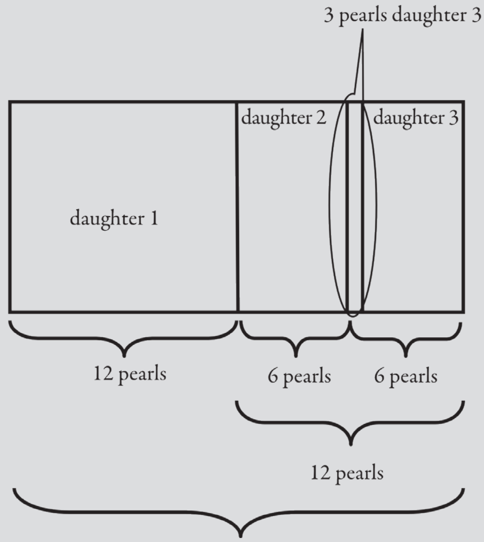
4 According to the reasoning presented, the fourth line of the notation should be: *25 number of pearls of the ~~third~~ first daughter*

Figure 14. Incorrect solution, answer 29 – work by Alexandra J.



Text translation

Since he gave the youngest half of half and 3, and then the casket became empty, this means that half of half is 6.

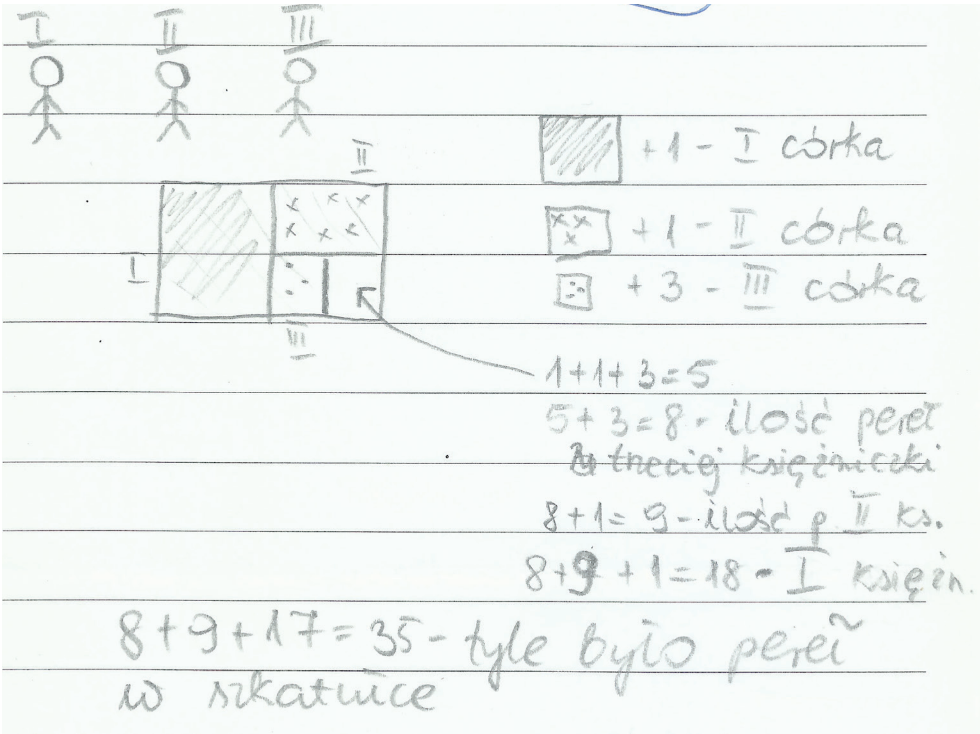


The Maharaja had	24 + 5 = 29 pearls.
1 daughter	12 + 1 = 13 pearls
2 daughter	6 + 1 = 7 pearls
3 daughter	3 + 3 = 6 pearls
	5 – 2 = 3 → that's how many pearls are left for the 3rd daughter.

She then added 5 to 24, this number 5 probably being the total number of pearls added (1 + 1 + 3 = 5), and considered the number of 29 thus obtained to be the number of all pearls. She determined the numbers of pearls received by her daughters as follows: the eldest 12 + 1 = 13, the middle 6 + 1 = 7 and the youngest 3 + 3 = 6, which she had already determined. It did not bother her at all that 29 is not divisible by 2, nor that the total number of pearls her daughters received was 26 (13 + 7 + 6 = 26), which is 3 less than the number 29 of all pearls.

The author of the next paper (Figure 15.), like her predecessors, incorrectly understands the situation described in the task. She also assumes that each younger sister gets 2 times less than the older sister, which she indicates in the figure: she catches three times, each time she catches the part that is still to be distributed, without taking into account that each new pot should be 1 less than what is shown in the figure.

Figure 15. Incorrect solution, answer 35 – work by Magdalena B.



Text translation

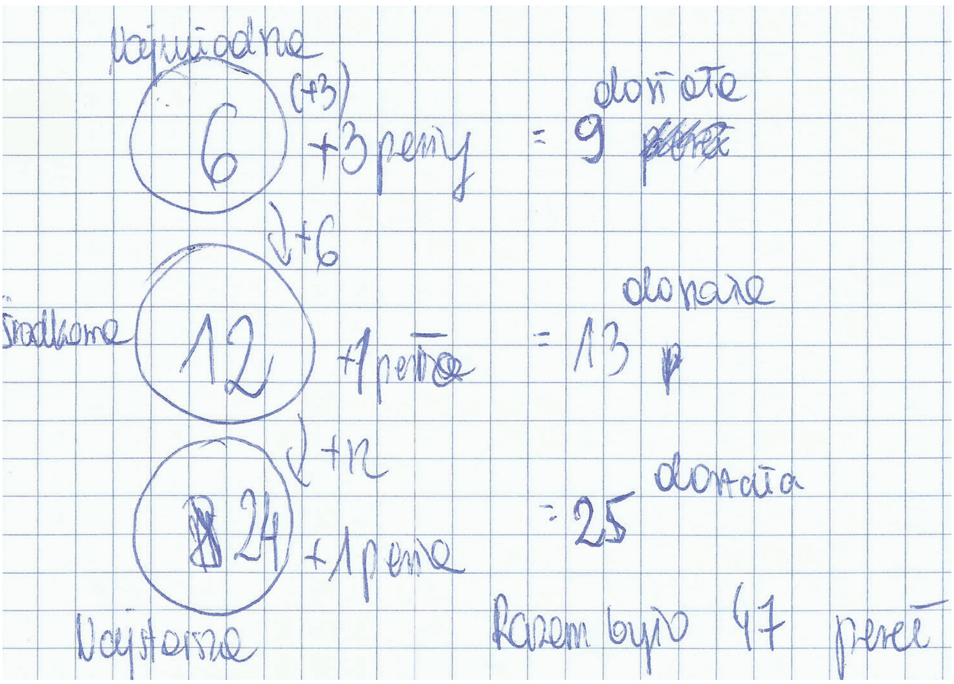
5 + 3 = 8 number of pearls 3rd daughter
8 + 1 = 9 number of pearls 2nd daughter
8 + 9 + 1 = 18 number of pearls 1st daughter
8 + 9 + 18 = 35 that's how many pearls were in the casket.

Thus, in the drawing of the rectangle, he distinguishes successively ½, ¼ and 1/8 of it as portions of pearls for the 1st, 2nd, and 3rd daughters respectively. He identifies the remaining 1/8 of the undivided part of the rectangle (1/8 of the whole pool) with the individual pearls added to the portions presented earlier and recognises that there are 5 pearls in this part (because 1 + 1 + 3 = 5). He then determines the number of pearls of each daughter starting with the youngest. He recognises that the youngest has 5 + 3, i.e., 8 pearls. He does not see any contradiction in the fact that the parts presented in the drawing are equal, so there cannot be 5 pearls in one of them and 3 in the other. He then recognises that since there are a total of 8 pearls in the lower right quadrant, there are also 8 in the upper right quadrant, and since the second daughter got one additional pearl, he calculates that she got

8 + 1, i.e., 9 pearls. She then determines that there are a total of 8 + 9 pearls in the “right half”, so presumably there must be the same number in the left half. Since the eldest daughter received an additional pearl, she must have received 8 + 9 + 1, i.e., 18 pearls. She determines the total number of pearls by adding the pearls distributed to the individual daughters 8 + 9 + 18, so she concludes that there were 35 pearls, and is not surprised by this odd number.

In the work of Fig. 16, the calculations are carried out in two ways. In the vertical arrangement, numbers are written on the graph such that each successive number is 2 times larger than the previous one: 6, 12, 24.

Figure 16. Incorrect solution, answer 47 – work by Kamila G.



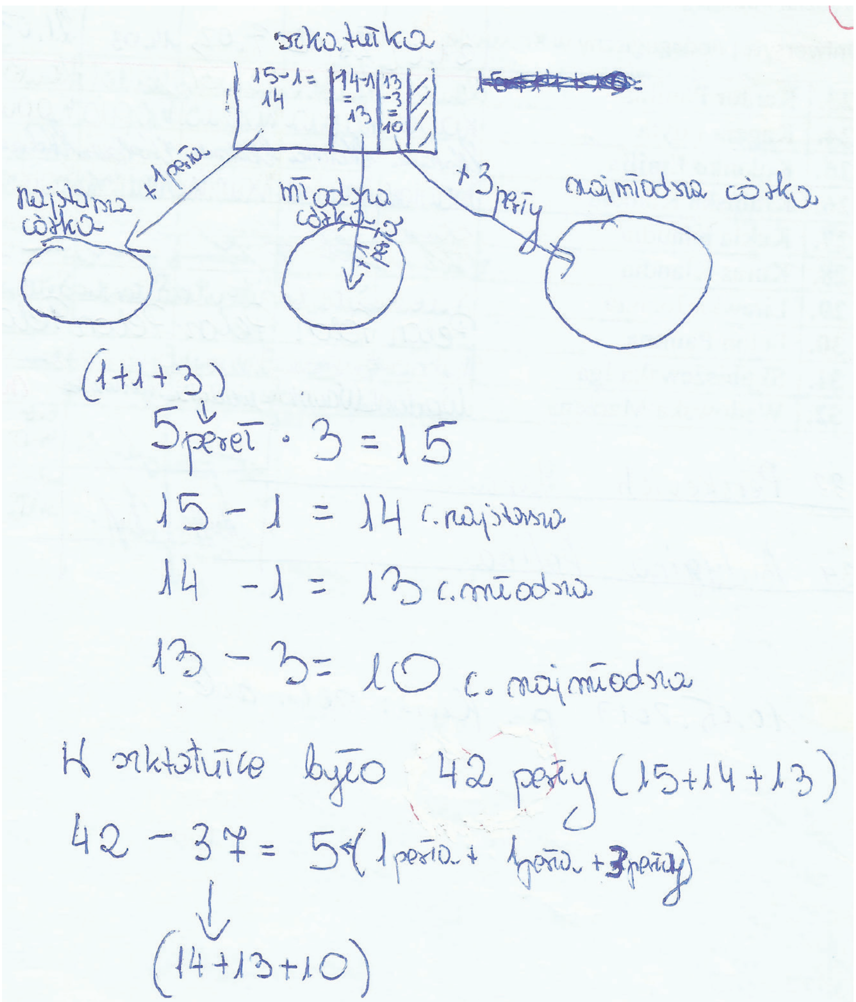
Text translation

The eldest 6 + 3 pearls = 9 got
↓+6
Average 12 + 1 pearl = 13 got
↓+12
The youngest 24 + 1 pearl = 25 got
Altogether there were 47 pearls

The numbers so written on the graph are part of the horizontal calculation used to determine how many pearls each daughter got. Starting from the top, the author recognises that the youngest daughter got 6 (a good number) only unfortunately she increases it by 3 more (6 + 3 = 9). She then assigns the middle one 2 times 6 and 1 more (12 + 1 = 13) and the eldest 2 times 12 and 1 more (24 + 1 = 25). Thus the total number of pearls is 47 (because 9 + 13 + 25 = 47). The pearls 3 and 1 and 1 added by her are apparently not included in the total pearl pool during the calculation. Only when determining the total number of pearls received by the daughters are they included in this pool. Clearly, no good solution can be found in this way.

There are some very twisted calculations at work in Figure 17. They are downright baffling. It is even difficult to see any idea in it.

Figure 17. Incorrect solution, answer 42 – work by Beata Ż.



Text translation

[The left branch of the graph]: + 1 pearl, the eldest daughter

[The middle branch of the graph]: + 1 pearl, the younger daughter

[The right branch of the graph]: + 3 pearls, the youngest daughter

(1 + 1+ 3) → 5 pearls · 3 = 15
15 – 1 = 14 eldest
14 – 1 = 13 younger
13 – 3 = 10 youngest
There were 42 pearls in the casket (15 + 14 + 13)
42 – 37 = 5 (1 pearl + 1 pearl + 3 pearls)

The author first determines the number of pearls added to her daughters: 1 + 1 + 3 and then triples this number. The number 15 obtained in this way is presumably taken as half of all the pearls (only then there would have to be 30 pearls) and it is not clear why she reduces it by 1 (15 – 1 = 14) and concludes that the eldest has received 14 pearls. The younger one 13 (14 – 1 = 13) and the youngest 10 (13 – 3 = 10). The total number of pearls, according to the author, is 42, because as she records: 15 + 14 + 13 = 42. The fact that the daughters received a total of 37 pearls (after all, 14 + 13 + 10 = 37) is not in doubt, because, as the author writes, 42 – 37 = 5, which is the total (1 + 1 + 3) additionally received by all the daughters.

This is an example of 'number juggling' (counting for the sake of counting, without reflecting on whether it makes any sense).

The second group – much more numerous than the previously discussed group with correct solutions – includes incorrect solutions (93 – excluding 14 partially correct solutions). Among the 10 examples of incorrect solutions, there are different types of errors and each has been analysed in detail, together with a description of the possible consequences of the chosen reasoning and calculation. The analysis of these examples leads to the conclusion that the students (after 15 years of schooling) do not apply the well-established methodology of solving text tasks in the didactics of mathematics of Polya (1993). The four stages of solving text tasks, the author presents and extensively analyses in his book "How to solve it". In a shorter version, these stages are discussed in an academic textbook for students, future teachers of mathematics, by Siwek (2005, pp. 116–131). The next stages are: 1. understanding the content (concepts and situation), 2. planning the solution (operations to be carried out sequentially), 3. writing down the solution (calculations, constructions), 4. "hindsight" (checking whether the results meet the conditions of the task).

As can be seen from the attached examples of incorrect solutions, in almost all of the female students' work, an error was made at the outset related to step 1 of Understanding the task. The students mostly only saw the "dividing in half" of the set of pearls; then they attempted to do something about the "addition of individual pearls". They did not know how to analyse the data and the conditions of the task and determine the subsequent operations: I divide the set of pearls in half and add 1; I divide the remaining remainder in half and add 1; I divide the remaining remainder in half again and add... now 3. And the difficult for many was to analyse the given information and conclude that the youngest daughter has 3 + 3 = 6 pearls and that the casket is empty – there are 0 pearls in it. This deciphering of the 'hidden' data gave a chance to solve the task correctly. The analysis of incorrect solutions shows how important it is in practice to know the methodology of solving text tasks and to be able to apply the activity method in solving mathematical problems.

5. Conclusions

The task was solved by 142 female students. The data presented in Table 1 shows that only 33 people solved the task correctly by reasoning. Of these, 16 were supported by an apt drawing and 3 of them used an arrow graph. In 14 cases, the solution contains pure reductive reasoning without any additional support. Unfortunately, as many as 107 (14 and 93) people failed to solve it. Two people found the answer to the task, but unfortunately obtained it using an inefficient trial-and-error method.

It is very encouraging that almost all students (except one) attempted to solve the task. Furthermore, almost all of them (except three who used fractions or equations) used a method available to early childhood education students in their solution. It is also important to note that only two used the trial and error method, which, although it may lead to a good result, is not a method that can be generalised and used in solving an analogous task. It is encouraging that the majority of respondents attempted to illustrate the relationships in the task, as the drawing is perceived globally (not sequentially like a written text), which makes it much easier to find a solution and sometimes the answer can be read directly from the drawing. Unfortunately, not all drawings were accurate and therefore could not support the solution.

The fact that only 33 students correctly solved the task by performing reductive reasoning shows that reduction is not an easy way of reasoning and, as can be seen, not everyone can apply it.

Table 1. Summary results of solving the pearl task

Method of solution / answers		Number of works	together	total	percentage
Correct solution	Correct drawing	16	33	35	25%
	Reasoning without drawing	14			
	Sagittal graph	3			
	Trial and error method	2	2		
Partially correct	Correct reasoning but wrong results due to calculation errors	14	14	14	10%
Wrong solutions	Answer: 40 (21, 11, 8)	39	63	93	65%
	Answer: 26 (13, 7, 6)	13			
	Answer: 27 (14. 7, 6)	7			
	Answer 44 (25, 13, 6)	4			
	Other answers e.g. 8, 12, 24, 25, 28, 29, 32, 34, 35, 38, 42, 47, 50	24	29		
	Equation written down but without solution	1			
	Incorrect fractions attempt	2			
	Strange calculations (other than above)	2			
No solution	No answer	1	1		
	Total	142	142	142	100%

Unfortunately, as many as 109 people failed to solve the task correctly. They either could not read the task with comprehension or could not mathematise the situation described in the task. They were unable to go beyond the learned patterns. They forcefully tried to apply any of the calculation schemes they were familiar with, hence the high number of "mindless calculations", which were not accompanied by any reflection. They also lacked the ability to check whether the numbers they obtained met the conditions of the task. Had they carried out such a check, they would probably have noticed that something was wrong and perhaps they would have spotted their mistakes and corrected them. Unfortunately, they did not do so.

The result: as many as 77% of female students failed the task, prompts reflection on both the effectiveness of mathematics education and the way in which candidates for the teaching profession are recruited. After all, the aim is for current students and future teachers to be able to develop logical, reflective and critical thinking in their students. That they teach them to take a variety of steps to make a sound assessment of the factual and objective state of affairs, that they teach them to take a variety of steps to exclude the over-hasty and hasty acceptance of unproven information or the results of uncertain solutions. After all, the key to success in education is the formation of skills of correct reasoning/inference, i.e., the formulation of accurate judgements on the basis of known premises[5]. After all, what we want is for schools to be filled with mature, thinking people, because only such people are able to meet all the challenges posed by the modern world and the future world, which we do not yet know, but in which the present students will live and work. Is it possible to achieve such a goal? Probably yes, provided the teacher is himself a logical and critical thinker. Rather, it is not possible to teach others what one does not know oneself.

5 This involves both deductive and reductive reasoning.

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Mirosława Sajka & Sławomir Przybyło
University of the National Education Commission

Chapter 14

DIAGNOSIS OF SCHOOL MATHEMATICS KNOWLEDGE AND SKILLS OF STUDENTS ENTERING UNIVERSITY TO BECOME MATHEMATICS TEACHERS

Summary: This work presents the results of a competency test administered to students beginning their studies in mathematics education. This test was held on the first day of the academic year for all first-year students starting their studies in mathematics at the undergraduate level. The intention was to diagnose their mathematics learning needs at the level between secondary school and university studies. The results raised concerns, as they are surprisingly low, but they also show the mathematical content that needs to be addressed in further mathematical university education of the research participants. They also present the difficulties the respondents encountered when solving tasks. The tasks of the test were prepared in a way as to test the same skills that are assessed in the secondary school state exam, called the matura, at the basic level of mathematics, which is compulsory for all students in Poland. The solution to each task was not numerically complex – the tasks tested the understanding of concepts rather than mastery of procedures. The results of the study showed a significant correlation with the results of the mathematics matura exam. Moreover, the results make it possible to diagnose gaps in the mathematical knowledge and skills of those entering mathematics studies and make lecturers and students aware of the reasons behind some of the misconceptions and reasoning. In the longer term, they will provide a basis for formulating remedial measures – the effective design of a course to fill these gaps, aimed at reducing the revealed issues. In the chapter, we present the general results of research on a sample of 78 people and make a brief quantitative summary and qualitative analysis of the answers to four tasks – with the best and the weakest results. The research reveals that at this level of mathematical education, it is necessary to emphasise careful reading of mathematical texts, to be aware of the existence and the operation of Systems 1 and 2 of fast and slow thinking, as well as to implement self-control, develop the habit of checking answers, implement the use of reductive reasoning in solving tasks, implement the methodology of justification and refutation of statements, and overcome psychological barriers associated with tasks that "seem" difficult. Furthermore, numerous misconceptions have been highlighted, for example those related to the properties of functions, and care should therefore be taken to correctly shape "concept images" in students.

Keywords: mathematics knowledge and skills, competency test, first-year mathematics education students, Dual-Process Theory, reductive reasoning, misconceptions, concept image, functions, equation.

1. Introduction

The motivation for undertaking the research described in this chapter is the need to effectively educate future mathematics teachers. In this context, we attempted an additional diagnosis of the knowledge of students at the beginning of their mathematics education, aiming to become teachers, in order to further address their educational needs in terms of mathematical knowledge and skills and adapt the university education process to them.

Undertaking this study also indirectly intended to update lecturers on the level of knowledge of secondary school students entering mathematics courses. There is a certain rigidity in university education, in that some lecturers expect the level of knowledge of students to be at the same level as it was a few or even a dozen years ago. In the meantime, many factors make the profile of the first-year student change dynamically. In this paper, we do not analyse these factors, but provide a selection of them in the summary of the discussion section.

2. Theoretical Background

The theoretical framework relates to the analysis of the results of the individual tasks presented in this chapter.

2.1. Dual-Process Theory

In our analysis, we draw attention to the fact that rapid, intuitive thinking is a hindrance when solving the tasks we discuss, which is why the **Dual-Process Theory** is being considered part of the theoretical background. However, the "Dual-Process Theory in cognitive psychology and mathematics education" is presented in Chapter 2 of this book (Sajka & Rosiek, 2025), so here we focus on the ability to think reductively (abductively, "from the end"). This type of reasoning is required to solve the discussed tasks. Moreover, in subsection 3, we present the theoretical foundations and principles of the construction of the secondary school exit exam (matura) in Poland, to which the design of our research tool refers.

2.2. Reductive Reasoning

From the point of view of implementing mathematics, the main type of reasoning is deduction. However, it is often difficult to use pure deduction or even local deduction in the context of school mathematics (Konior, 1989). Even when proving theorems and solving tasks, we often use reasoning that is in some sense inverse, starting to solve the task by analysing the theorem's thesis, or solving the task from the final state provided in the text. We then use reasoning that is referred to in the literature as: reductive, abductive, or "from the end to the beginning".

We can understand reductive reasoning classically (Krygowska et al., 1957) in the context of justifying the reasoning scheme $Z \Longrightarrow T$:

We look for such sentences $T_1, T_2, T_3, \ldots, T_n$, to show successive implications: $T_1 \Longrightarrow T$ (given the assumptions and previously proved theorems), $T_2 \Longrightarrow T_1$, $T_3 \Longrightarrow T_2$, etc. $T_n \Longrightarrow T_{n-1}$, i.e.:

$$T_n \Longrightarrow T_{n-1} \Longrightarrow \ldots \Longrightarrow T_4 \Longrightarrow T_3 \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T.$$

If T_n is a assumption or one of the theorems on which reasoning is already allowed, then obviously the given theorem $Z \Longrightarrow T$ is true (Krygowska et al., 1957, p. 100).

Therefore, in reductive reasoning in the classical view, we are looking for sufficient conditions for a claim T and we want to show that they consist of the assumptions of the theorem and the previously justified sentences. In this context, as part of teaching mathematics, it is a reliable method for proving theorems or solving tasks.

However, reduction can also be understood differently. For example, Bocheński (1992) describes reduction as reverse reasoning to deduction, of the type: $Z \Longrightarrow T$ *and T occurs, so Z occurs*. In this view, reductive reasoning is unreliable and verification is necessary. Bocheński adds that we can introduce types of reduction in two ways, for example: progressive and regressive reduction, and inductive and non-inductive reduction. When performing a progressive reduction, one starts from an as yet unknown predecessor and leads the reasoning to a known and ascertainable successor. The opposite is the case with regressive reduction, where one starts from a known successor and leads to an unknown predecessor. The distinction between inductive and non-inductive reduction is based on the type of predecessor – if it is a generalisation of the successor, then this type of reduction is called inductive; if this is not the case, then we speak of non-inductive reduction (Bocheński, 1954; 1992).

Peirce (1932; 1935) introduced the term "abduction", which is described as: *to start with an observation B and then infer A from B and* $A \Longrightarrow B$ (Peirce, 1932). Abduction was therefore defined analogously to regressive reduction according to Bocheński (1992), so in this view, it is fallacious reasoning.

However, *The Stanford Encyclopedia of Philosophy* points out that the term abduction exists in philosophical literature in two related, but different senses:

> In both senses, the term refers to some form of explanatory reasoning. However, in the historically first sense, it refers to the place of explanatory reasoning in generating hypotheses, while in the sense in which it is used most frequently in the modern literature it refers to the place of explanatory reasoning in justifying hypotheses. In the latter sense, abduction is also often called "Inference to the Best Explanation" (Douven, 2017, Abduction, *Stanford Encyclopedia of Philosophy*).

In both approaches, abduction is a kind of reasoning that begins with an observation of a situation, followed by attempts to explain that observation.

Research in mathematics education has explored the role of abductive reasoning in the context of making assumptions and proving. A review of the literature on this topic was provided by Komatsu and Jones (2022), and their study contributed to the literature by using abduction to analyse the process of discovery and to deal with refutations of hypotheses in the form of providing counter-examples.

Another terminology in the context of teaching mathematics is proposed by Pólya (1975), who refers to the deductive method that moves from beginning to end as the "direct method", and the reductive method from end to beginning as "backward reasoning" or "backward problem-solving", or the analytical method. He emphasises that, in the direct method, we draw logical conclusions from assumptions and data, each step being necessary to reach a conclusion, while in the reductive method, we look for sufficient conditions to justify the conclusion. Pólya also stresses that in reasoning, we use the so-called "mixed method", which combines elements of inductive and deductive reasoning. He describes this method as working alternately from the beginning and the end of a problem, a technique he calls the "method of alternate movement". Pólya emphasises that this method allows for a more flexible approach to problem-solving by integrating different strategies and perspectives.

In this current discussion, we either use the term "reduction" in its classical sense (Krygowska et al., 1957) or the term "backward reasoning" (Pólya, 1975) in situations where we start by observing a given situation (a thesis or end state in a task), then justify and explain it by presenting conditions sufficient for it to occur, which also corresponds to one of the descriptions of abduction.

Applying deductive and reductive reasoning to school practice, one can cite the example of Treliński (1985, p. 27). In a task where one has to justify the formula for the area of a trapezoid (knowing the formula for the area of a triangle), one can proceed deductively by making a drawing, carrying out an auxiliary construction, justifying that the different triangles are congruent and coming to a final conclusion. When reasoning reductively, we ask: W*hat would this result from? What figure would you need to consider to have an area equal to* $\frac{a+b}{2} \cdot h$*? It would be sufficient, for example, to have a triangle with the sides of the length and height . Therefore, we construct such a triangle using the given trapezoid and obtain the auxiliary construction.* Starting "from the end" allows us to plan the next steps of reasoning and to understand their purposefulness.

Treliński provides examples of questions that guide the different methods of reasoning:

- Reasoning deductively, we ask: What is given? What do we know? How do we use this information? What is its result? What can I get from it? How do we transform the given information? Which theorems to apply?

- Using reductive reasoning, we ask: What are we looking for? What is the unknown? How do we find an unknown of this kind? What information is needed for this? What is enough to know? What does it come from? What should we know in order to obtain the unknown? (Treliński, 1985, p. 27).

Reductive reasoning is often much more convenient, as proof by deduction is not as natural. Proofs of algebraic theorems that are most convenient to carry out through reductive reasoning appear regularly in the mathematics matura exam at elementary level, e.g.,

Prove that the geometric mean of two non-negative numbers is no greater than their arithmetic mean.

Deductive proof:

$$(a > 0 \text{ and } b > 0) \Longrightarrow (\sqrt{a} > 0 \text{ and } \sqrt{b} > 0) \Longrightarrow (\sqrt{a} - \sqrt{b})^2 \geq 0 \Longrightarrow$$

$$\Longrightarrow a - 2\sqrt{a}\sqrt{b} + b \geq 0 \Longrightarrow a + b \geq 2\sqrt{ab} \Longrightarrow \frac{a+b}{2} \geq \sqrt{ab}$$

Reductive proof:

$$\frac{a+b}{2} \geq \sqrt{ab} \Longleftrightarrow a + b \geq 2\sqrt{ab} \Longleftrightarrow a + b - 2\sqrt{ab} \geq 0 \Longleftrightarrow (\sqrt{a} - \sqrt{b})^2 \geq 0$$

$$- \text{ it is always true for } a, b > 0$$

Reductive reasoning is not only about proof, but can and should be used to solve various tasks.

3. On the Core Curriculum and the State *Matura* Examination in Poland

The matura exam in mathematics is a written state exam, taken at two levels: basic (elementary) and advanced (extended). In our publication, we will focus only on the basic level; moreover, the information provided below refers to the type of exam that was taken by the participants of the analysed competency test. Since 2010, this exam has been compulsory for all secondary and technical school graduates. The students who took part in the examination filled out a worksheet, with a time limit of 170 minutes (this has now been extended to three hours). A score of 30% was required to pass the examination.

This exam tested knowledge in the areas described by the national education core curriculum in mathematics for secondary schools at the basic level (MEiN, 2008), namely, the following topics were defined (applicable until 2021):

a. Real numbers,
b. Algebraic expressions,
c. Equations and inequalities,
d. Functions,
e. Sequences,
f. Trigonometry,
g. Planimetry,
h. Geometry in the Cartesian plane,
i. Stereometry,
j. Elements of descriptive statistics. Probability theory and combinatorics.

The content that is included in these sections has been revised several times in the almost 20 years of the matura exam, but these have not been major changes.

In the 2010–2020 matura exams, the points were split approximately in half between closed and open-ended tasks. The only type of closed tasks were multiple choice tasks (A, B, C, D) with only one correct answer. Many of these tasks can be solved by methods other than the method used for solving open tasks – for example by elimination of incorrect answers or performing substitution, which requires different skills than solving open-ended tasks. What is more, the open tasks often remain similar over the years, for example, the task of solving a quadratic inequality has been repeated at every exam for many years. The consequence of this is that even a student with only average proficiency in mathematics is able to achieve a result of more than 80 or even 90% in this exam.

4. Methodology

4.1. Purpose of the Study

The aim of the study was to diagnose the learning needs in the context of mathematical content and the proficiency in school mathematics of students entering mathematics studies to become teachers. The following research questions were posed:

Q1. What is the knowledge and skills at the basic level of secondary school mathematics of first-year mathematics education students?

Q2. What misconceptions in the understanding of school mathematics can be distinguished among the first-year mathematics education students?

4.2. Method and Research Tool

The research method was the students' individual paper-and-pencil answers to a *Research Worksheet,* also called a *competency test,* constructed in order to answer the research questions. The level of mathematical content examined and the mathematical skills needed to solve the tasks did not exceed the content obligatory to the matura exam at the basic level, and the calculations required to solve these tasks were elementary. On the other hand, the tasks were not typical school tasks and they did not require solely a schematic solution nor algorithm, but necessitated critical thinking, often requiring a change in the imposed strategy. The examples of such tasks are described further, in the analysis of task descriptions.

The *Research Worksheet* contains tasks that were previously used as research tools and proven in the role of diagnostic tools (Tasks: 1, 6, 10, 11, 16), while the rest are sourced from task collections at secondary school level or are the authors' own modifications. The worksheet consists of 17 tasks.

Each task was designed to answer the research question Q1 in relation to specific content defined in accordance with the requirements for the basic level matura exam in the basic range of content defined for the national mathematics core curriculum. In addition, due to the timing of the pandemic and the reduction of the syllabus and content range for the matura exam, some content appearing in the core curriculum, but not applicable to the 2021 matura, was also excluded from the study. These contents included, in particular, issues related to the application of acquired skills in a practical context, sketching graphs of exponential functions and inverse proportionality, angles in a prism, cross-sections of solids, and descriptive statistics (MEiN, 2020).

The research diagnosed the knowledge and skills related to the content assessed in the matura exam, as presented in subchapter 3. Table 1 presents the content labelled (a) to (j) tested in the individual, consecutive tasks of the *Research Worksheet*.

Table 1. Content of worksheet tasks

Task	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Content	a	j	e	b	a	a	c	c	f	d	d	g	g	h	c	d	i

The written output of the students' work was then analysed quantitatively and qualitatively.

4.3. Participants and Research Procedure

The participants were students entering mathematics education studies at a university engaged in training future mathematics teachers, most of whom had chosen the teaching specialisation.

A total of 78 people took part in the research. The study was announced several weeks in advance and took place at the inauguration meeting for the academic year, on the first day. The students wrote their answers to tasks printed on A4 sheets, in a large congress hall, at separate tables. One person was visually impaired and was given a sheet with adaptations to suit her needs. The duration of the study was limited to 75 minutes. Participants were informed prior to the research that the study was designed to diagnose their educational needs and that the results would be discussed and analysed at a course entitled "Fundamentals of Higher Mathematics", created to repeat content that is considered the basis of higher mathematics and to compensate for the deficiencies in secondary school mathematics. The participants were also informed that they would be graded for this study, but would be notified of their score (the number of points they obtained).

5. Results and Their Analysis

As the scope of the research is very broad, in this chapter we focus only on selected aspects of the analysis of the results.

5.1. Presentation and Analysis of General Results

5.1.1. General Results and Task Difficulty

The general results of the study are presented in Table 3, where for each task we present the number of points possible to obtain in a particular task (1 or 2) and its difficulty (relative percentage score), which is defined according to Niemierko (1999) as the quotient of the sum of points obtained for a task or set of tasks by the sum of points possible to obtain for this task or set of tasks (Table 2). We consider this indicator important, as it also allows to assess the degree of difficulty of both the mathematics matura exam in Poland as a whole as well as the examination sheets of particular editions of the examand its tasks. In the case of 1-point tasks, this indicator is the same as the so-called success rate.

Table 2. Difficulty of task according to Niemierko (1999)

Difficulty	0,00 – 0,19	0,20 – 0,49	0,50 – 0,69	0,70 – 0,89	0,90 – 1,00
Worksheet/Task	Very difficult	Difficult	Moderately difficult	Easy	Very easy

Table 3. Difficulty particular tasks of the test

Task	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	Σ
Points	1	1	1	2	2	2	2	2	2	1	1	1	1	2	2	1	2	26
Difficulty	0.36	0.26	0.26	0.10	0.08	0.29	0.28	0.28	0.24	0.05	0.08	0.19	0.14	0.24	0.04	0.38	0.17	-

The difficulty levels of the tasks are provided by rounding to two decimal places. According to the interpretation given in Table 2, the entire examination sheet ranked on the borderline between difficult and very difficult, as its difficulty level amounted to 0.20.

The modal value of the results for the entire test was 4 points, and the average percentage score obtained was around 20%.

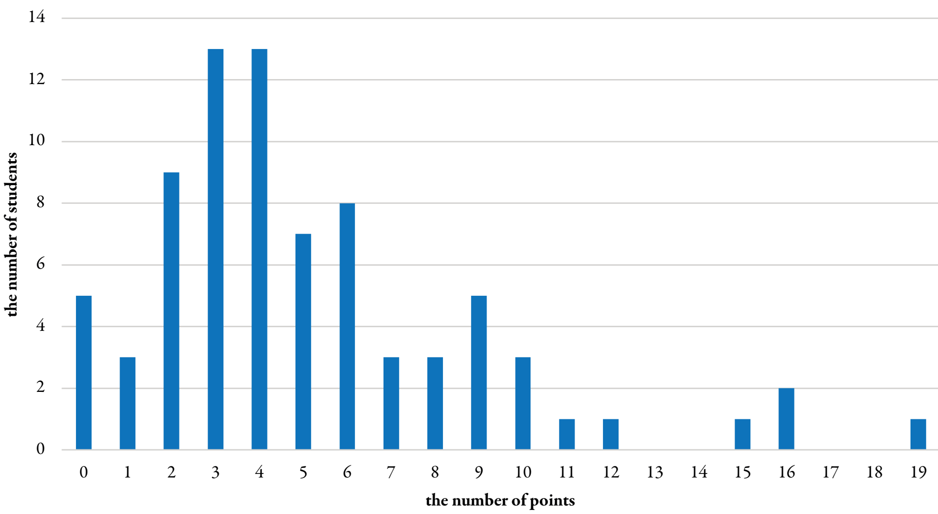
According to the results, the *Research Worksheet* did not contain moderately difficult, easy, or very easy tasks at the level of the participants, and all some tasks were considered difficult or very difficult.

Eight tasks were very difficult for the participants: 4, 5, 10, 11, 12, 13, 15, 17. The remaining tasks were ranked as difficult: 1, 2, 3, 6, 7, 8, 9, 14, 16.

Tasks no. 15 and 10 were the most difficult, and tasks no. 1 and 16 were the easiest. The results for these four extreme tasks will be analysed in the following section.

In total, 26 points could be obtained by solving all the tasks correctly. The graph in Figure 1 shows how many people obtained a particular number of points (from 0 to 19, as the best score obtained was just 19 points).

Figure 1. Students who obtained the corresponding number of points (0-19)



5.1.2. Correlation of Test Results with Mathematics Matura Results

The recruitment system for students entering mathematics education studies allowed us insight into the results of the basic mathematics matura exam for all respondents. The content of our *Research Worksheet* and the basic level matura in mathematics was the same, therefore we investigated whether there was a correlation between these results. Despite the long summer break between the exam and the study (almost 5 months) as well as the short time spent working on the tasks from our *Research Worksheet* and the difference in the type of tasks, the Pearson correlation coefficient for this data set was 0.48, indicating a moderate correlation.

5.2. Qualitative Analysis of the Solutions to the Tasks Where Respondents Were Given the Correct Answer Most Often

In this category we consider two multiple-choice tasks with one correct answer. These were Task 16 (*Stone Problem*) with a success rate of 0.38 and Task 1 (*Duckweed Problem*) – success rate 0.36, which are presented alongside the results below.

5.2.1. Duckweed Problem – Description, Results, Analysis, and Discussion

Duckweed Problem description

The content of the *Duckweed Problem* is as follows:
Task 1 (1 point). *The pond is overgrowing with duckweed*. The area covered by duckweed doubles every two days. The whole pond became overgrown in 64 days. How many days did it take for $\frac{1}{4}$ of the pond to become overgrown? Indicate the correct answer.*

A. It cannot be solved
B. After 4 days
C. After 16 days
D. After 60 days
E. Another answer …

**) Duckweed – a kind of small aquatic plant*

From a mathematical skill point of view, the *Duckweed Problem* can be solved even by primary school pupils, as it requires elementary skills: dividing even natural numbers smaller than 100 by 2, subtraction of 2, and an understanding of the fractions $\frac{1}{2}$ and $\frac{1}{4}$. However, the task presents two major difficulties.

The prototype of the task is Kahneman's lily task (Kahneman, 2011, p. 54), described as an example of activating the operation of System 1. In order to solve the task correctly, it is necessary to overcome the imposing operation of System 1, which prompts a simple, immediate and, at the same time, erroneous association: since the whole pond was overgrown after 64 days, $\frac{1}{4}$ will be overgrown after 16 days, since $\frac{1}{4} \cdot 64 = 16$ (or $64 : 4 = 16$). In addition, the school experience of forming the concept of directly proportional quantities perpetuates this intuitive approach and activates the wrong solution scheme. In order to break the activation of System 1, it is necessary to activate System 2.

The second major difficulty is methodological in nature. The task is non-standard in relation to the tasks solved at school, as it requires the activation of "backward reasoning" (Pólya, 1975) or reductive reasoning in the sense of Krygowska, Kulczycki and Straszewicz (1957).

The *Duckweed Problem* has been used in other research – among middle school students (e.g. Sajka & Rosiek, 2015a) and future teachers, and has thus been validated as a research tool using eye tracking.

When solving the task, it is convenient to consider the end state as the starting point, i.e., to start the analysis of the task from the fact that the pond becomes completely overgrown after 64 days, and to perform two steps of reasoning backwards in time using the data from the content of the task skilfully and reversing the operation. The duckweed doubles every two days (there is twice as much of it), i.e., two days earlier there was half of it, i.e., on day 62 it occupied half the area of the pond and on day 60 it occupied $\frac{1}{4}$ of the pond. This reasoning can be effectively supported by the figure.

It is worth noting that although in the proposed solution to the task we perform reasoning backwards in time and use reverse operations which can be interpreted as reductive reasoning or "backward reasoning", from a logical point of view, we use the data in the task and arrive at the solution through deduction. Reverse reasoning, on the other hand, is of the opposite nature.

The task was closed, so the respondents were all the more able to approach its solution by choosing a hypothetical answer and verifying it. In this case, the refutation or confirmation of the hypothesis followed the direction of the passage of time of the situation described in the task and was consistent with the description of the operations taking place there, that is, in this sense, "from the beginning to the end". On the other hand, from a logical point of view, this direction is reductive, because we start from the answer to the task, which we then verify, rather than starting from analysing the content of the task.

However, the task can be solved both fully deductively through elementary means as well as "from beginning to end" – both logically as well as from the situation depicted in the task in the context of time. For example, an unknown can be introduced:

x – number of days after which the pond is ¼ overgrown
+2 | $\cdot 2$
$x + 2$ – number of days after which the pond is ½ overgrown
+2 | $\cdot 2$
$x + 4$ – number of days after which the pond will be fully overgrown

Therefore $x + 4 = 64$, hence the answer is 60.

Although the resulting equation is very simple to solve, it seems secondary school students rarely have the opportunity to solve this type of equation; they are more likely to do so at earlier stages of their education. In secondary school, on the other hand, students repeatedly solve tasks by composing equations based on their knowledge of arithmetic or geometric sequences. By realising that, due to the doubling of the overgrown area, we could model this situation using a geometric sequence – a solution could, e.g., be as follows:

> Let a_n represent the area overgrown by the duckweed after n two-day periods. Since the whole pond was overgrown in 64 days, therefore from the general formula for the n-th expression of a geometric sequence we get $1 = a_1 \cdot 2^{31}$. Hence, we calculate $a_1 = \frac{1}{2^{31}}$. Since we want to count how many two-day periods have elapsed until a quarter of the pond is overgrown (let's denote this number by k), we get the equation: $\frac{1}{4} = \frac{1}{2^{31}} \cdot 2^{k-1}$. From here we calculate $k = 30$, so 60 days are needed.

The solution presented here, although it appears to be a typical geometric sequence task, has several difficulties. Firstly, it operates on the notion of a two-day period. This is because the duckweed doubled its area not every day, but every two days. This interpretation of the task may be difficult for students to realise. Paradoxically, if the task had stated e.g. "every seven days" instead of "every two days", it might have been easier for the students, because they could have considered the period of seven days as 1 week, which would have simplified the solution. Another difficulty is that it is not obvious what the values of the successive expressions of the sequence are, namely that the 32nd expression is the whole, i.e. 1, and the k-th expression is $\frac{1}{4}$. In attempting to solve this task by means of an equation, therefore, the participants could encounter considerable difficulty.

Duckweed Problem – analysis of results and discussion

In this task, the respondents achieved the second-best score compared to the other tasks of the test, but the task was nevertheless difficult for the participants.

It is a closed task in which only answer D was correct. The distribution of the respondents' answers is as follows:

Table 4. Answers to the *Duckweed Problem*

Answer	A.	B.	C.	D.	E.			No answer
Item formulation	*It cannot be solved*	*After 4 days*	*After 16 days*	*After 60 days*	*Another answer*			
					62	5	8	
Number of participants	4	9	30	28	4	1	1	1

Among the answers that appeared under "E. another", four people answered 62, and answers 5 and 8 appeared once.

Although this task caused the least difficulty for the participants in comparison to other tasks from the test, the fact that it is multiple-choice means that we do not know how many of the answers were random; the participants, not knowing the solution to the task, may have marked the correct answer by chance. Note that despite the fact that this task can be solved already by primary school students, the majority of the respondents did not give the correct answer.

It is particularly noteworthy that the answer most frequently selected by the subjects was "C. after 16 days", rather than the correct answer. Although the respondents were not asked about the motivations behind their answers, hypothetical reasons can be assumed.

The most likely reason for this is that the writers considered, according to our description of the task, that the overgrowth of the pond occurs evenly, that is, exactly the same amount of the pond overgrows on each day. If this were the case, then, as stated earlier, it would be sufficient to divide 64 by 4 (i.e., multiply 64 by $\frac{1}{4}$), obtaining the result of 16. Such reasoning demonstrates the activation of System 1 – thinking quickly on the basis of intuition and ignoring the crucial second sentence of this task. It is important to note that while numbers and operations appear in this sentence, they are not denoted with numbers and operation signs, but with words (two, doubles). It is possible that, for some people, this was the reason for omitting these two pieces of relevant information.

In an earlier eyetracking study whose participants were middle school students ($n = 52$), visual attention when reading the content of the task was focused on numbers written as digits, as presented by the visual attention heat map:

Figure 2. Visual attention of lower secondary school students solving Task 1 in previous eye tracking study (Sajka & Rosiek, 2015a, p. 1755)

Another potential reason is similar – it is possible that the subjects concentrated only on the third sentence for the same reasons as before, but did not consider the rate of overgrowth of the pond at all, instead performing an operation that, in their education so far, they used most often when provided the numbers 64 and $\frac{1}{4}$, or similar, in a task (that is, a natural number and a fraction whose denominator is a divisor of a natural number) – multiplication. Unfortunately, such an approach to text-based tasks is very often observed in pupils who do not think at all about the content of the task, but mechanically perform actions on the numbers appearing in the task. Often, these actions are arbitrarily chosen because the pupil remembers that, given similar numbers, he/she has most often performed such an action, or because this action seems to him/her the easiest to perform at that moment. This is also compounded by the fact that this is a task in which the student is provided potential answers, and would be therefore likely to reject operations such as $64 \pm \frac{1}{4}$. In this context, the fact that one of the possible answers that the respondent had been given was the number 16 was a deliberate difficulty and not a facilitation, as is usually the case when one has to choose an answer from a list.

Although both reasons are mainly present at an early stage of mathematics learning, they are unfortunately also present in people who could have likely coped with the task without problems if only they had properly concentrated on its contents. This is related to the aforementioned fast thinking (System 1) and its influence on the answer.

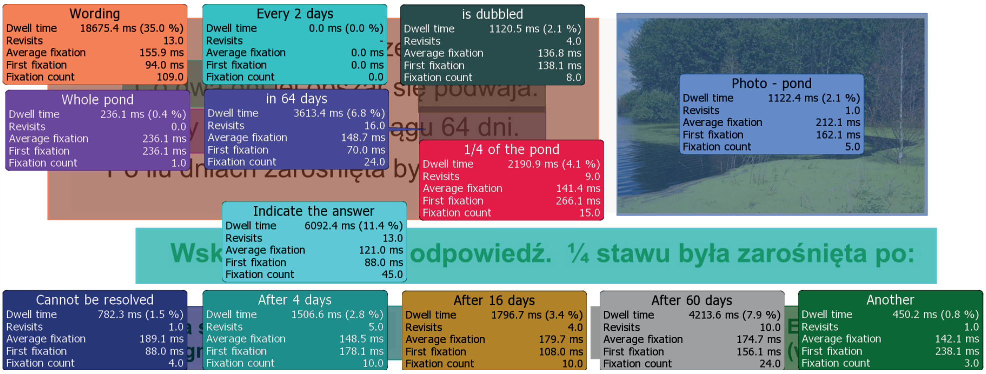
Other wrong answers are also worth noting. A relatively large number of people gave the answer “B. after 4 days” (12%). The reasons for this could be assumed by pointing out hypothetical student reasoning. It is possible that, as in the case of the reasoning leading to answer 16, the respondents took into account the second sentence again, but in such a way that, since there are two pieces of information related to the number 2 in it, the number 16 should be divided twice by 2, obtaining a result of 4. It may also have been the case that the subjects completely ignored the numbers 64 and $\frac{1}{4}$, and, solely on the basis of the second sentence, by multiplying the 2 that appeared in the task, obtained the result of 4.

Several participants also answered: "A. cannot be resolved". The reason for this answer is most likely due to difficulties in activating reductive thinking or "backward reasoning" in the context of the task. The writers failed to recognise that the task should be analysed "from the end", i.e., from the information that the whole pond became overgrown in 64 days. By trying to start "from the beginning" they may have concluded that they do not know where to begin, in which case the task appears much more complicated than it actually is. Answer A was chosen by 4 people.

It is also worth considering the number that was most frequently provided as part of the answer "E. other:", the number 62, which appeared in four works. Since this is the answer that was indicated by the respondents themselves, we can therefore assume that it was not random, but given as a result of some type of reasoning. It was probably the result of correct reasoning, in which the student omitted only one piece of information, namely "every two days", implicitly assuming that the doubling of the overgrown area occurred every day. This reasoning is also supported by a study that was conducted among prospective mathematics teachers (fourth-year students) who solved this task in the context of eyetracking methodology and were subsequently interviewed. In that study, answers consisting of the number 62 occurred due to omitting, when reading or processing, the phrase "every two days", and implicitly assuming that the overgrown area doubles in size every day, as documented, for example, by the measurement shown in Figure 4, where the top-left rectangle in turquoise (Area of Interest "two") shows that the respondent omitted the entire phrase of the words "every two days" from their visual analysis, which was also confirmed during the interview.

In summary, the students in our study performed comparably to lower secondary school students, whose success rate was 0.33 (Sajka & Rosiek, 2015a; 2015b), and made similar mistakes as future mathematics teachers (Sajka & Rosiek, 2016).

Figure 3. Areas of Interest for the visual attention of a pre-service teacher. The phrase "every two days" was not perceived while reading Task 1 in a previous study (Sajka & Rosiek, 2016)

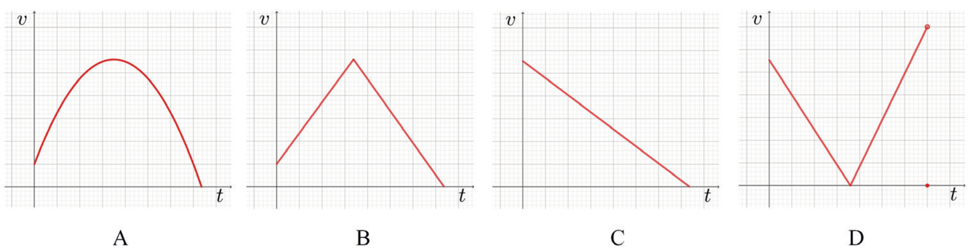


5.2.2. *Stone Problem* – Description, Results, Their Analysis, and Discussion

Stone Problem description

The content of the *Stone Problem* is as follows:
Task 16 (1 point). *A stone was thrown vertically upwards. Show the motion graph which illustrates its speed (v) versus time (t). (Omitting air resistance)*

Figure 4



What doubts do you have about this task? What do you think about it?

……………………………………………………………………………………………

The *Stone Problem,* from a mathematical skill point of view, can be solved even by primary school pupils, as it is closed-ended, with only four answers to choose from. A version of the graph in the shape of a parabola with the arms pointing upwards has deliberately not been added to the list of answers to avoid the necessity of using one's knowledge of physics. In this version of the task, it is sufficient to note the biphasic nature of the stone's movement (which eliminates answer C) and, when imagining the movement, that the speed of the stone's movement first decreases to zero, and then increases. It is therefore sufficient to recognise the correct monotonicity of the function modelling this phenomenon.

However, the task contains fundamental difficulties, which we will refer to in the presentation and analysis of the results. The *Stone Problem* has been used by researchers many times, in different versions and methodologies. Different strategies for solving the task and the causes of errors in wrong strategies was investigated in the context of eyetracking methodology in particular. A description of the distinguished strategies and reasons for wrong answers based on research with a diverse sample of 210 people and in different methodologies can be found in the paper by Rosiek and Sajka (2019).

Stone Problem – analysis of results and discussion

Table 5 shows the distribution of answers in this study. Thirty people out of 78 indicated the correct answer D. It is worth noting that the answer "other" was spontaneously provided by the participants in the space intended for them to share their doubts and was not among the proposed answers (in contrast to the *Duckweed Problem*). The vast majority of the respondents ($n = 56$) did not comment on the task – all 22 comments are quoted below. When quoting the participants' statements, their data is coded as [Pxy], meaning Participant with the code "xy".

Table 5. Answers to the *Stone Problem*

Answer	A.	B.	C.	D.	Other	No Answer
Number of participants	23	7	5	30	2	11

Two people decided not to pick any of the answers A-D and provided their own in the commentary field. The first self-proposed answer emphasises the monophasic nature of the movement and the strong association of the graph with the trajectory of the movement:

- [P75]: "No graph fits. Over time the speed should increase".

The second, most likely, proposes a vertical line as a graph, which also demonstrates a strong association with the stone's movement trajectory:

- [P14]: "The slope of the throw also seems important to me. But more reliably, when we throw something vertically, it usually falls in the same track – we get hit on the head with this stone".

Similarly, two others who chose answer A suggested that this was the closest to the correct answer and the graph should be vertical:

- [P41]: "A stone thrown vertically upwards will fall vertically downwards (it is impossible to throw a stone in a perfectly straight, vertical line)".
- [P50]: "There is no vertical line among the drawings".

One person answering A revealed his/her misunderstanding of the standard acceleration of gravity, thinking that it is not the same in both phases of motion (uniformly decelerated and then accelerated), and therefore expressed doubt about the pointed shape in the graph:

- [P07]: "When a stone is thrown upwards, it seems to me that it is slower to rise than it is to fall, and when it reaches its highest point, it is not likely to fall abruptly as if it has bounced off something".

Three students answering A or B specified other doubts:

- [P08]: "I have doubts about the point at which the stone stops flying upwards and starts falling downwards, I am not sure about the relationship between speed and time".
- [P23]: "My doubt is what the mass of the stone is, because I think the bigger it is, the faster the stone will fall".
- [P21]: "It depends on the height and whether we are also observing it falling down".

In contrast, one person who answered B only drew attention to the physical context of this task:

- [P52]: "Isn't this physics?"

Taking into account the two people who gave their own answer and those who chose graph A or B, we find that in their answers, 32 people out of 78 revealed the "picture" misconception, well-known in didactic literature and related to the interpretation of graphs (Leinhardt et al., 1990). It involves the identification of a graph of motion with the trajectory of the object's movement. In our task, this is an "up and down" association. Again, as in the *Duckweed Problem*, this has to do with the activation of System 1 based on a fast, intuitive response. Such a response is furthermore compounded by the common experience of the subjects' everyday life, as everyone has had the experience of performing and observing an upward vertical throw. This experience reinforces the temptation to choose the "up-down" shape of the graph. A second, hypothetical reason for indicating an A-B response could involve the implicit indication of a distance-time graph instead of a speed-time graph; such mistakes have also occurred in earlier eyetracking studies when a participant did not read or unconsciously assumed a different relationship to be presented. There may have been other reasons, but since they appeared sporadically in other studies, we do not cite them in this work (Rosiek & Sajka, 2019).

Previous research indicates that as many as about half of the examined high school students, mathematics education students, and even general mathematics students choose up-down answers on their first attempt at the *Stone Problem* (Rosiek & Sajka, 2019), therefore their amount being 41% in the current study is an improvement – it should, however, be noted that 11 people did not provide any answer to this task and their answer is therefore considered unknown.

The C answers have a different basis for their choice. Only one person among those providing this answer wrote a comment. This was a question that accurately reflected their reasoning and was the most likely reason for those who chose this answer:

- [P29]: "Did the stone fall?"

Through this question, his/her answer could be counted as correct, as he/she only analysed the first phase of the stone's movement. Only 5 students gave such an answer, but random answers cannot be ruled out due to it being a multiple-choice task with the only one correct answer. It can certainly be said that if the respondents consciously analysed one phase of the described movement, then they overcame the fundamental difficulty of the task and such an answer can therefore be considered a correct solution. In an earlier study (Rosiek & Sajka, 2019), this type of argumentation appeared only among mathematics students who gravitate towards abstract reasoning; students of other subjects, however, assumed using common sense that the stone must fall.

The correct answer D was accompanied by comments from 12 respondents. One category of contributions consisted solely of comments, for example, that the task has a physical context, which was positively perceived (by 3 people), stated neutrally (1), or poorly perceived (1). The comments also referred to the fact that the task is interesting and tricky and that the respondents have no doubts (3) or that they do because of the aforementioned reasons (2). We quote these statements below:

- [P54]: "This task is interesting, unusual for a mathematician – would suit a physicist".
- [P51]: "Very cool task, even a bit physics-based, and I really like physics and feel comfortable with it, so that's great! [heart emoji]".
- [P77]: "Cool, physics-based, tricky".
- [P59] "A physics-related task".
- [P65]: "Physics is not something mathematicians like!"
- [P12]: "It is puzzling and gives some food for thought, but at the same time it is interesting".
- [P56]: "Interesting, tricky (I hope my answer is good)".
- [P16]: "I don't seem to have [doubts], but that's probably because I did something wrong".

One person indicated his/her time-related problem:

- [P22]: "All in all, probably none, because I have 5 minutes left and I'm selecting the first thing that came to my mind".

Only three people raised factual concerns. One of these concerns involved the initial speed, which according to the respondent should be 0:

- [P28]: "Why do none of the answers start at zero speed when the stone is held right before being thrown at speed?"

The second concern involved, among other things, that the correct shape should be parabolic, i.e., resembling the letter U:

- [P63]: "It is not known to what height it was thrown, so in a certain case its fall speed would have reached maximum at some point. Also, this graph should be more parabolic".

One person wrote explicitly about their struggles with the movement path of the graph:

- [P62]: "I'm not sure if this concerns the trajectory of the flight or the changes in speed in relation to time".

5.3. Qualitative Analysis of the Solutions to the Two Most Difficult Tasks in the Test

Two tasks, both open-ended, ranked almost ex aequo in this category. These were Task 15 – success rate 0.04, and Task 10 – success rate 0.05.

5.3.1. Equation with Absolute Values Task – Description, Results, Their Analysis and Discussion

Task 15 description

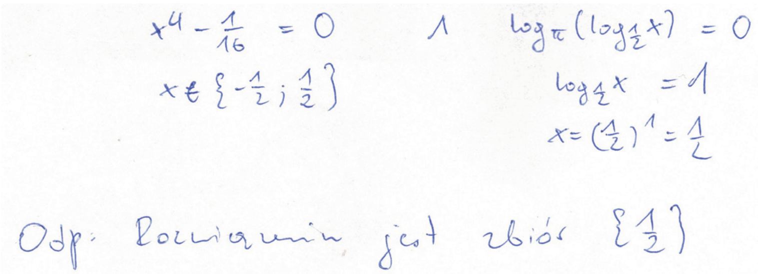
The content of *Task 15* is as follows:
Task 15 (2 points). Solve the equation:

$$|x^4 - 16^{-1}|^{17} + |\log_{\pi}(\log_{\frac{1}{2}} x)| = 0$$

To solve this equation, it was necessary to take advantage of the fact that the sum of two non-negative expressions is equal to zero when both expressions are equal to zero. Consequently, it was necessary to solve the two equations formed by equating the expressions under absolute value to zero and selecting such solutions that satisfy both equations or solve one of them, and then check which of these solutions are also solutions of the other one.

An example of a correct solution is shown in the following scan of the work of one of the respondents:

Figure 5. Fully correct answer to Task 15 (2 points, [P60])

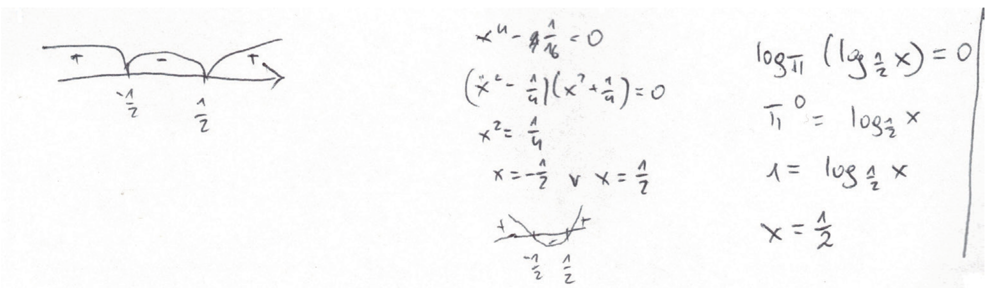


Text translation

Answer. The solution is the set $\{\frac{1}{2}\}$

Correctly solving this task resulted in 2 points, while 1 point was awarded if the solution was not carried through to the end, but significant progress was made, as can be seen e.g. in the following work:

Figure 6. Partially correct answer to Task 15 (1 point, [P26])



This solution lacked the conclusion that only the number ½ can be the solution to the initial equation.

This task was solved fully by only 3 people, while 1 point was scored by only one person, whose work is quoted above (Figure 6).

The categorisation of 0-point solutions is shown in Table 6.

Table 6. Categories of responses to task 15 scored at 0 points

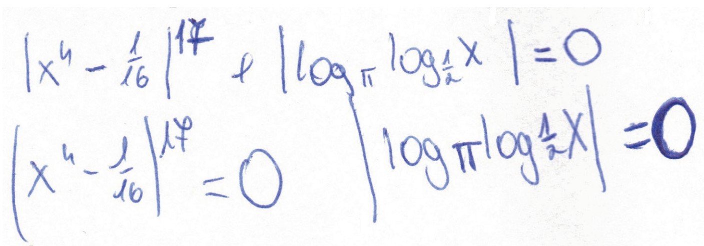
Response categories:	Number of responses
A. No notes of any kind	46
B. Copied excerpt from task content	8
C. Performing minor transformation of expressions under absolute value	13
D. Equating expressions under absolute value to zero	2
E. Using wrong formula or relation	3
F. Other error	2

Category A and B. More than half of the respondents (46 out of 78) did not attempt to solve this task at all, likely not making a mental attempt to analyse it either, but it is not known whether they skipped it or not. Eight further respondents only rewrote part of the task and made hardly any attempt to solve it, although it is known that they did not skip it.

Category C. Among the thirteen solutions in which minor transformations appeared, the predominant entries were those in which the expression 16^{-1} was replaced by the expressions $2^{4^{-1}}$ or $\frac{1}{16}$. In total, therefore, as many as 67 people did not attempt any solution to this task or made only a minor modification.

Category D. Two respondents equated the expressions under the absolute values to zero (each of them separately), as in the correct method, but stopped there. It is also unclear whether their thinking was correct, as no logical conjunction was written between the two equations, as a scan of one of the solutions shows.

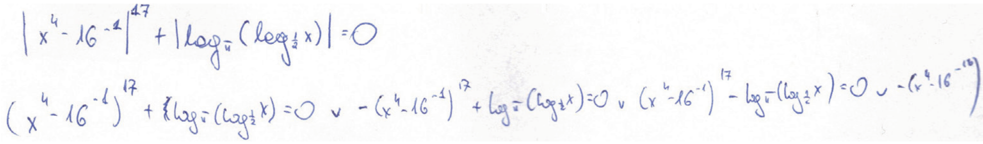
Figure 7. Example of a solution to Task 15 of category D [P35]



Most likely, both respondents correctly noted the need to switch to solving the two equations separately with the absolute value, but they did not know how to perform the exponentiation with such a large power (17), or how to simplify the calculation or transformation of a logarithm with base π, in which the logarithmic expression is another logarithm.

Category E. Incorrectly applied formulas related to multiplication and an incorrect understanding of absolute value also appeared among some works. Below (Figure 8) is an example of such an occurrence, in which the student likely applied an algorithm familiar to him/her from solving equations with several absolute values, trying to consider the cases but not considering the assumptions.

Figure 8. Example of a solution to Task 15 of category E [P01]



The weak performance in regard to this task may have a number of reasons.

Among the solutions to this task, it is worth noting the potential difficulties of the participants in "backward reasoning" (Pólya, 1975). An effective approach to this task can be considered an aspect of reductive thinking, because the last operation to be performed, the addition of absolute values, should be analysed first. One has to pose the question "When can the sum of two absolute values give zero?", i.e., to apply Treliński's (1985) question "What would be enough to know in order to solve this equation?" If the solver does not look holistically at the expression to the left of the equals sign, i.e. at the fact that it is a special equation of the form $|a| + |b| = 0$ and that 0 plays a decisive role here (if it were replaced by another number, the way of solving would have to be completely different), his/her chances of solving correctly are very small.

Another important psychological reason may have been that the task seems to give the impression of being difficult. It contains a lot of mathematical symbols which students associate with causing difficulties. The absolute value in the equation alone tends to cause a lot of problems for students, which would be compounded by the presence of two absolute values. The task also features logarithms, with the number π as the base of the logarithm, as well as an exponentiation to the power of 17, which could lead the students to think that, for example, if they do not remember how to raise an expression to the 17th power or how to solve equations containing a logarithm with a base of π (as the majority of respondents most likely never had to solve such problems), they will not have a chance to solve this task. That is, without a qualitative approach to solving the equation – without noticing the method described above – the students would not really know how to start working on this task. The additional accumulation of symbols and their associations with hardship may have caused a kind of psychological block and a lack of attempts to solve the problem – such an effect is observed in almost 70% of respondents.

The technical aspect of the test may also have been an important factor – the task was one of the last in the whole test with limited time remaining, so some of the subjects might have lacked the time to tackle it, and those who took up the task experienced additional stress.

This task, incidentally, allowed us to discern 3 people who solved the task by demonstrating a preliminary inclination for mathematics studies.

5.3.2. *Task on Injective Function* – Description, Results, Their Analysis, and Discussion

Task 10 was the second-most difficult task of the entire test, therefore we will present its analysis.

Task 10 (1 point)
Judge whether this statement is true:

A numerical function is a one-to-one (injective) function if and only if it is only an increasing function or only a decreasing function in its entire domain.

Justify your answer.

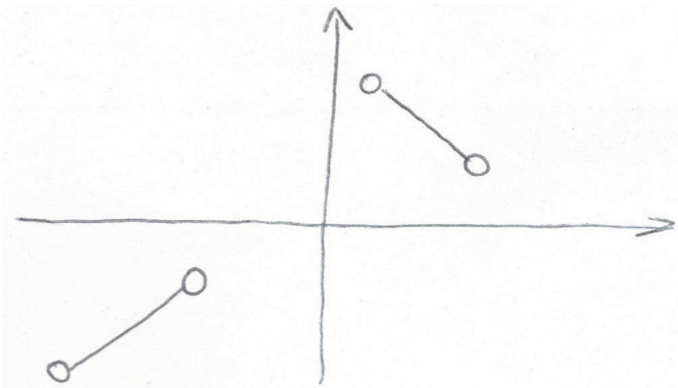
The types of student responses given and their amount are shown in Table 7.

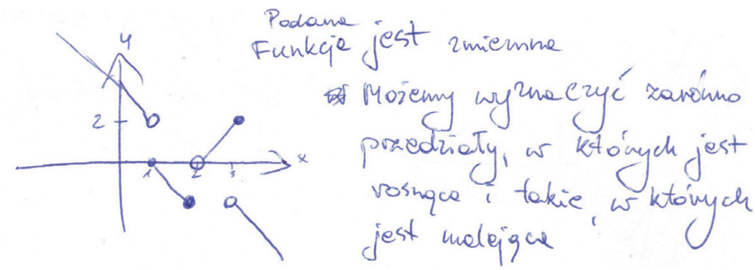
Table 7. Answers to the task on injective function

Description	Number of responses
Answer NO & proper justification	4
Answer NO & attempting justification	20
Answer YES & attempting justification	30
Only answer NO	7
Only answer YES	0
No answer	17

"NO" answers with an incomplete justification were not evaluated positively – 1 point could only be obtained if the answer was supported by correct justification. Only four people answered correctly in full. Three of them provided a counter-example in the form of a graph as justification, which are shown in Figure 9. In the third answer, we note the inconsistency between the graph and the description of the set of values of the function given next to the graph, however, determining the set of values of the function provided as a counter-example was not the focus of the task.

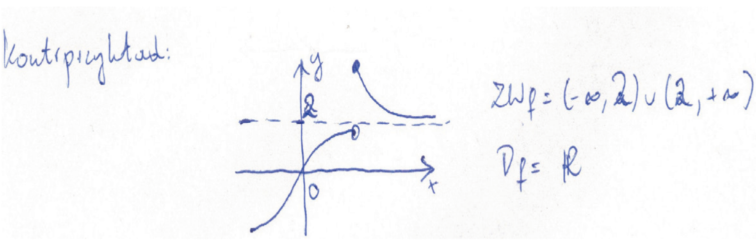
Figure 9. Three answers to Task 10 justified by a counter-example in the form of a graph ([P22], [P48], [P60])





Text translation

The provided function is a variable function. We can determine both the intervals in which it is increasing and those in which it is decreasing



Text translation

Counter-example:

In contrast, the fourth student gave a descriptive answer highlighting another property, namely emphasising that the function does not need to be continuous. Although this answer lacked a counter-example, we considered this solution to be correct, in favour of the student, due to the fact that the provided description captured the essential features of the counter-example. The respondent wrote:

- [P36]: "A numerical function can decrease in some interval of its domain, or increase in some interval of its domain. The numerical function whose statement we have to evaluate is not necessarily continuous, so it can decrease and increase in certain intervals of the function and still be an injective function".

The response score alone does not say much about the mathematical knowledge of the participants of the study due to the high difficulty level of the task. The task indeed proved to be very difficult for the study participants, and not by chance. It was previously used as part of an exam in the course "Didactics of Mathematics" for future teachers (4th year of mathematics studies), where its solvability was below 20%. Moreover, even a textbook approved for school use for the first year of secondary school and technical school contained an incorrect statement (Kalina et al., 2000, p. 99).

The task is objectively very difficult, requiring well-formed concepts such as function and its representations and a correctly formed *concept image* of function (according to Vinner, 1983) among the respondents. Furthermore, it requires knowledge of concepts such as the injective function, the monotonic function, and the relationship between them. It also requires linking the different elements of knowledge together and overcoming the imposing image of a continuous function on the real domain. In a certain sense, this is also where System 1 interferes, suggesting examples from a typical school experience, as the Polish curriculum provides successive classes of continuous elementary functions: linear, quadratic, polynomial, rational (including homographic), trigonometric, power, exponential, and logarithmic functions.

An additional difficulty is the methodological nature of how to justify correct/incorrect statements. The respondent in this task has to demonstrate the ability to refute a false statement, which is made even more difficult because it is in the form of an equivalence in which one implication is true. This true implication catches the participants' attention. A counter-example must be constructed, which requires some creativity and the choice of a suitable and convenient function representation. If, in the *concept image* (according to Vinner, 1983) of the respondent, there is a common belief that functions "should" be provided as formulas – then the difficulty level of the task increases even further.

With these difficulties in mind, we further categorised the respondents' answers according to two additional criteria: methodological, concerning the quality of the justification attempts, and in terms of the revealed misconceptions.

Attempts at justification

In the context of the type of attempted justification, we distinguished 11 categories, which are presented in Table 8. However, it should be noted that 24 people did not provide any justification, and the intention could not always be read in the remaining entries. Table 8 additionally provides information on the context, whether the answers justified “YES” or “NO”, and their frequency of occurrence.

Table 8. Categories of attempts at justification for injective function task

Type of justification attempted for each answer	No	Yes	n
1. Counter-example or its description	**5**		**5**
1.1. Proper justification: counter-example given as graph	3		3
1.2. Proper justification: description without graph	1		1
1.3. Attempt at providing a counter-example in the form of a graph (detaching from the continuous functions given on the set of real numbers)	1		1
2. Negation of monotonicity and conclusion	**1**	**11**	**12**
2.1. Negation of monotonicity and “YES” answer based on tacit assumption of continuity		7	7
2.2. Negation of monotonicity and “YES” answer due to contradiction: constant function cannot be injective		4	4
2.3. Negation of monotonicity and answer “NO”	1		1
3. Provide statement intended to refute sentence	**11**		**11**
3.1. False claim of relationship between injection function and monotonicity	2		2
3.2. Formulation of other false statement	9		9
4. Only reference property of injective function	**2**	**5**	**7**
5. Justification by explaining only one true implication		**12**	**12**
6. Pseudo-justification	**3**	**2**	**5**
6.1. Pseudo-justification: repetition of statement		2	2
6.2. Pseudo-justification: formulation of true (non-relevant) statement	3		3

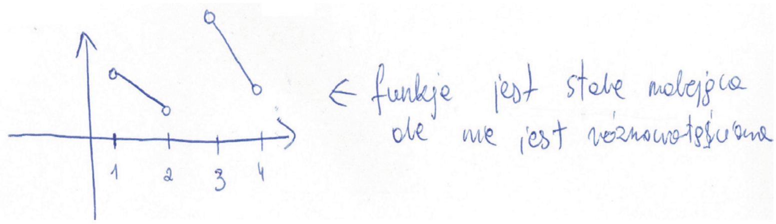
Re 1. Counter-example or its description (n = 5)

The first two categories consist of the responses of those who solved the task correctly (Figure 9), i.e., those who demonstrated their ability to refute false statements by providing a counter-example.

A total of 5 people gave or described counter-examples, but one of them [P77] made a different error in its construction (Figure 10, see also misconception E, Table 9) related to a lack of correct understanding of the monotonic function.

Person [P77] (Figure 10) chose the correct method to refute the statement and at the same time overcame one of the fundamental difficulties in the task, which is to detach from the imposing examples of continuous functions given on the set of real numbers.

Figure 10. Example of the use of a counter-example revealing an incorrect understanding of the monotonic function in person [P77]



Text translation

← the function is constantly decreasing, but it is not injective.

Re 2. Negation of monotonicity and conclusion (n = 12)

Another method of justification was to make an attempt to negate monotonicity. The respondents made an assumption: suppose a function is neither increasing nor decreasing.

By negating monotonicity, the seven respondents made a contradiction based on the implicit assumption that the function is always continuous, which in their view confirmed the truth of the statement:

- [P17]: "Yes, because if it were not just decreasing or increasing it would take the same values for the two arguments".
- [P68]: "Yes. If a function is not either increasing or decreasing in its entire domain, only constant or of various monotonicity, then some of the arguments have the same value".

Four people inferred that the function must be constant and that the constant is not injective, which also ended contradiction-based reasoning and, in the respondents' opinion, justified the truth of the sentence. For example:

- [P10]: "Yes. Because the constant function takes the same values regardless of x".
- [P34]: "Yes. Because the constant is not injective".

Methodologically, this is the correct way of searching for a counter-example or contradiction, however, gaps in knowledge prevented correct conclusions. This revealed a misconception (see J, Table 9) in which the participants of the study are convinced that all functions are given on the whole set of real numbers and continuous, or one in which they are convinced that all functions are either increasing or decreasing or constant (see I, Table 9). The second misconception may also have been rooted in the triggering of System 1, as the respondents – being used to the fact that in the context of monotonicity of continuous functions given on the whole set of real numbers, apart from increasing functions, only decreasing functions are injective – mechanically repeated the third type of monotonicity (constant functions), which was reinforced by the linear function being their first known family of functions.

An attempt to negate monotonicity with a similar misconception also played a role in justifying the negative answer:

- [P06]: "No, because when a function takes the same values, i.e., it is a constant function, it cannot be labelled an injective function".

In this case, the reasoning is not methodologically correct.

Re 3. Provide a statement intended to refute the sentence (n = 11)

Two people tried to justify their negative answers with (false) statements about the relationship between monotonic and injective functions:

- [P33]: "No. The statement is not correct because the injectivity of a function does not depend on whether it is increasing or decreasing".
- [P46]: "No. Not every increasing (or decreasing) function is injective".

A further nine respondents attempted to justify their negative answers by providing other misconceptions, examples of which are given in Table 9. It is worth noting that methodologically, this is the correct approach, but the misconceptions made it ineffective.

Re 4. Only reference the property of the injective function (n = 7)

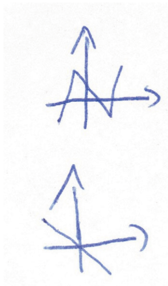
Further attempts at argumentation are difficult to interpret in the context of the methodology, as they consisted only of referring to the differentiability of the function. Among the 7 who gave such answers, 5 referred to the justification of a positive answer, for example:

- [P24]: "Yes, because every y is assigned one x".
- [P39]: "Yes, because no value can be repeated".
- [P40]: "Yes. This function is injective because it takes on different values for different arguments".

In contrast, 2 people tried to justify their negative answer in the same way:

- [P35]: "No. Because aninjective function takes different values throughout its domain for all arguments". [The person has included the following graphs in their answer:]

Figure 11



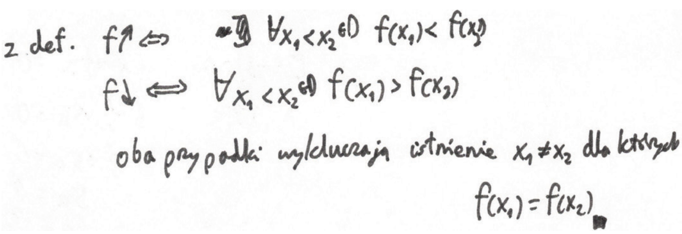
- [P42]: "No, because an injective function is only possible if the values do not repeat".

Re 5. Justification by explaining the only one true implication (n = 12)

Twelve students responded "YES", justifying it with one implication – if a function is increasing or decreasing, it is injective. Some examples of this justification are given below:

- [P59]: "Yes. When a function is either only increasing or only decreasing in its entire domain, no two function values are the same for different arguments".
- [P47]: "Yes. For *f.* increasing and *f.* decreasing it takes on different values, so it is an injective *f*".
- [P64]: "Yes. In an increasing function, the values of the respective independents will keep increasing, so there will never be the same value across the domain. The same goes for decreasing".
- [P63]: "Yes".

Figure 12



Text translation

Both cases exclude the existence of $x_1 \neq x_2$ for which $f(x_1) = f(x_2)$

The respondents correctly diagnosed the truth of one implication occurring in the provided sentence and focused all their attention on it.

Re 6. Pseudo-justification (n = 5)

Two people provided a pseudo-justification by repeating, almost verbatim, the evaluated sentence. In contrast, four people provided statements that express their belief but do not provide justification, for example:

- [P30]: "No. Other functions can also be injective functions".
- [P38]: "No. It depends what the function is".

One person misread the instruction (having all values equal[1]) and answered "NO" (i.e. correctly), writing down the following reasoning, which is correct in this context:

- [P01]: "No. A numerical function is an equal-valued function if and only if, over its entire domain, it is a constant function".

In conclusion, it should be noted that 17 people out of 78 did not provide any answer to the task, and 7 people did not justify their answer "NO". The majority of respondents who attempted to justify the correctness of the statement chose the correct methods to do so, which were: searching for a counter-example and attempting to refute – checking the correctness of the statement (Categories 1-3), which according to Komatsu and Jones (2022) is a manifestation of abductive thinking. A total of 28 out of 54 revealed such a skill, which was, however, not always realised correctly due to numerous revealed misconceptions.

It is certain that the error of providing a positive answer and justifying it with a single implication (Category 5) is methodological in nature and related to a lack of skill in proving statements or refuting hypotheses – but such an error does not show suggest profound difficulties in understanding concepts. To excuse these participants, it should be noted that in school learning, proving statements formulated in the form of an equivalence is rarely undertaken, even more so the ability to refute statements in general, especially equivalence statements.

Methodologically concerning are the missed attempts at justification ($n = 7$) and pseudo-justification ($n = 5$), as well as justifications from Category 4 ($n = 4$) that do not reveal the reasoning.

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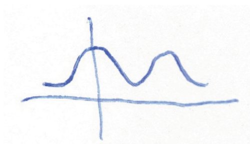
[1] In Polish these two words differ by only one letter: injection (ró**ż**nowartościowa) and function "having all values equal, equal-valued"(ró**w**nowartościowa), although a different word is used in mathematics in this context: constant (stała), as written by the participant.

Disclosed misconceptions

Qualitative analysis of the responses revealed a variety of misconceptions among the students, the categories of which, together with the number of students in whom they were revealed, are presented in Table 9.

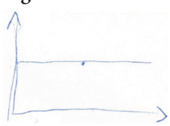
Note that several misconceptions could sometimes be found in a single statement, as in the case of the statement of subject [P62], who tried to justify his/her answer "NO" by stating "Every function is injective" and presenting the following graph:

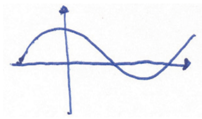
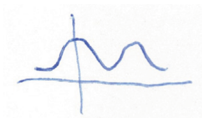
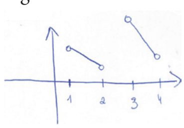
Figure 13



This justification both reveals a belief that aninjective function has variable values and changeable monotonicity, which the respondent presents on the graph (category C). Furthermore, the respondent stated that every function is injective, which in turn either shows a lack of consideration of the constant function or a recognition of the constant function as injective (category E).

Table 9. Disclosed misconceptions and their examples

	Misconception	Example(s)	No. of ans. Yes/No	Total n
A	Misunderstanding of injective function: confusing injective condition with definition of function	[P21]: "Yes. Because only one y is assigned to each x. This is the basic condition of an injective function".	Y-2	2
B	Misunderstanding of injective function: "Non-constant"	[P31]: "Yes". **Figure 14**  "A numerical function will not be an injective function when it is constant, although it will be in other cases".	Y-1	1

C	Misunderstanding of injective function: "Has variable values or variable monotonicity"	[P15]: "No. Functions taking different values can be decreasing or increasing in given intervals, e.g. the quadratic function". [P74]: "No. A numerical function is an injective function when its values are different, i.e. when it is increasing and decreasing". [P23]: "Yes. Since the injective function is defined by the formula": **Figure 15**  "Therefore it either has to increase or decrease". [P62]: "No. Every function is an injective function:" **Figure 16** 	N-7 Y-2	9
D	Misunderstanding of injective function: "Must have positive and negative values"	[P76]: "No. Injective functions take positive and negative values in their domain".	N-2 Y-1	3
E	Other difficulties in understanding of injective function	[P62]: "No. Every function is an injective function". [P54]: "No. Because a function can be a constant and can have different values".	N-5 Y-1	6
F	Confusing non-monotonic function (monotonic on intervals) with monotonic function	[P77]: "No. The function is continuously decreasing but not injective". Figure 17  [P03]: "No. When there is an increasing function and a decreasing function in a numerical function, such a function is also an injective function". [P67]: "Yes. The statement is correct because if the function were increasing and decreasing within the domain, it would not be an injective function".	N-3 Y-2	5
G	Belief that there is no or incorrect relationship between monotonicity and injection of functions	[P33]: "No. The statement is incorrect because the injection of a function does not depend on whether the function is increasing or decreasing". [P46]: "No. Not every increasing (or decreasing) function is an injective function".	N-2	2
H	There are only 3 types of functions: All functions are either increasing, decreasing, or constant	[P09]: "Yes. If it were not increasing or decreasing, it would have to be constant, in which case it would not be injective". [P34]: "Yes: Since the constant function is not injective". [P06]: "No, because when a function receives the same values, i.e. it is a constant function, it cannot be labelled an injective function".	N-1 Y-10	11

I	Implicit assumption that function is continuous on real number set	[P68]: "Yes. If a function is not either increasing or decreasing in its entire domain, but constant or variable, then some arguments have the same value". [P17]: "Yes. Because if it were not just decreasing or increasing, it would take the same values for 2 arguments".	Y-6	6
J	The function includes several functions	[P08]: "No. It is an injective function when it contains both functions in its entire domain".	N-2	2
K	Other	[P73]: "No. The function is differentiable ⇔ For every argument in the domain, its opposite element can be found".	N-1	1

The task – despite the fact that it caused the most difficulties for the research participants – turned out to be a good diagnostic task, revealing as many as 11 misconceptions in the respondents, as well as various types of methodological and conceptual difficulties with a great deal of variation. At the same time, however, it made it possible to identify 4 participants who have good methodological knowledge and skills alongside a substantive basis in the scope of the concept of function.

6. Discussion and Conclusions

6.1. Conclusions Concerning Solutions to the Easiest Tasks

The tasks concerning the duckweed and the stone were the easiest in the whole set. We note several reasons for this.

1. Their solution did not require sophisticated or advanced knowledge of mathematics. Basic school level mathematics were sufficient to solve them,
2. A qualitative analysis of the situation and the use of common sense were sufficient to solve them,
3. Both tasks had a non-mathematical context which referred to familiar situations from everyday life,
4. The fact that both tasks were closed-ended certainly increased the score, as we cannot exclude the possibility of correct answers being provided at random,
5. Due to the design of the task, it was possible to activate reductive reasoning in the logical sense, i.e. to take the given answers as a hypothesis and test them. In this way, it was possible to avoid the trap of giving a quick, intuitively wrong answer imposed by System 1,
6. Several methods can be used to approach these tasks.

At the same time, both tasks contained significant difficulties that needed to be overcome, therefore, their success rate is not satisfactory.

The majority of respondents succumbed to System 1-based quick thinking, intuition, and initial associations when solving both tasks. In the *Duckweed Problem*, this involved the use of proportionality, which was not relevant to the task, or operations on numbers involving the numbers 64, $\frac{1}{4}$, 16. The task with the stone involved two main reasons. The first was the imposition of a very common misconception related to the identification of the speed-time graph with the trajectory of motion, an obstacle that is natural and present at all levels of mathematical education and is also related to the operation of System 1. The second reason behind the wrong answers was the possible confusion between the types of graphs – the analysis of "distance-time" instead of "speed-time".

In the *Duckweed Problem*, it was easy to check whether the answer given (e.g., 16) was correct. If the respondents had performed a simple check after giving the wrong answer, they would most likely have discovered their error. It is therefore plausible to hypothesise that, having given the wrong answer, they were confident in their choice, which is characteristic of the operation of System 1. The habit of verifying answers was therefore lacking, which is an area for further work.

With both tasks, another reason for failure may have been the omission of important parts of the content of the task, as observed in the results of previous eye-tracking studies, where, in the *Duckweed Problem*, the numbers given verbally were omitted when reading the task content, and in the *Stone Problem*, the analysis of the type of relationship and the axis description analysis was omitted.

The analysis of the results of the investigation based on these tasks indicates that, in further work with these students, attention should be paid to three main elements:

1. Accuracy in reading a mathematical text,
2. Awareness of the existence and operation of Systems 1 and 2 and the need to implement self-monitoring, checking System 1 by consciously activating System 2, including developing the habit of checking answers,
3. Devoting attention to practising the use of reductive reasoning (abductive or "backward reasoning") when solving tasks.

6.2. Conclusions Concerning Solutions to the Most Difficult Tasks

Tasks 15 and 10 were the most difficult tasks in the entire set. A number of reasons for this can be identified, as discussed when analysing the solutions to these tasks. The most important of these, in our opinion, are presented below.

1. Both tasks were open-ended, so it was not possible to indicate the answer at random or to apply the method of verifying the provided options.
2. In the solutions to both tasks, mathematics knowledge at secondary school level had to be demonstrated and, in contrast to the easiest tasks, common sense and knowledge at primary school level was not sufficient to solve them – the context of the task was purely mathematical:
   a. Task 10 required referencing a discontinuous function or a function given on a domain other than the set of all real numbers, as well as knowledge and understanding of the general properties of functions. It also required an understanding of sentence equivalence and proving. It therefore required not only factual knowledge, but also a more sophisticated, methodological knowledge of mathematics – the understanding of an equivalence-type statement and the ability to refute it.
   b. Task 15 required knowledge of concepts and symbols such as absolute value, logarithm, irrational numbers, power of 17, and the ability to solve a non-trivial equation.
3. Both tasks were non-standard in relation to tasks solved at school:
   a. In Task 10, the correctness of the given sentence had to be assessed. Students in Polish schools are rarely put in situations involving evaluating sentences, even more so in such a general context, which is additionally complicated logically. At the basic level of secondary school mathematics teaching, little time is generally spent analysing the general properties of functions, especially those that are not given on the set of real numbers or are discontinuous, although students do encounter such examples, e.g., in the context of transforming graphs of functions.
   b. In Task 15, an equation with absolute value, not typical of those encountered at school, had to be solved.
4. Both tasks activated the operation of System 1 and required overcoming it.
   a. The first approach to solving Task 10 activates the designators of continuous functions in the real domain and a quick, intuitive affirmative answer. Students at basic secondary school level learn the properties of specific classes of continuous functions, given mainly on the real numbers set (e.g., polynomial – with particular emphasis on linear and quadratic; exponential), which all the more reinforces the idea that functions are usually continuous and given on the set of real numbers. The discontinuity of functions is considered in more detail at the advanced level.
   b. Task 15 activated the method, familiar to the students, of solving linear equations with two absolute values from the definition of the absolute value by considering cases at intervals, which was not an effective strategy in this task.
5. Both tasks required, in some sense, the activation of "backward reasoning" (Pólya, 1975), abductive thinking (Komatsu & Jones, 2022).

   a. As written in the summary of the results of Task 15, this required a holistic assessment of the expression to the left of the equals sign, i.e. the realisation that it is a special equation in the form of, and asking a reduction-type question: "What would be enough to know to solve this equation?" (Treliński, 1985).
   b. Constructing a counter-example and attempting to refute – checking the correctness of the statement according to Komatsu and Jones (2022) is a manifestation of abductive thinking.
6. Another important psychological reason may have been that both tasks gave the impression of being difficult and may have caused a kind of psychological block, which was not the case with the easiest tasks.
   a. Task 10 is a proof task, which usually causes difficulties for students.
   b. Task 15 contains a lot of mathematical symbols and concepts learnt in secondary school (highlighted in section 1), all of which cause difficulties for students and are considered by students to be problematic.
7. The technical side of the test may also have been an important factor – tasks 10 and 15 were placed towards the end of the entire time-limited worksheet, so some of the respondents ran out of time to tackle them, and those who took up the task were accompanied by additional stress.

6.3. Answers to the Research Questions

In this section of the chapter, we will attempt to answer the subsequent research questions.

RQ1. What is the knowledge and skills at the basic level of secondary school mathematics of first-year mathematics education students?

All tasks from the worksheet proved to be difficult or very difficult for the research participants. However, significant correlation with the results of the matura exam shows that the knowledge of the examined students is quite stable, despite the unfavourable timing of the test (see section 6.5. for the reasons behind the weak results).

Task 10 revealed a number of methodologically correct attempts to verify and refute the given statement. However, this task was too difficult for many of the participants to succeed in verifying the given sentence correctly.

Unfortunately, there are many weaknesses in the knowledge of the respondents.

Four students achieved an extremely negative score of 0 points. A lack of knowledge or understanding of basic mathematical concepts and poor calculation skills were revealed in many works.

There was an alarming number of papers where no attempt was made to provide an answer, which happened in the case of all of the tasks, even the easiest ones, i.e. choosing one of the provided answers. One of the lowest response rates was in Task 15, where 54 of the respondents either failed to provide any response or merely rewrote the content of the task (Categories A and B).

Multiple misconceptions were revealed. Using Task 10 alone as an example, 11 misconceptions related to monotonicity, injective function, and continuity of functions were highlighted, which were revealed as many as 47 times in a sample of 78 respondents; unfortunately, they were revealed in the majority of respondents, with an additional 17 respondents giving no answer and a further 7 responding "YES" without any attempt at justification.

Methodological difficulties were also revealed. Using Task 10 as an example, the main categories of justification given in Table 8 were distinguished, of which 5 were incorrect. Twenty-eight subjects attempted to contradict the given statement by looking for a counter-example, or attempted to give a statement refuting the one in the task, but only 4 did so correctly; 24 did so methodologically incorrectly (gave a statement or pseudo-substantiation, or positively verified only one of the two implications) and a further 24 did not provide any justification at all.

The tasks that had the highest success rate also provided further information on the deficiencies of the subjects. These are listed in section 6.1. "Conclusions concerning solutions to the easiest tasks".

The fact that, throughout this text, most of the content is devoted to analysing the students' difficulties and gaps in their knowledge may give the impression that the test went poorly for all participants. However, this is not the case – several examined participants achieved a score in the test that can be considered very good. Therefore, the test also highlighted those individuals who could potentially become very good at mathematics.

The best score on the entire worksheet of the research was 19 points, and it is the only such score (person number P63) that significantly deviates from the mean (5.18) and modal (3 and 4) values. It is also worth noting that this student achieved a score of 100% in the basic level mathematics matura exam. Participant P63 is also among the outstanding students in this group. The next highest scores were 16 (achieved by two people) and 15 (one person), and all three achieved the same very high score in the basic mathematics matura exam, at 98%. Therefore, the research identified four students who performed significantly better than their peers.

Among the students who participated in the research, in addition to the four mentioned above, there were 18 who obtained a score of 100% or 98% in the basic mathematics matura exam, while in our research, the average score in this group was only 32%, and the modal values were 4 and 5.

Based on the results of the research, it can be concluded that the matura exam in mathematics puts an emphasis on standard, algorithmic reasoning, which is understandable, as it is a compulsory exam for all high school graduates.

RQ2. What misconceptions in the understanding of school mathematics can be distinguished among the first-year mathematics education students?

The analysis of Task 10 alone presented 11 misconceptions, shown in Table 9, which occurred 47 times in total, and consisted of:

Misunderstanding of injective function:

a. Confusing injective condition with definition of function,
b. Misunderstanding of injective function: "Non-constant",
c. Misunderstanding of injective function: "Has variable values or variable monotonicity",
d. Misunderstanding of injective function: "Must have positive and negative values",
e. Other difficulties in understanding injective function,
f. Confusing non-monotonic function (monotonic on intervals) with monotonic function,
g. Belief that there is no or incorrect relationship between monotonicity and injection of functions,
h. There are only 3 types of functions: All functions are either increasing, decreasing, or constant,
i. Implicit assumption that function is continuous on real number set,
j. Function includes several functions.

There was also an off-topic statement which was entirely nonsensical (K).

Most of these misconceptions have an indirect origin in fast, System 1-aligned thinking, as these kinds of beliefs have formed spontaneously based on school experience, from examples most commonly present in school practice, and have influenced intuition and formed the wrong ideas regarding these concepts. It is highly probable that the respondents, when solving tasks involving a certain mathematical concept, such as function, did not consider the definition and the conditions declared therein (which they may not have even remembered), but about the examples most frequently used during lessons. Their reasoning was therefore based on a wrong idea of the concept, rather than the analytical reasoning characteristic of System 2 activity. The respondents did not perform analyses based on the definition of monotonic and injective functions as well as analyses of the statement in terms of logic (as a sentence in equivalence form). In this context, it would be interesting to check how the same students would have solved Task 10 if they had been provided the definitions of the concepts used in the task. Some, e.g., [P01] (see Re 6. Pseudo-justification) gave their answers as a result of anentirely literal, quick, and inattentive reading. Many of the given misconceptions resulted from ignoring or omitting task-relevant data, which is a characteristic of System 1 (for example, omitting equivalence and unconsciously considering only one implication – see Re 5. Justification by explaining the only true implication, (n = 12)).

6.4. Conclusions – Areas for Further Study

The analysis of the respondents' answers to the four selected tasks made it possible to distinguish areas for further study involving students:

1. Noting the care put into reading mathematical texts,
2. Awareness of the existence and operation of Systems 1 and 2 and the need to implement self-monitoring, checking System 1 by consciously activating System 2, including developing the habit of checking answers,
3. Implementing the use of reductive reasoning (abductive or "backward reasoning") when solving tasks,
4. Overcoming psychological barriers associated with tasks that seem difficult,
5. Implementation of a methodology of justification and refutation of statements, mastering mathematical proving skills,
6. Revising selected mathematical content from secondary school, working on the correct formation of particular *concept image*s in students, such as the concept of a function, by analysing different designations of concepts with different properties in order to eliminate misconceptions that have arisen as a result of limited examples analysed in secondary school.

6.5. Conclusion – Reasons for Poor Performance

It is worth considering the reasons why the respondents scored so low in the study. Certainly, one can distinguish between factors related to the worksheet itself and the way of conducting the research:

1. The timing of the study was very unfavourable for the respondents. The students took the questionnaire after their longest summer holidays – a 5-month break from studying (early May to October 1).

2. The duration of the test was relatively short (75 minutes).
3. The tasks were non-standard in comparison to school and exam tasks and did not test the ability to perform typical procedures – they contained traps designed to activate System 1 and sometimes required the use of very well formed concepts (as in Task 10).
4. The tasks were significantly more difficult than those in the matura exam.
5. The respondents were stressed by having to fill this worksheet on the first day of their academic year.

Further factors include the evident mismanagement of time by the participants of the study – many did not provide correct answers despite spending a long time on selected tasks, which was evident through the amount of notes.

On the other hand, global factors are also important. Generational changes are evident at this stage of education, which are affecting the way young people learn. In a world that seems to be accelerating, this often involves very fast learning. Students increasingly do not pay attention to learning the definitions and properties of mathematical concepts, and limit themselves only to learning basic algorithms. Any non-standard situation, other than a repeatedly reproduced procedure or algorithm, may cause difficulties and attempts to refer to things that are better known. Another reason may also be the concentration problems that have been increasingly noticed in young people. Furthermore, in times when we are constantly bombarded with information (e.g., advertising or news), the way to cope is often to perform a rapid selection of the provided content. Such selection in the case of a mathematical text, in which every word and symbol has meaning, usually leads to the misinterpretation of tasks.

In addition, an important factor related to the respondents' education was the COVID-19 pandemic and its associated remote learning affecting the last period of schooling, where the scope of learning content was reduced. In addition, a general decline in mathematical skills has been observed in the years preceding the study.

It is possible that one of the factors behind the fact that mathematics studies are sometimes chosen by students who are severely mathematically deficient is the fact that they achieve high results in the basic level of the matura exam in mathematics.

One of the aims of our work was also to draw attention to the problem occurring at schools of the students not being able to form basic intuition regarding concepts and the relationships between them, as, according to many teachers, "there is no time for that, because lots of calculation problems, often very algorithmic, have to be solved instead".

The research has made it possible to distinguish the difficulties that prospective students have and to provide a basis for the implementation of a course for mathematics education freshmen. The current course is designed to repeat the teaching content at the advanced level of the matura exam. Our research shows that there is a need of implementing a new type of course for new students – a course where, for example, various relationships between concepts and dependencies, and interesting (e.g., borderline) cases would be analysed, but in which there would be (almost) no calculations whatsoever.

Advancing one's level of education is always the cause of various difficulties – this applies even more so to moving from secondary school to university level. At this stage, students, who have become adults, learn to take complete responsibility for the results of their learning and they learn to organise their way of learning. This is something they need support with.

7. Limitations of the Study

The undertaken research has numerous limitations. A selection of these, related to time, the flow of the research, and the design of the worksheet, are mentioned in section 6, in which, among other notions, we analysed the reasons for poor responses.

Another main limitation of the research is that it was only carried out in one of the universities engaged in training prospective mathematics teachers, which makes it impossible to generalise the results. Future research would need to be implemented among various universities to observe possible global trends.

The research methodology in the form of analysing the students' written output on the *Research Worksheet* also limited the amount of possible conclusions. In future research, the methodology would need to be extended to include, for example, interviews related to selected work.

These and other limitations would need to be considered in future studies.

8. Follow-up

Immediately after the research was carried out, the results were partly used in the implementation of the "Foundations of Higher Mathematics" course in mathematics studies, dedicated to filling in gaps and repeating secondary school content. This course was followed by a "post-test", an analogue of which may be the subject of further studies.

As mentioned, there is a need to implement yet another strand to the course for new students, a course in which, for example, various relationships between concepts and dependencies, and interesting (e.g., extreme) cases would be analysed, but it would involve (almost) no calculus. After such a course, its effectiveness would have to be tested with a similar tool.

It would also be worthwhile to carry out an analogous *Research Worksheet* in subsequent year groups as well as longitudinal and comparative studies.

It would also be very interesting to conduct such a study in other universities in order to be able to draw more generalised conclusions.

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PART IV

Professional Preparation of Future Mathematics Teachers

Marlene Kafui Amusuglo & Antonín Jančařík
Charles University

Chapter 15

COLLEGES OF EDUCATION EARLY GRADE MATHEMATICS CURRICULUM AND NATIONAL KINDERGARTEN MATHEMATICS CURRICULUM IN GHANA: A COMPARATIVE ANALYSIS

Summary: Mathematics as a subject is taught in all basic schools and all the forty-six Colleges of Education (CoEs) in Ghana. This research provides a comparative analysis of the mathematics curriculum of the CoEs early grade program and the national kindergarten mathematics curriculum in Ghana as very little is known when it comes to such comparative analysis. The content of the CoEs mathematics curriculum is not integrated, while that of the kindergarten curriculum is an integrated curriculum for all subject disciplines taught in kindergarten and not specifically mathematics. The national kindergarten (KG) curriculum, however, contained no description of topics relating to mathematics. It only indicates the aspect to be taught. For example, in the KG curriculum, the topic for sub-strand 2 for KG I, Term 1, is 'The Parts of the Human Body and their Functions'. Under this sub-strand, teachers are expected to teach the four subjects in the curriculum (Language and Literacy; Numeracy; Creative Arts, and Our World and our People) using the human parts as exemplars, which is not situated in the CoE curriculum. This could be challenging for newly qualified teachers who are used to the kind of structure of curriculum used at their CoEs.

Keywords: Mathematics, curriculum, teacher education, kindergarten, early grade.

1. Introduction

Achieving Sustainable Development Goals 4 (SDG) and the other 16 SDGs has become the hallmark of most educational systems, and there has been a continuous discussion about how to achieve these SDGs (UNESCO, 2021). In the process of working towards the achievement of SGD 4, many changes have been made in most school curricula. In Ghana, the basic school curriculum – basic school, also referred to as elementary school, constitutes the initial stage of formal education for children before progressing to secondary school (MoE 2019) – has been going through a number of refinements. The vision of the pre-tertiary teacher education program is to:

> Prepare teachers to teach in basic and second-cycle schools and develop and nurture these pre-service teachers to become reflective and proficient practitioners capable of providing quality education for Ghanaian children (Ministry of Education [MoE], 2012, p. 8).

Similarly, the provision of accessible and quality education for all to meet the country's needs is the core of the basic school curriculum; it is based on this that the new curriculum sets out the learning areas that need to be taught and how they should be taught and assessed. It provides a set of core competencies and standards that learners have to know, understand and demonstrate as they progress through the curriculum (MoE, 2019). The concept of early grade has been defined differently from different perspectives. The United States National Association for the Education of Young Children (NAEYC) defines early grade as the age before the age of eight. It is the period from birth to 8 years (UNESCO, 2000). Early grade education plays a key role in the formative years for children when they start developing their cognitive and non-cognitive skills (Rao et al., 2019). Early grade education is important in several domains of development, including learning skills (Conger et al., 2019), educational achievement (Cortázar, 2015) and employment performance (Wilson, 1995). Coury et al. (2014) assert that early-grade education is an influential factor in improving the child's development. It is, therefore, important to pay attention to how teaching and learning are done at the early grade stage. Curriculum implementation requires the teacher to build a relationship with the students and promote individual learning. This relationship should inspire students to innovate and help them confidently take risks in learning (Young, 2011). Similarly, Begg (2005) asserts that the curriculum is "all planning for the classroom". This implies that the curriculum provides a design that enables learning to take place. It defines the learning that is expected to take place during a course or program of study in terms of knowledge, skills and attitudes. It specifies the main teaching, learning and assessment methods and indicates the learning resources required to support the effective delivery of the course (MoE, 2019). The focus of this work will be comparing the Colleges of Education (CoE) early grade bachelor of education (B.Ed) mathematics curriculum and the National Kindergarten (KG) mathematics curriculum in Ghana.

2. System of Basic Education in Ghana

With the recent education reforms, formal Basic Education for all Ghanaian learners is from KG 1 to SHS 3 (Grade 12), and it is put into five phases: Phase 1: Foundation level comprising Kindergarten 1 and 2, Phase 2: Lower primary level made up of B1 to B3, Phase 3: Upper primary level of B4 to B6, Phase 4: Junior High School (JHS) level of B7 to B9, and Phase 5: Senior High School (SHS) level comprising SHS1–SHS3 (MoE, 2018). Education at Phase 1 starts at age 4 with Kindergarten (KG) education and links with Lower Primary education up to age 8. KG education predisposes children to conditions of formal schooling, giving them the mental attitude for learning during future years. The Upper Primary phase (age 9-11 years) attempts to lay a strong foundation for inquiry, creativity and innovation, and lifelong learning in general, and to provide building blocks for higher levels of education (Anamuah-Mensah, 2002). The third phase of basic education is the three-year Junior High School or JHS (age 12-15 years), which is lower secondary education and provides the opportunity for pupils to discover their interests, abilities, aptitudes and other potentials. The final phase of basic education is the three-year SHS (age 15-18 years), which is upper Secondary education and allows learners to specialise in any one of the following programs: Science, General Arts, Technical and Vocational, Business. SHS education is the platform that delivers the extensive scope of academic knowledge and skills required for entry into further education and training in the tertiary institutions of Ghana and elsewhere. In this context, after sitting and passing the West Africa Secondary School Certificate Examination (WASSCE) conducted by the West African Examination Council (WAEC), SHS graduates may gain direct employment or admission into tertiary institutions like universities, polytechnics, colleges of education, nursing training, or undertaking a specialised program such as Ghana Police Command, Ghana Institute of Journalism, and others (MoE, 2019).

2.1. Basic Education in Ghana, including Curriculum Matters

There have been several curriculum reforms in Ghana. For example, in 2017, the Government of Ghana tasked the National Council for Curriculum and Assessment (NaCCA) to review the pre-tertiary curriculum in Ghana to respond to international best practices (Stephen, 2021). In September 2019, the government of Ghana implemented the new curriculum in basic schools. The new curriculum is aimed at addressing the loopholes in the old curriculum, which included content overload, limitations of the objective-based curriculum, and the failure of the assessment system to provide enough data on which teaching and learning could be styled (Aboagye & Yawson, 2020). The new curriculum is purposely designed to improve the acquisition of reading, writing, arithmetic and creativity skills across the entire primary curriculum while strengthening the teaching of mathematics (Aboagye & Yawson, 2020). Addai-Mununkum (2020) added that this new curriculum is also intended to promote the acquisition of 21st Century skills such as critical thinking and problem-solving, creativity and innovation, communication and collaboration, cultural identity and global citizenship, personal development and leadership, as well as digital literacy.

2.2. Teacher Education in Ghana with Emphasis on Preschool and Elementary Level

Ghana has been working on teacher education before independence (Antwi, 1992). In the Gold Coast era (Antwi, 1992; McWilliam & Kwamena-Poh, 1975), European merchants assisted in the training of individuals to become teachers so that they could assist with interpretation to aid their companies (Antwi, 1992; McWilliam & Kwamena-Poh, 1975). The objective of Ghanaian teacher education, according to Adegoke (2003) and Benneh (2006), is to provide a comprehensive teacher education program through pre-and in-service training that will develop competent, committed, and dedicated teachers who will improve the quality of teaching and learning. Teacher Training Colleges (TTC), now known as Colleges of Education (CoE), initially offered 2-year Post-Middle Certificate "B" programmes, followed by 4-year Post-Middle Certificate "A" and 2-year Post-secondary Certificate "A" programmes. In the 1980s, the 2-year post-secondary programme was extended to a 3-year programme but ran alongside the 4-year certificate "A". The first legislation is the passing of the 2008 Education Act (Act 778). Under the Act, Section 9 called for the creation of a National Teacher Council (NTC), which has since been established. The NTC is mandated to establish professional practices and ethical standards for teachers and teaching, and registration and licensing of individuals seeking to enter the teaching profession (Buabeng et al., 2020). The second legislation is the Colleges of Education Act 847 to upgrade CoE into a tertiary institute.

Following this legislation and with effect from October 2018, the CoE was upgraded to four-year degree awarding institutions and no longer three-year diploma awarding Colleges. The introduction of the degree program is to enable prospective teachers to specialise in the programs pursued in the CoE, namely the Early grade program, the Primary Education program and the Junior High School program. This means that prospective teachers, ultimately, will be licensed to teach at very specific grade bands within the basic school levels. Some of the anticipated benefits from such a move are to enable 1) prospective teachers to acquire deep knowledge within a specified grade band and 2) extensive knowledge and experiences within the chosen grade band or specialisation (MoE, 2017). The CoE train teachers to teach from KG 1 to JHS 3 in Ghana. With the upgrade, all the CoE in Ghana has been affiliated with the five (5) public Universities in Ghana, namely, the University of Cape Coast, the University of Ghana, the University of Education, the University for Development Studies and Kwame Nkrumah University of Science and Technology. The passage of the Colleges of Education Act 2012, Act 847, has provided legal backing to their new elevated status. The CoE are now under the Ghana Tertiary Education Commission (GTEC), which is a government body responsible for the regulation of tertiary education institutions in Ghana by Act 1023.

3. Problem Statement

Mathematics continues to be regarded as the most challenging subject in the curriculum in Ghana, according to research by Eshun (2004) and Eshun-Famiyeh (2005); this general perception is reflected in students' performance. Recent studies say students perform poorly in Mathematics in Ghanaian schools (Fletcher, 2018; Hagan et al., 2020). It has been reported that only 6% of primary school learners attained the desired standard for numeracy in the results of the Early Grade Literacy and Mathematics Assessment (EGLMA) (MoE, 2019). The results from the Criterion Reference Tests (CRT) conducted by the Primary Education Programme (PREP) of the Ministry of Education with the support of the United States Agency for International Development (USAID) for primary six learners from 1992 to 1996 showed that fewer than 10% of the pupils had the standard score in mathematics and that scores rose steadily beginning from the base year of 1992 by 0.8% with the intervention from USAID (CRDD, 2001). However, not much has been done in research to explore how mathematics is taught at the early grade stage in Ghana. Specifically, there appears to be an apparent lack of literature comparing the national kindergarten mathematics curriculum and the curriculum used for training pre-service teachers pursuing early-grade programmes in Ghana who will eventually be posted to teach in these kindergarten schools. This research provides a comparative analysis of the two curricula.

4. Methodology

A qualitative content analysis was used to compare the mathematics curriculum of the Colleges of Education (B.Ed. Early Grade) mathematics curriculum and the national KG curriculum. According to Wallen and Fraenkel (2001), content analysis examines document contents, whether textual or visual. In addition, Graneheim et al. (2017) argued that content analysis should serve a meaningful function in research, contributing vital knowledge to the subject of study or generating information beneficial in assessing and improving social or educational activities. It is a technique for objectively extracting the characteristics of the information from a document's content. The analysis was done with cognizance of the following research questions: Is there any difference between the objectives and course description of the Colleges of Education (B.Ed. Early Grade) mathematics curriculum and the national KG mathematics curriculum? In what ways do the contents and pedagogical approaches of the curricula for both CoE and national KG curricula differ? The content analysis was done using the Kindergarten Curriculum for preschools developed by the National Council for Curriculum and Assessment (NaCCA) under the Ministry of Education and the four-year Bachelor of Education Degree Eight Semester Initial Teacher Education Curriculum (B.Ed. Early Grade program) developed by four universities in Ghana – namely, University of Ghana, University of Education, University for Development Studies and Kwame Nkrumah University of Science and Technology – with their affiliated Colleges of Education in Ghana. The objectives, course description, contents and pedagogical approaches of both curricula were examined and analysed qualitatively.

5. Results

5.1 Objectives of the CoE Early Grade Curriculum and the national KG curriculum

In relation to the objectives, the CoE B.Ed. Early Grade Curriculum had no specific objectives for mathematics; the objective was general for all subjects. The National KG curriculum had a specific objective for mathematics. In common for both curricula is the emphasis on the use of ICT, creativity and critical thinking skills. Table 1 shows the objectives of both curricula.

Table 1. Objectives of the CoE and the National KG curricula

The objectives of the CoE B.Ed. Early Grade Curriculum	The objectives of the national KG Curriculum
The curriculum is designed to prepare teachers who:	The overriding aim for the KG integrated thematic curriculum is to promote early literacy and numeracy as well as the requisite social skills that equip young learners with effective foundational language, literacy and numeracy to enable them to do the following:
Are equipped with professional skills, attitudes and values, secure content knowledge as well as the spirit of enquiry, innovation and creativity that will enable them to adapt to changing conditions, use inclusive teaching strategies, engage in life-long learning and demonstrate honesty, integrity and good citizenship in all they do.	to acquire the six essential skills in language and literacy (phonemic awareness, concept of print, alphabetic knowledge and phonics, vocabulary, comprehension, fluency) and use them effectively in their everyday reading and writing activities, i.e. to communicate orally and read fluently with understanding in both the Ghanaian languages and English and also be able to write.
Understand the subject, pedagogy and progress in learning across specialism areas, and promote critical thinking, problem-solving, and communication through the learning environment they create.	to develop essential numeracy (counting, basic number operations, shapes, data collection) and generic and analytical skills that would enable them to solve their everyday mathematical problems.
Know how to use ICT; have technology and information literacy and are able to integrate technology into teaching,	to develop the spirit of curiosity, creativity, innovation and critical thinking for understanding and developing themselves and their local and global environment.
Source: University of Ghana 2018 B.Ed. Early Grade Curriculum	Source: Republic of Ghana 2019 KG Curriculum

5.2. Course Description and Learning Outcomes of the Two Curricula

Another area of comparison is subjects/course description and learning outcomes. Here, a comparison was done to find out whether detailed descriptions were given to the courses/subjects to be studied. Regarding course description, the national KG curriculum contained no description of topics relating to mathematics. This makes it difficult to ascertain the specific content of mathematics that is required to be taught. For example, in the national KG curriculum, the topic for sub-strand 2 for KG I, Term 1, where mathematics is first introduced, is 'The Parts of the Human Body and their Functions'. Under this sub-strand, teachers are expected to teach the four subjects in the curriculum (Language and Literacy; Numeracy [mathematics]; Creative Arts; and Our World and our People) using the human parts as exemplar. A description was not provided about the topics or areas of the subjects that are to be taught. Instructions were only given about the pedagogy approach. In the case of mathematics, the following instruction was stated;

> Count the number of the names of the body parts in songs through clapping on the rhythm. Learners sing three different songs, clap to the rhythm and count the number of parts they hear in the songs. E.g., My head, my shoulder, my knee (3 names). Help them understand that the last number names are the number of objects or items counted (MoE, 2019, p. 4).

Although this suggests that teachers are to teach counting numbers using the part of the body as an exemplar, the topic (Numbers) is also not stated explicitly in the curriculum. The CoEs B.Ed. Early Grade Curriculum is different as it includes descriptions for all the courses in mathematics content. Numbers and Algebra, for instance, have the following as course descriptions and learning outcomes clearly stated in the curriculum; specifically, for numeracy (Mathematics), there is the need to do auditing of subject knowledge to establish and address student teachers' learning needs, perceptions and misconceptions in Numbers and Algebra. Knowledge, skills and understanding of the fundamental concepts of Numbers and Algebra, as well as the ability to identify one's characteristics (culture, ethnicity, religion, family constellation, socio-economic background, disability), can lead to a student teacher's ability to apply these two areas of mathematics in patterning, generalisation and algebraic reasoning in reminding the student teachers of the role of deductive reasoning in developing mathematical ideas. Topics in Number and Algebra include

> recognising and developing patterns, using numbers and number operations, properties of numbers, the concept of sets, number bases and modulo arithmetic, and algebraic expressions. In addition, student teachers will explore operations on algebraic expressions and apply mathematical properties to algebraic equations and functions. Using many examples of different local and global contexts, student teachers will solve mathematical problems using equations, graphs and tables to investigate linear and quadratic relationships (MoE, 2019, p. 17).

From the above, a description is given of the course and the various topics within numbers and algebra that need to be taught in the CoEs (B.Ed. Early Grade) curriculum. However, this is not explicit in the national KG curriculum, which makes it difficult to understand the mathematics content that the teacher is required to teach the KG learners.

5.3. Comparison of the Content of the CoE and the National KG Curricula

The study also ascertained whether the nature and contents of the national KG curriculum are in line with or differ from the CoE (B.Ed. Early Grade Curriculum). The pedagogical approaches of the two curricula were also examined. Table 2 shows the content of both curricula.

Table 2. Content of the CoE and the national KG curricula

Content of the B.Ed. Early Grade Curriculum	Content of the KG Curriculum
Year 1 has the following courses: ▪ Introduction to learning and applying numbers and algebra ▪ Learning, teaching and applying geometry and handling data Year 2: ▪ Theories in the learning of numeracy in early grade ▪ Teaching and assessing numeracy for early grade ▪ Year 3: ▪ Teaching and assessing numeracy ii for early grade	KG 1 and KG2 have the following scope of content. ▪ Number ▪ Algebra ▪ Geometry and Measurement ▪ Handling data
Source: University of Ghana 2018 B.Ed. Early Grade Curriculum	Source: Republic of Ghana 2019 KG Curriculum

From the analysis of the two curricula, the contents have the same scope, that is numbers, algebra, geometry, measurement and handling data. This provides consistency in the content of the two curricula, making the B.Ed. Early Grade Curriculum a reflection of the national KG curriculum. It can be deduced that the B.Ed. Early Grade Curriculum is meant to produce teachers who will be well-trained to master the subject matter knowledge in areas such as numbers, algebra, geometry and handling data.

Table 3. Comparison of the pedagogical approaches of the CoE and the National KG curricula

The pedagogical approaches for the CoE B.Ed early grade curriculum	The pedagogical approaches for the national KG Curriculum
Interactive pedagogy: Student teachers will be prepared to base the pedagogy they use on the social constructivist view, which sees teacher education as the co-construction of knowledge. They will be able to use differentiated instruction and assessment strategies.	The curriculum emphasises: that the use of relevant active play-based methods in the curriculum delivery will be paramount as research has established that learners learn better through play.
Pedagogical Knowledge, including general pedagogical knowledge, assessment strategies, introduction to and development of cross-cutting issues, education studies, preparation for supported teaching in school, classroom enquiry and research, Inclusion and equity, SEN and ICT.	A thematic integrated method will be used to integrate experiences from the various learning areas, as research indicates a child's brain is not compartmentalised. Subject teaching should, therefore, not be used at the kindergarten level. ▪ the positioning of inclusion and equity at the centre of quality teaching and learning. ▪ the use of differentiation and scaffolding as teaching and learning strategies ensures that no learner is left behind. ▪ the use of Information Communications Technology (ICT) as a pedagogical tool ▪ the integration of assessment into the teaching and learning processes as an accountability strategy
Source: University of Ghana 2018 B.Ed. Early Grade Curriculum	Source: Republic of Ghana 2019 KG Curriculum

Table 3 shows some similarities in the stated pedagogical approaches for both curricula. The use of ICT as a pedagogical tool, differentiated instruction, inclusion of learners with special educational needs (SEN), equity and creation of learner-centred classroom learning are areas where there appears to be agreement in the pedagogical approaches of the two curricula. Although the curricula appear to be in sync with each other, there are variations in some aspects of their pedagogical approaches. Generally, the KG national curriculum is play and activity-based, which is clearly seen in how the mathematical play activity should be conducted in the classroom, spelling out the role of the teacher and the learners. For example, in teaching the concept of addition and subtraction (algebra) to KG 2 learners, the pedagogical approach stated in the curriculum was entirely play and activity-based. The teaching approach was stated in the KG curriculum as follows:

> Prepare a shopping list, and use the money to shop for ingredients for the festival's special meal. Exemplar: Learners apply the concept of addition and subtraction as they use real money to go shopping for some essential ingredients in the classroom store. Count the number of people in the family and buy enough food for them. Solve addition and subtraction word problems during the week (MoE, 2019, p. 12).

This gives room for the learners to come out with creative and innovative thinking while the teacher guides them through scaffolding. Although play as a teaching approach is briefly mentioned in the content of the B.Ed. Early Grade Curriculum. For example, under the Numbers and Numeration in the B.Ed, the pedagogical approach is to be play-based. However, it is stated very briefly without detailing how the play should be initiated. The learning activity for the topic is stated

> as using various collaborative activities, including think pair, share, group work and role play, that will lead to the development of the numeration system (MoE, 2019, p. 3).

6. Discussion and Conclusion

The content of both curricula is related. Both curricula had a similar scope of content, including numbers, algebra, geometry, measurement, and handling data. The pedagogical approaches such as problem-solving approaches, scaffolding, creativity, and play-based activities were inculcated into teaching mathematics. The problem-solving approach gives students numerous opportunities to connect mathematical ideas and develop conceptual understanding, according to Suurtamm, Quigley, and Lazarus (2015). Sinay and Nahornick (2016), who examined ways to make space for students to think mathematically, recommended that it is important to present problems in the mathematics classroom that are complex and rich, allowing for multiple entry points, different approaches, scaffolding, and engagement without imposed procedural steps. This problem-solving approach is well integrated into the KG and B.E.d curricula, which will help learners use mathematical concepts to solve real-world problems.

Scaffolding was also another pedagogical approach that was found in both curricula. Researchers have described instructional scaffolding as the cornerstone to assisting struggling learners in accessing the core curriculum (Coyne, Kame'enui, & Carnine, 2011). To determine the amount of instructional scaffolding to provide during an instructional task, Clarke et al. (2015) recommended that teachers consider whether learners have the background knowledge required to accomplish the task. They further stated that in situations where learners are less prepared or the task is complex or novel, teachers would have to provide greater support to engage learners in key mathematics content deeply (Clarke et al., 2015). This is important because both the KG and B.E.d curricula emphasise the inclusion of learners with mathematics learning disabilities. The inclusion of a scaffolding approach to instructional delivery will enable teachers to meet the learning needs of all learners. The play-based instructional approach took the centre of the two curricula. Although the CoE B.E.d early grade curriculum did not give a detailed description of how play-based instruction should be delivered, it did state it as part of the teaching approach for all mathematics to be studied by initial teacher education students. The KG curricula, on the other hand, provided many details of how the play activities should be done. Play can be defined as activities that 'are fun, voluntary, flexible, involve active engagement, have no extrinsic goals, involve active engagement of the child, and often have an element of make-believe' (Weisberg et al., 2013). Gasteiger (2015) recounts that early learning needs should be based on play. He added that innovative approaches to early mathematics should not only be developmentally adequate and effective but also compatible with kindergarten pedagogy, which is play-based (Gasteiger, 2015). As kindergarten learners are highly motivated to learn, but not in a formal, instructional way, play can be regarded as a powerful vehicle for learning (Hauser, 2005). Since both curricula are restructured, it is recommended that the KG mathematics curriculum be integrated into the CoE early-grade mathematics curriculum so that pre-service teachers will have an idea of exactly what they will experience in practice.

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Agnieszka Bojarska-Sokołowska
University of Warmia and Mazury in Olsztyn

Chapter 16

THE USE OF AN INTERACTIVE FORM OF CLASSES TO MOTIVATE PRE-SERVICE TEACHERS OF EARLY CHILDHOOD EDUCATION TO SOLVE MATHEMATICAL PROBLEMS

Summary: The study presents a way of motivating students majoring in early childhood education to discover and solve mathematical problems on their own. It was carried out in the form of interactive activities, during which the learners solved mathematical problems on individual thematic "stations". While learning, they could be helped through object manipulation. This article presents selected problems and aspects from the conducted research. The study was carried out in the form of an action research procedure. The research results were described in four scopes of the researcher's activities: preparation of the learning environment (research organisation), observation of the respondents, communication with the students (interviews with the students), and application of the results.

Keywords: mathematical education, prospective early childhood teachers, interactive learning, motivating pre-service teachers.

1. Introduction

To carry out this research, I was prompted by the attitudes of early childhood education students (i.e., students majoring in early childhood education) towards mathematics education activities. From the statements of most of the students, it would seem that they do not like mathematics and undervalue their own 'strengths' and abilities when solving mathematical problems. This is likely related to the difficulties they had in mathematics at school and the bad memories of learning the subject. These fears and attitudes towards school mathematics are also confirmed by research conducted on middle school students by researchers Oszwa and Szablowska: "Many students believe that they will never manage to understand mathematics, at most they are able to learn it in such a way as to give the illusion that they understand it" (Oszwa & Szablowska, 2018, p. 70). The authors also point out that maths anxiety lowers students' self-esteem (after Oszwa, 2018, p. 76). Two types of maths anxiety have been identified in the literature. The first is "caused by a mental block in the process of learning mathematics: it refers to various triggers, such as symbols or concepts occurring during mathematics learning" (after Oszwa, 2018, p. 75). This is because the "process of learning mathematics has its own specificity – learning mathematics requires systematicity and patience, and mathematical knowledge is cumulative" (Baczko-Dombi, 2017, p. 39). The second type of anxiety is the result of sociocultural influence: "It appears as a consequence of prevailing cultural beliefs about mathematics. Some parents even make excuses for their children, saying 'I couldn't do maths either and I always had problems with it'". Similarly, W. Sawyer believed, writing: "The fear of mathematics is a tradition handed down from generation to generation from the times when most teachers knew little about human nature and nothing about the nature of mathematics itself" (Sawyer 1988, p. 8).

One may ask whether university students can be motivated to solve mathematical tasks, and if so, how this can be achieved.

The terms "motivation" and "motivating" come from the Latin word movere and means to move, to set in motion, to encourage someone to do something, to stimulate (Gasiul, 2007, p. 222). According to Brophy, motivating pupils means finding ways with which the teacher can encourage them to accept the goals of their work and learn knowledge and skills, regardless of whether this activity gives them pleasure and whether they would undertake it if they did not have to (Brophy 2007, p. 14). Okoń distinguished between two types of motivation: intrinsic and extrinsic. Intrinsic motivation stimulates action by having intrinsic value; an example of this is an interest or love for something. Extrinsic motivation, on the other hand, creates an incentive to act that is rewarded in some way or that avoids punishment (Okoń, 2005, p. 178)". G. Pettie distinguished between long-term and short-term motivating factors. Among the former, the author included: "What I am learning is useful and the qualifications I gain from learning are needed". Among the short-term ones, on the other hand: "I am usually successful at school and it improves my self-esteem"; "I will gain acceptance from teachers and peers if I learn well"; "I expect the consequences of neglecting learning to be unpleasant (and I will experience them very quickly)"; "What I am learning is interesting and arouses my curiosity"; and "Learning activities are enjoyable" (Petty, 2015, pp. 50–52).

Ideally, a person should be intrinsically motivated to learn a subject and thus satisfy their natural curiosity and delight in the world. However, it is not always possible to get a student to be interested in a subject, in which case it would be good if at least the teacher's activities were interesting, i.e., innovative, fun, light-hearted, encouraging self-expression, or creative thinking.

Researchers Getzels and Thelen, who looked at classroom life as a social system, also saw the communicative aspect of personal learning: "The main determinants of students' behaviour are their personal needs and individual interests brought into the school and into the social group that develops as the interactions between students and with the teacher increase" (Arends 1998, p. 135).

Morańska draws attention to the fact that "one of the most important factors determining the educational results achieved is stimulating the motivation to learn". The researcher also adds that "nowadays in academic didactics not only 'What?', i.e., the content of education, but also 'How?', i.e., the way it is presented, determines the effectiveness of the educational process. On the other hand, the most common questions posed by the students are 'Why am I learning this?', 'What is the point?'. Considering the answers to the mentioned questions is a key prerequisite for planning learning and the lecturer's arrangement of learning situations" (Moranska, 2019, p. 62).

2. Interactive Form of Activities to Support Pupil / Student Motivation

Nowadays, various forms and methods are being sought to motivate and enhance the effectiveness of pupils'/students' learning. One of these may be the form of interactive learning presented below. Its origins can be traced back to 19th-century museums with displays, private collections and offices opened to the public for the purpose of introducing them to the achievements of science (Kruk, 2005, p. 195). The concept of an interactive way of learning was born in museums, science centres, which, adapting to the modern audience, transformed themselves from purely collectors' establishments into institutions for education and experimentation – "Don't watch, interfere!" in the words of Hacking (after Karwasz & Kruk, 2012, p. 19). Interactive museums/exhibitions attract visitors through innovative ways of creating exhibitions, which can include six elements: viewer-centredness, narrative, emphasis on education, interactivity, freedom of interpretation and multi-perspective, variety of media. The central place in the exhibition is occupied by the object, through which teaching-learning takes place. The exhibits provide the viewer with diverse access to the topic, the issue, while influencing the visitor's interests in accordance to his or her knowledge. The content of the exhibits can be discovered by the visitor according to his or her own predisposition and ability.

"Interactivity" (from the Latin "interactus" – mutual act) is the "ability of communicating parties to interact with each other"[1]. This word consists of two elements: activity and cooperation/communication. It involves two aspects. The first relates to the active conception of learning (Aeblie), i.e., the research activity of a learner who learns and constructs knowledge (Karwasz & Kruk, 2012, p. 16). The second aspect relates to communi-

[1] https://pl.glosbe.com/pl/pl/interaktywno%C5%9B%C4%87, accessed 13.06.2018.

cation: "learners intensify cognitive processes by interacting with each other, age does not play a role in this case" (Karwasz & Kruk, 2012, p. 17). Kruk, the creator of interactive didactics, points out that interactivity should not be limited only to relationships with other people, but also with objects. The researcher defines interactivity in education as a "relationship with one's object of attention during which the object is shaped as an object of perception with an extended meaning, as the process of interaction proceeds" (Karwasz & Kruk, 2012, p. 17). The visitor's perception, knowledge and experience are the result of a number of factors that determine the final interpretation and the way the interaction carries out during the exhibition. This process is outlined by J. Kruk in four steps: "The viewer's experience – What am I dealing with? What do I want to understand? The exhibit – What is the object? What message does it contain? Perception (interaction) with the exhibit – Reconstruction of meaning. Knowledge construction – Interpretation of the message" (Karwasz et al., 2011).

The form of classes described above was used to create such conditions for students while learning mathematics so that they could discover and get to know mathematics from a slightly different perspective, i.e., as a field of knowledge useful in life, and at the same time quite interesting and, above all, with enough effort, possible to understand and providing the satisfaction of solving it on one's own. When constructing the research tool, i.e., the topic stations, care was taken to ensure that the mathematical skills to be applied were not difficult and the mathematical problems presented were not "boring school knowledge"; moreover, the topics of the problems were selected in such a way that they could be used by the students in further studies or future work.

3. Methodological Basis of the Research

Action research can be used to improve the quality of education. It aims to change the practice by combining learning with the changing of educational practice. Situational context is important in this research, as the solution to the problem relates to this context. Knowledge gained from research is meant to allow for a change in practice (Czerepaniak-Walczak, 2010, p. 325). In my study, I focused on preparing the implementation of classes on mathematics education, communicating, and making use of the results to introduce changes into educational practice. Organising the described forms of conducting mathematics education classes allowed to develop and conduct classes for early childhood pupils in mathematics education, in an interactive form, by the students participating in the research. However, after supplementing them with additional issues, they will be used to write and publish a script with lectures and exercises in the subject of Didactics of Mathematics for students of early childhood education. I consult my observations and comments on an ongoing basis with professional early childhood education teachers and researchers conducting their research in mathematics education.

As part of the undertaken research, I formulated the research question: Does the interactive form of classes motivate early childhood education students to solve mathematical problems independently? The subject of the research involved task solving and the opinions of the surveyed students. The determinants of the motivation were the number of students who attempted to solve the given problems, the correctness of the solutions, as well as the opinion of the students about what they learned, what they managed to do, what interested them, what more they would like to know about a particular topic and what they paid attention to while dealing with a particular issue. In the study, I used the free observation method, school achievement tests (tool – worksheets, at particular workstations) and the diagnostic survey method (technique used – categorised interview).

The research was conducted in three series of topics, and a selection of examples from fifteen conducted in three series is presented below. The research was conducted on 92 early childhood education students (53 part-time students, 29 full-time students). Each was divided into smaller groups so that everyone could move freely through the prepared stations. Each session for all groups lasted 100 minutes. At each stand (a given thematic issue), in addition to the objects to be manipulated, a real-life application of the issue was presented. This was intended to make the students more curious about the particular issue. In addition, each proposed topic could, if presented and described appropriately, be used to guide mathematics activities for children in grades 1-3.

The table shows the problem topics in each of the three study series.

Table 1. Problem topics in the three study series

First study series	Second study series	Third study series
Station 1: Tying a tie	Station 1: Taxi metrics	Station 1: Sudoku
Station 2: Laying parquet	Station 2: Fibonacci sequence	Station 2: Mobius strip
Station 3: Decoding a suitcase	Station 3: Credit cards, bar strips	Station 3: Cube grid
Station 4: The shortest route	Station 4: Logic puzzles	Station 4: Fractals
Station 5: A game of roulette	Station 5: Mathematics in art	Station 5: Squares

Tables 2 and 3 present the students' skills in particular stations as shown in already published research (Bojarska-Sokołowska, 2019; 2022a; 2022b).

Table 2. Description of the mathematical skills of the students during their work on problems from a particular workstation, from the first series of the study

Station 1: Tying a tie	The student is able to read the symbolic notation of the successive actions to be performed and to write down symbolically the algorithm of the observed tie being tied. He/she is able to create his/her own algorithm, taking into account the conditions that allow for correct execution. Can arrive at the discovery of the rules for tying a tie correctly.
Station 2: Laying parquet	Student knows how to fill out a rectangle, with different types of shapes, predicts filling, uses symmetry, translation, rotation. Developing flat geometric imagination.
Station 3: Decoding the suitcase	The student can read numbers written in binary and ternary into decimal and vice versa. In addition, he or she can solve a logic puzzle involving manipulating the jockeys in such a way as to place them on horses without cutting or bending them.
Station 4: The shortest route	The student knows how to draw a broken line, measure and estimate length results, calculate route lengths.
Station 5: Roulette game	The student performs monetary calculations, learns to manage his/her funds. Anticipates and chooses the likeliest hands, analyses and interprets random situations on an ongoing basis. In addition, he/she realises that "good fortune" does not last long – the law of large numbers.

Source: own study.

Table 3. Description of the mathematical skills of the students during their work on problems from a particular workstation, from the second series of the study

Station 1: taxi metrics	The student knows how to determine different broken lines with given properties and lengths. The student finds all possibilities that satisfy the conditions of the task.
Station 2: Fibonacci sequence	The student knows how to fill out a rectangle, with squares of different sizes, and to calculate its area in two ways. The student finds the rule for finding consecutive numbers in a sequence.
Station 3: credit cards, barcodes	The student knows how to apply the given algorithm to calculations. The student knows the terms: even number, odd number, remainder from dividing by 10. The student knows how to solve an equation.
Station 4: logic puzzles	The student is able to diagrammatically record the steps of individual boat crossings.
Station 5: mathematics in art	The student knows how to encode and decode images.

Source: own study.

In the remainder, I describe the research results for four out of the 15 problems found in three series of surveys, presenting the research tool and briefly describing the results from the surveys.

In the first series of the study, in Station 1, the students tied a tie. At the beginning, an explanation was provided on how to symbolically write down the example steps of tying a tie.

Tying the tie can be done according to the algorithm presented in the table, the moves are made according to the grey part of the tie.

Step 1. Left front-L(F)

Step 2. Right from behind-R(B)

Step 3. Left front-L(F)

Step 4. Centre below neck-C(N)

Step 5. Centre-C

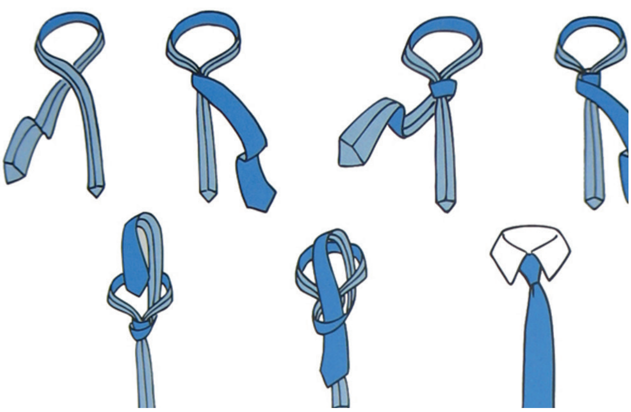
L(F) R(B) C(N), C-step designations

The algorithm for tying a tie in this way can be written in short as: L(F) R(B) L(F) C(N) C

Below are the three problems the pre-service teachers of early childhood education were asked to tackle.

Problem I.1.1. Tie a tie using the algorithm given above. Show the instructor.

Problem I.1.2. Create an algorithm to tie a Kelvin knot

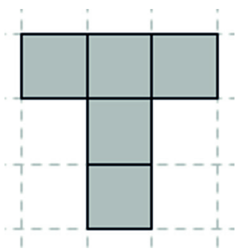


Problem I.1.3. Create your own algorithm for tying a tie. Write an arrangement of six steps/a sequence of letters. Following the rules, there must not be two of the same letter in a row in the tie knot sequence, e.g. L(R) L(R). The tie knot sequence must start with the letter L with a F-front or B-back/behind in parentheses. The tie node string must end with the sequence of letters RLC or LRC with letters in parentheses.

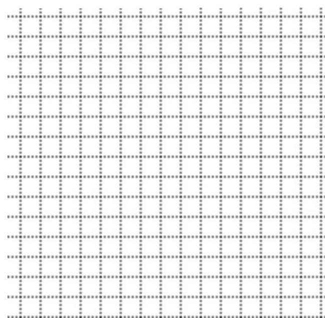
Alongside this exposition, in addition to the information concerning the scientific description of the process of tying a tie by two mathematicians, Thomas Fink and Young Mao, the rules the algorithm must fulfil for one to be able to tie a tie are also given. In addition, several algorithms (out of 85 possible) for other methods of tying ties are shown.

In Station 2, students laid out and drew parquet floors. They had the following three problems to solve:

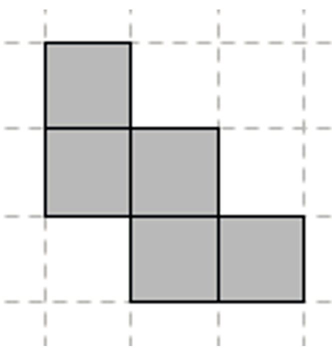
Problem I.2.1. Draw a parquet floor made up of type blocks only.



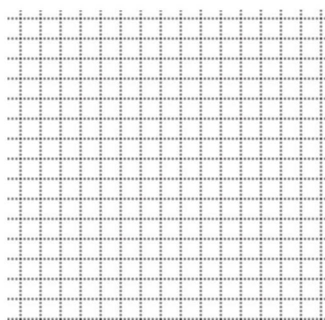
Remark. You can help yourself by manipulating the blocks in this shape.



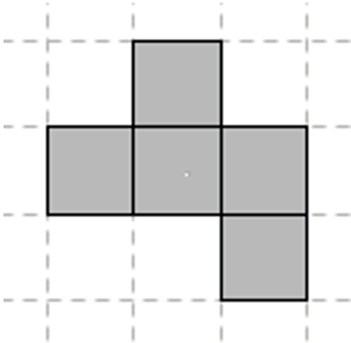
Problem I.2.2. Draw a parquet floor made up of only blocks of the following type:



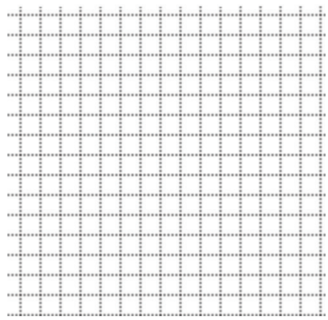
Remark. You can help yourself by manipulating the blocks in this shape.



Problem I.2.3. Draw a parquet floor made up of only blocks of the following type:



Remark. You can help yourself by manipulating the blocks in this shape.



In the theoretical part, parquetry was referenced. The principles for the construction of parquets and their types, e.g., Platonic parquets, Archimedean parquets, etc. are presented, also describing the principles for their creation.

In Station 1 of the second series of the study, students explored the properties of the "taxi metric"[2]. Prior to the introduction of the tasks, a necessary explanation of the comparison between Euclidean and taxi metrics was provided.

Look at the drawings.

Figure I

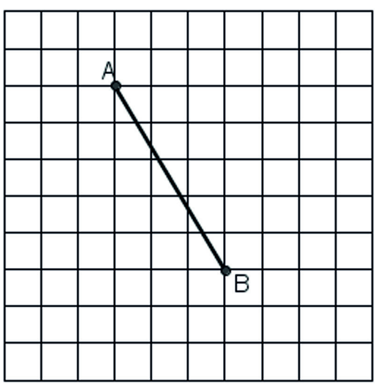
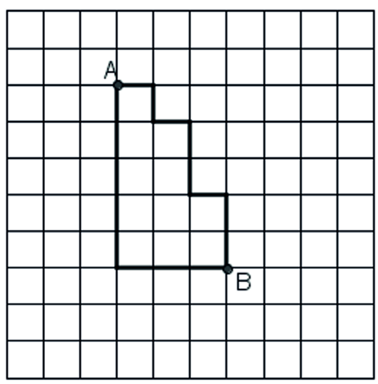


Figure II



Finding the shortest path from point A to point B is very simple (Figure I), it involves connecting these points with a straight line. However, we cannot do this in an urban real-

2 Compiled from the books: Gómez (2012, pp. 11–24); Moscovich (2009, p. 35); Alsina (2012).

ity, because this would mean that we are moving through buildings, bushes, etc., not pavements or streets. Therefore, the shortest route from A to B is 8 units (small grids)-Figure II. In this task we use the urban (taxi) metric.

Station 1 contained geoplans with points A and B, D and S marked on them (content of task 1-picture 1,2) and rubber bands. The students could use these objects while figuring out the solution of the two tasks.

Figure 1. Explanation on how the taxi metric differs from the Euclidean metric

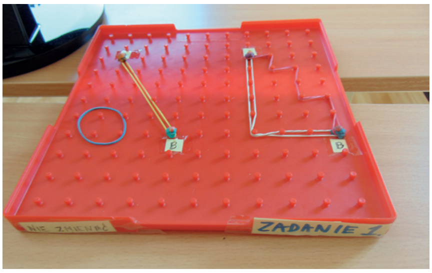
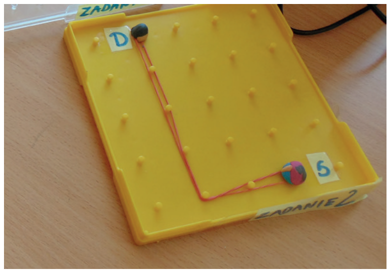
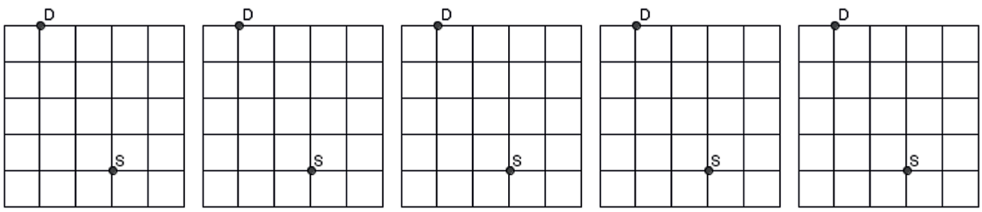


Figure 2. Geoplan to be manipulated while solving Problem 1



Problem II.1.1. Draw some of the shortest routes leading from point D to point S, in urban metric

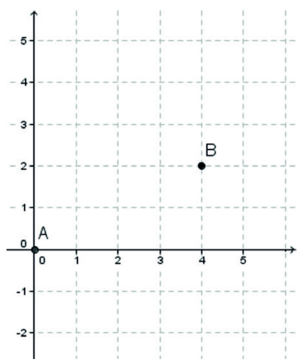


Problem 2.

Imagine that you have to build a highway in a city to connect two neighbourhoods. The most important places for the residents are points A and B. Your highway must also meet two conditions:

1. **Any vehicle travelling on the highway should have the same distance to A and to B.**
2. **As few buildings as possible must be demolished (the buildings are inside the grid squares).**

Draw the solution to the problem in the diagram.



With this display, in addition to information on the properties of the taxi metric, a formula is also included for calculating the amount of the shortest paths consisting of n steps upwards and m steps in one direction.

In Station 2 of the second series of the study, the students were asked to tackle the Fibonacci mystery of rabbit reproduction. The station provided a calendar with twelve months and rabbits for the students to use when solving Problem 1.

Problem II.2.1.

Face the Fibonacci mystery[3]:

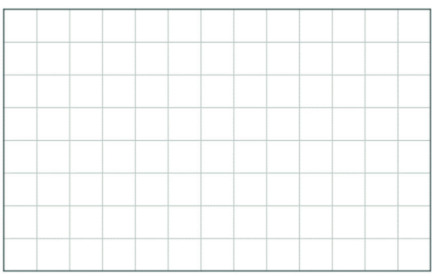
Calculate how many pairs of rabbits you will have after one year if:

- **each couple becomes fertile after 2 months,**
- **each couple gives birth to one new pair every month,**
- **the rabbits never die.**

Month number	**Number of pairs**	**Sum of pairs of rabbits**
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		

Problem II.2.2. had to do with Fibonacci numbers and concerned the golden rectangle:

Problem II.2.3.
We have a rectangle sized 13 cm by 8 cm. Draw squares with the largest possible area, then calculate the sum of the areas of these squares and them compare with the area of the starting rectangle (use centimetres as the unit).



[3] Compiled from the book by (Moscovich, 2009, pp. 43–45).

In addition to information on the Fibonacci sequence and the golden number, the theoretical section also includes their omnipresent occurrences in nature, architecture, etc.

4. Analysis of the Research Results

4.1. The First Series of the Study

In Station 1 of series one, the first task was solved by 53% of all students. All succeeded in tying the tie correctly. 85% managed to symbolically write down the algorithm shown in the drawings for tying a tie. 65% provided the correct sequence for tying the tie. No one managed to provide all of the rules of the algorithm for correctly tying a tie during the class. Some took the topic home with curiosity. From the students' statements, it was evident that they had learnt "to tie a tie according to my idea (S2)"[4], "to teach how to tie a friend's tie (N23)" at this station. The problem generated a lot of curiosity among the students: "that there are many ways to tie a tie and that an algorithm can be created from each of them (N45)", "that I was able to present my suggestions for tying a tie (S29)".

The second station in series one (photo 4) had 63% of all students working. 62% drew the floor correctly (without holes) for all four types of blocks. From the students' statements, it was clear that while working at this station they learned to "assemble geometric figures" (N21). Some people also mentioned what interested them while solving the problems at this station, the problem itself: 'that there are different combinations and you can create different parquet patterns (N36)' as well as the idea behind the task: 'the way the task is presented (N45)', 'that simple blocks can be arranged in many patterns (N11)'.

Figure 3. The parquet floor stand

Figure 4. Drawing parquet

Most of the students tried to assemble a parquet floor from the blocks that were available at the station and then draw on the card. However, there was no one (as among the middle

[4] Explanation of coding of responses: S-students of full-time studies, N-students of part-time studies.

school students) who would, for example, assemble a smaller piece from two or three blocks and then, through translations or rotations, fill in the whole plane around it.

The future teachers, however, showed more surprise and curiosity in comparison to the surveyed middle school students, both regarding the problem posed as well as the blocks they thought helped them arrive at a solution.

4.2. The Second Series of the Study

In station one of series two, the first task was solved by 53% of all students. All 15 possibilities in Problem 1 were drawn by 31%. The students either repeated roads already drawn or did not find all of them. Problem two was solved correctly by only 17%. Students found two different correct solutions to route this highway. From the respondents' statements, it was clear what they had learnt while working at this station: "finding the shortest routes (S2)[5]", "there are many ways to get to the destination (N23)", "there exists a taxi metric (N25)". Both full-time and part-time students noted what effort it takes to determine all the routes in the taxi metric, furthermore: "that you can always find the shortest route (N44)", "that you can do a lot of tasks on the geoplan (S18)", "you can adapt these tasks to the age of the children (N47)", "these types of tasks are interesting (N51)", "that it is great fun (S14)".

The teaching students' strategies of drawing the shortest paths in problem one did not feature the strategy of ordering these possibilities by way of coding, as in the solutions conducted on middle school students.

However, the surveyed female students showed, in relation to the pupils, more surprise and curiosity about both the problem posed and the blocks, which they thought helped them find all possible paths.

Station 2 in the second series of tests (photos 5 and 6) had 63% of students working. 42% correctly calculated the number of all pairs of rabbits after one year; they also managed to figure out how to find the consecutive numbers of the Fibonacci sequence. The most common errors were that the students did not include all pairs of fertile rabbits in a given month, taking only one, probably the one that was becoming fertile.

[5] Explanation of response coding, S-students full-time, N-students part-time.

Figure 5. Solving the task about rabbit reproduction

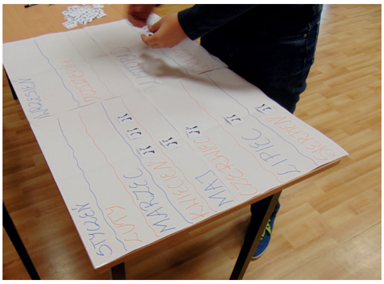


Figure 6. Solving the task about rabbit reproduction

56% of students solved task two correctly by calculating the area of the rectangle, by multiplying its dimensions and by adding the sums of the areas of the designated squares: 8 by 8, 5 by 5, 3 by 3, 2 by 2, and two with dimensions of 1 by 1. Those who failed only divided the rectangle into squares but did not calculate the areas. There was also a person who did not divide the rectangle into the largest possible squares, but into unit squares. Based on the students' statements, while working on problem one at this station, they had learnt that: "there is a Fibonacci sequence (N27)", "the rule of adding an increasing number to a previous number (S13)" and for problem two: "the area of a figure is equal to the sum of the areas of the other figures that fall within it (N21)".

In addition to motivating the students to try to solve mathematical problems, the form and topic of these activities were also intended to inspire the students to plan and carry out activities for children in Grades 1-3. And so it happened, two students used the inspiration of the problem of tying a tie to prepare an activity for the children involving tying different coloured strings according to an applied algorithm. Seven of the university students offered the children a station where they had to fill a piece of paper with different figures (different types of polygons), tracing a given type of block or checking to see if it could be done (circles, semicircles). The work also consisted of filling in in such a way that the symmetries of particular floors appeared. Two teaching students, inspired by the taxi metric problem, prepared an activity for children to determine the length of the perimeter of different rectangles without calculating them. Three, meanwhile, used rubber bands and a geoplan for the problem of finding different routes to get from one point to another. Two suggested a station where the children had to fill a given rectangle with different figures, i.e., different regular convex polygons, circles, and other shapes, e.g., star-shaped concave polygons. During these activities, the children were provided with sheets of A4 paper and cut out figures which they could outline on the sheets.

5. Conclusions of the Study

According to the research, the interactive form of learning evoked motivation to try to solve mathematical problems in the majority of the students. The students liked the fact that they could decide which problem, in which order, for how long, and with whom they could solve. In addition, they appreciated that they were praised even for small steps when solving, and for simply trying to solve the problems. After the class, they were still able to work on the solution e.g. at home, and to show off their results at the next class, when the individual tasks were discussed, without evaluating them (only the proposal and execution of the interactive activities for the students were evaluated). After the classes, the students described them as enjoyable and referred to their childhood memories. They added that mathematics was not that difficult so far. Similar statements from the respondents (secondary school students) are described by Boczko-Dombi, "they described their first interactions with mathematics as joyful, interesting, exciting" (Boczko-Dombi, 2017, p. 44). Moreover, it was evident from the students' statements that the proposals for solving the geometric problem surprised the respondents – "mathematics is not just about counting (S17)". The students also liked the form of the classes – "this form of exercise is interesting (N52)" and the fact that they use aids during the classes – "mathematical tasks are much simpler when we can assist ourselves with a model (N8)". In their statements, the students emphasised the emotional colouring, which Santrock also pointed out in his research. Writing that "students put more or less effort into learning depending on whether the teaching environment and the specific teaching situation are pleasant or unpleasant" (after Arends, 1998, p. 139). The students' suggestions for children's activities, following the interactive workshops, were mostly considered more inspiring than before in formulating interesting research problems for children.

In conclusion, one can quote Krajewska's words that skilful teaching today is active because it provides a conducive learning environment; opportunities; interactions; tasks that shape deep learning. (...) The concept of active learning often also refers to the use of group-based, collaborative learning, which particularly exposes the contribution that social interactions can make (Krajewska, 2021, p. 13). Shaping a social environment conducive to effective learning, an environment in which university students/pupils would manifest high motivation to solve mathematics tasks is a great challenge. Teachers' motivational competences play an important role in motivation, covering a wide range of tasks in organising the didactic process, taking into account the contemporary generations of university students/pupils being educated. As far as mathematics education is concerned, this requires further in-depth research.

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